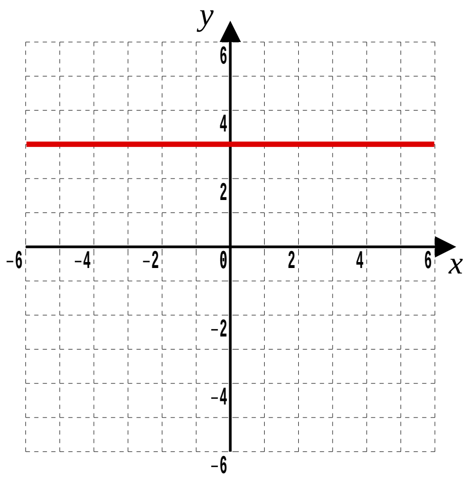
9.2 Answers to 9.1 Revision

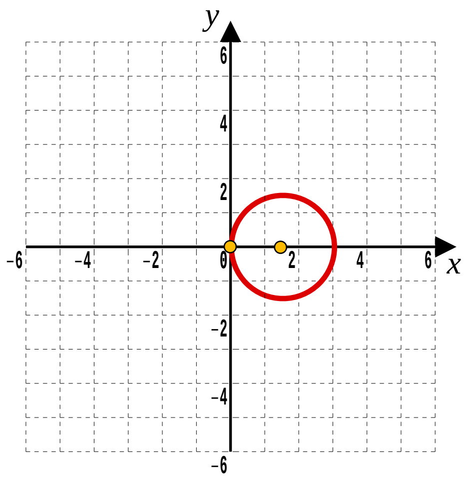
Further A-Level Pure Mathematics, Core 2

Polar Coordinates

Answer 1

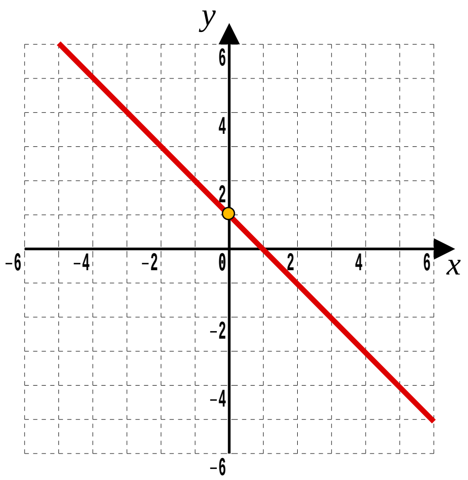


(i) $r = 3\, csc\, \theta$

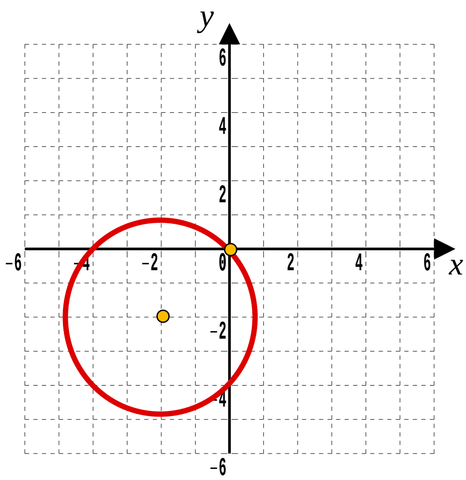


(ii) $r = 3 \cos \theta$

[2, 2 marks]



(iii) $r = \dfrac{2}{\cos \theta + 2 \sin \theta}$



(iv) $r = 4\sqrt{2} \sin (\theta - 45°)$

[2, 2 marks]

Answer 2

(i) The curve and the line will intersect when,

$$\left(\frac{3\sqrt{6}}{2}\,sec\,\theta\right)^2 = 9\,sec\,2\theta$$

$$\frac{9 \times 6}{4\,cos^2 x} = \frac{9}{cos\,2\theta}$$

$$\frac{cos\,2\theta}{cos^2\theta} = \frac{2}{3}$$

$$\frac{cos^2\theta - sin^2\theta}{cos^2\theta} = \frac{2}{3}$$

$$1 - tan^2\theta = \frac{2}{3}$$

$$tan^2\theta = \frac{1}{3}$$

$$tan\,\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \pm\frac{\pi}{6}$$

[4 marks]

