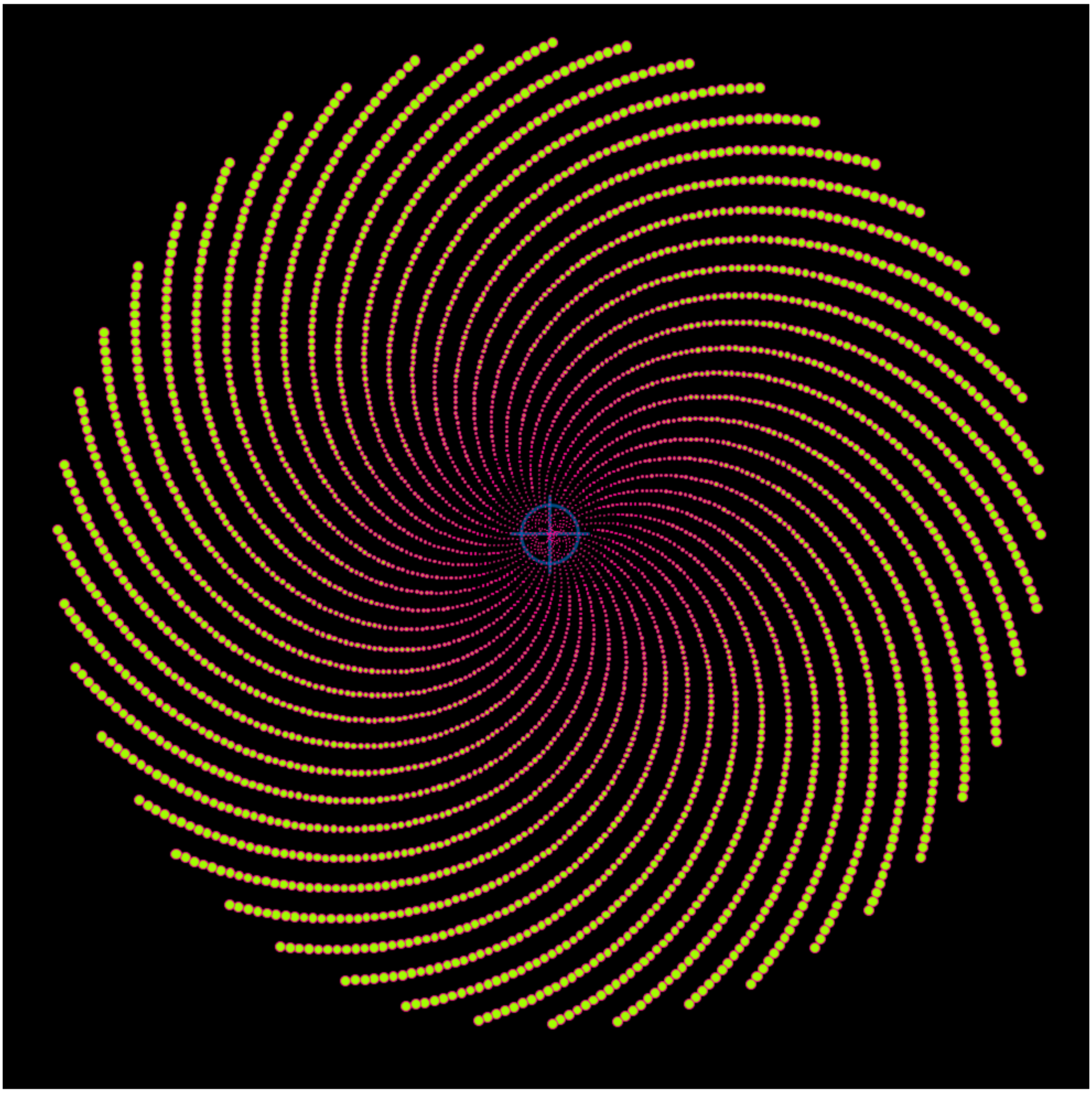
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

POLAR COORDINATES



Artwork made using Polar Coordinates

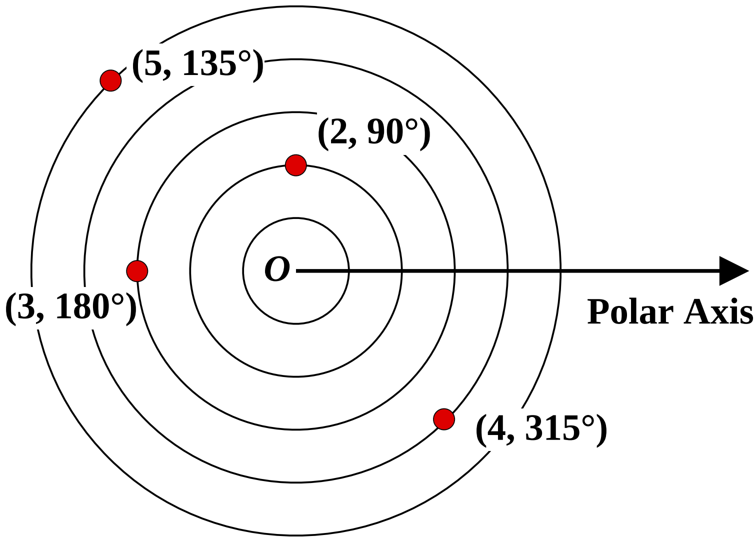
P O L A R   C O O R D I N A T E S

Lesson 1

Further A-Level Pure Mathematics, Core 2
Polar Coordinates

1.1 The Polar Coordinate System

The polar coordinates of a point describe its position in terms of a distance, r, from the origin, O, (called the "pole") and an angle, θ, measured anticlockwise from the polar axis. Usually the polar axis is in the same direction as the positive x-axis when using Cartesian coordinates.