(ii) Area $P = \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{6}} 9\,sec\,2\theta\,d\theta$

$$= \frac{9}{4}\int_0^{\frac{\pi}{6}} 2\,sec\,2\theta\,d\theta \quad \text{(Setting up a chain rule backwards)}$$

$$= \frac{9}{4}\left[\,ln\left|\,sec\,2\theta + tan\,2\theta\,\right|\,\right]_0^{\frac{\pi}{6}} \quad \text{(from formulae booklet)}$$

$$= \frac{9}{4}\left[\,ln\left|\,sec\left(\frac{2\pi}{6}\right) + tan\left(\frac{2\pi}{6}\right)\right| - ln\left|\,sec\,0 + tan\,0\,\right|\,\right]$$

$$= \frac{9}{4}\left[\,ln\left|\,\frac{1}{cos\left(\frac{\pi}{3}\right)} + tan\left(\frac{\pi}{3}\right)\right| - ln\left|\,\frac{1}{cos\,0} + tan\,0\,\right|\,\right]$$

$$= \frac{9}{4}\left[\,ln\left|\,2 + \sqrt{3}\,\right| - ln\,1\,\right]$$

$$= \frac{9}{4}ln\left(2 + \sqrt{3}\right) \quad \text{which is about 2.963...}$$

[5 marks]

(iii) When $\theta = \dfrac{\pi}{6}$, $r = \dfrac{3\sqrt{6}}{2} sec\left(\dfrac{\pi}{6}\right)$ (using equation of line)

$$= \frac{3\sqrt{2}\sqrt{3}}{2} \times \frac{2}{\sqrt{3}}$$

$$= 3\sqrt{2} \quad \text{(hypotenuse of triangle)}$$

$$\text{Area } \Delta = \frac{1}{2} a\, b\, sin\, C$$

$$= \frac{1}{2} \times \frac{3\sqrt{2}\sqrt{3}}{2} \times 3\sqrt{2} \times sin\left(\frac{\pi}{6}\right)$$

$$= \frac{9\sqrt{3}}{4}$$

$$\text{Area } Q = 2 \times (\text{ Area of right angled triangle } - \text{ Area } P\,)$$

$$= 2\left(\frac{9\sqrt{3}}{4} - \frac{9}{4} ln\left(2 + \sqrt{3}\,\right)\right)$$

$$= \frac{9}{2}\left(\sqrt{3} - ln\left(2 + \sqrt{3}\,\right)\right) \quad \text{about } 1.8679...$$

[4 marks]

Answer 3

To get the angles associated with the points of intersection,

$$(1 + \sin\theta) = 3(1 - \sin\theta)$$
$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

The same limits work on both curves,

$$= 2 \times \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \sin\theta)^2 - 9(1 - \sin\theta)^2 \, d\theta$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + 2\sin\theta + \sin^2\theta - 9 + 18\sin\theta - 9\sin^2\theta$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 20\sin\theta - 8 - 8\sin^2\theta \, d\theta$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 20\sin\theta - 8 - 8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$
$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 20\sin\theta - 12 + 2(2\cos 2\theta) \, d\theta$$
$$= \left[-20\cos\theta - 12\theta + 2\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= [0 - 6\pi + 0] - \left[-10\sqrt{3} - 2\pi + \sqrt{3}\right]$$
$$= 9\sqrt{3} - 4\pi$$

That is, $p = 9$ and $q = 4$

[9 marks]

Answer 4

(a) We require there to be four solutions to $\dfrac{dx}{d\theta} = 0$

$$x = r\cos\theta$$
$$= a(p + 2\cos\theta)\cos\theta$$
$$= ap\cos\theta + 2a\cos^2\theta$$
$$\frac{dx}{d\theta} = -ap\sin\theta - 4a\cos\theta\sin\theta$$

$$ap\sin\theta + 4a\cos\theta\sin\theta = 0$$

Divide through by a as we know $a \neq 0$

$$p\sin\theta + 4\sin\theta\cos\theta = 0$$
$$\sin\theta\,(p + 4\cos\theta) = 0$$

Either $\sin\theta = 0$ giving two solutions, $\theta = \{0, \pi\}$

or $(p + 4\cos\theta) = 0$

As there are four solutions $(p + 4\cos\theta) = 0$ must yield two more solutions

$$\text{so } \cos\theta = -\frac{p}{4}$$

Keeping in mind that p is a positive constant greater than 2 and

$$-1 \leqslant \cos\theta \leqslant 1$$

$$\text{gives } 2 < p < 4 \qquad \square$$

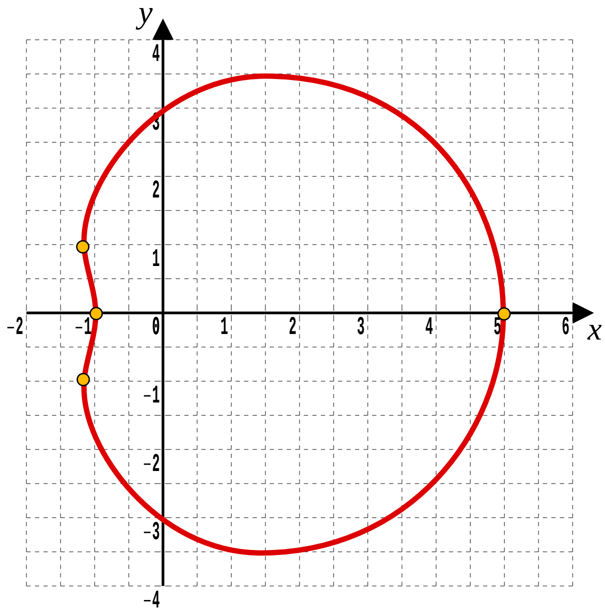
[5 marks]

Graph Plotting :

Out of interest, to plot the limaçon $r = 3 + 2\cos\theta$ on a graph plotter you may need to convert it into Cartesian form (the a is an enlargement scale factor);

$$r = 3 + 2\cos\theta$$
$$r^2 = 3r + 2r\cos\theta$$
$$x^2 + y^2 = 3\sqrt{x^2 + y^2} + 2x$$
$$x^2 + y^2 - 2x = 3\sqrt{x^2 + y^2}$$
$$\left(x^2 + y^2 - 2x\right)^2 = 9\left(x^2 + y^2\right)$$

(b)



Part (a) has provided hints on how to draw this limaçon.
Two of the four perpendicular tangents are when when $\theta = 0$ or π which is on the x-axis. Putting those values of θ into the curve's equation gives the Cartesian points (5, 0) and (– 1, 0).
With $p = 3$, the other two perpendicular tangents occur when,

$$\cos\theta = -\frac{3}{4} \text{ from which } \sin\theta = \pm\frac{\sqrt{7}}{4}$$

The associated points are given by,

$$(x, y) = (r\cos\theta, r\sin\theta)$$

$$\begin{aligned} x &= r\cos\theta \\ &= (3 + 2\cos\theta)\cos\theta \\ &= \left(3 - \frac{2 \times 3}{4}\right)\left(-\frac{3}{4}\right) \\ &= -\frac{9}{8} \\ &\quad (-1.125) \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= (3 + 2\cos\theta)\sin\theta \\ &= \left(3 - \frac{2 \times 3}{4}\right)\left(\pm\frac{\sqrt{7}}{4}\right) \\ &= \pm\frac{3\sqrt{7}}{8} \\ &\quad (\text{about } \pm 0.992...) \end{aligned}$$

All four of these points are plotted above and as there are no other points with a perpendicular gradient, the simplest curve through those four points is what is required.

[1 mark]

(c) Realising the shape has mirror symmetry in the x-axis suggests using limits of 0 and π and doubling the area found from that.

$$Area = 2 \times \frac{1}{2} \int_0^{\pi} (\, 20\,(\, 3 + 2\cos\theta\,))^2 \; d\theta$$

$$= \int_0^{\pi} 400\left(\, 9 + 12\cos\theta + 4\cos^2\theta\,\right) \, d\theta$$

$$= 400 \int_0^{\pi} 9 + 12\cos\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta \right) d\theta$$

$$= 400 \int_0^{\pi} 11 + 12\cos\theta + 2\cos 2\theta \, d\theta$$

$$= 400\left[\, 11\theta + 12\sin\theta + \sin 2\theta \,\right]_0^{\pi}$$

$$= 400\left[\, 11\pi + 0 + 0 - 0 - 0 - 0 \,\right]$$

$$= 4400\pi \ \text{cm}^2$$

So the volume is $4400\pi \times 90 = 396000\pi \ \text{cm}^3$

$$\text{Time} = \frac{396000\pi}{50000}$$

$= 25$ minutes (as requested, to the nearest minute)

[7 marks]

(d) In practice, John may not want the pond to be filled to the brim and even if he did, it would need to have a perfectly built with, for example, a perfectly horizontal brim.
Lots of other answers are possible and acceptable.
Anything sensible gets the mark.

[1 mark]