The diagram shows four point along with their polar coordinates.
Here, degrees have been used but often radians are preferred.

1.2 Plotting a Polar Curve

On a Cartesian graph the points are of the form (x, y) and equations are formed with x and y in them, these then being graphed. On a Polar graph the points are of the form (r, θ). Mirroring what is done with the Cartesian system, polar equations can be formed with r and θ in them. And these can be graphed, but on polar graph paper, rather than Cartesian.

Here is a polar equation which is to be graphed.

$$r = 12\cos^2\theta - 4\sin\theta$$

To graph this polar equation, complete the table provided and then plot the polar coordinates obtained on the polar graph paper.

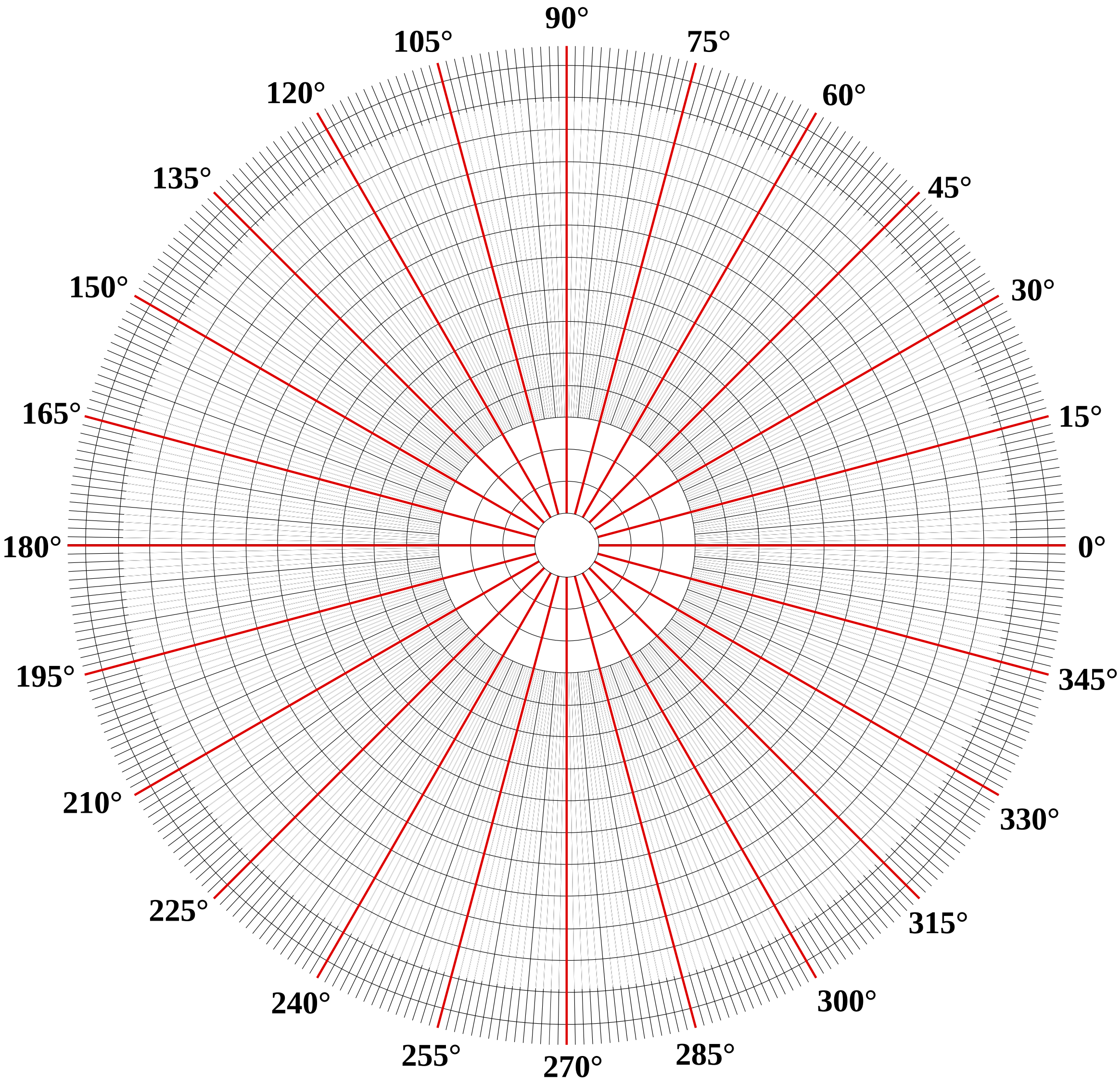
Work to 1 decimal place.

θ (in degrees)	0	15	30	45	60	75	90
$r = 12\cos^2\theta - 8\sin\theta$							

θ (in degrees)	105	120	135	150	165	180
$r = 12\cos^2\theta - 8\sin\theta$						

θ (in degrees)	195	210	225	240	255	270
$r = 12\cos^2\theta - 8\sin\theta$						

θ (in degrees)	285	300	315	330	345	360
$r = 12\cos^2\theta - 8\sin\theta$						



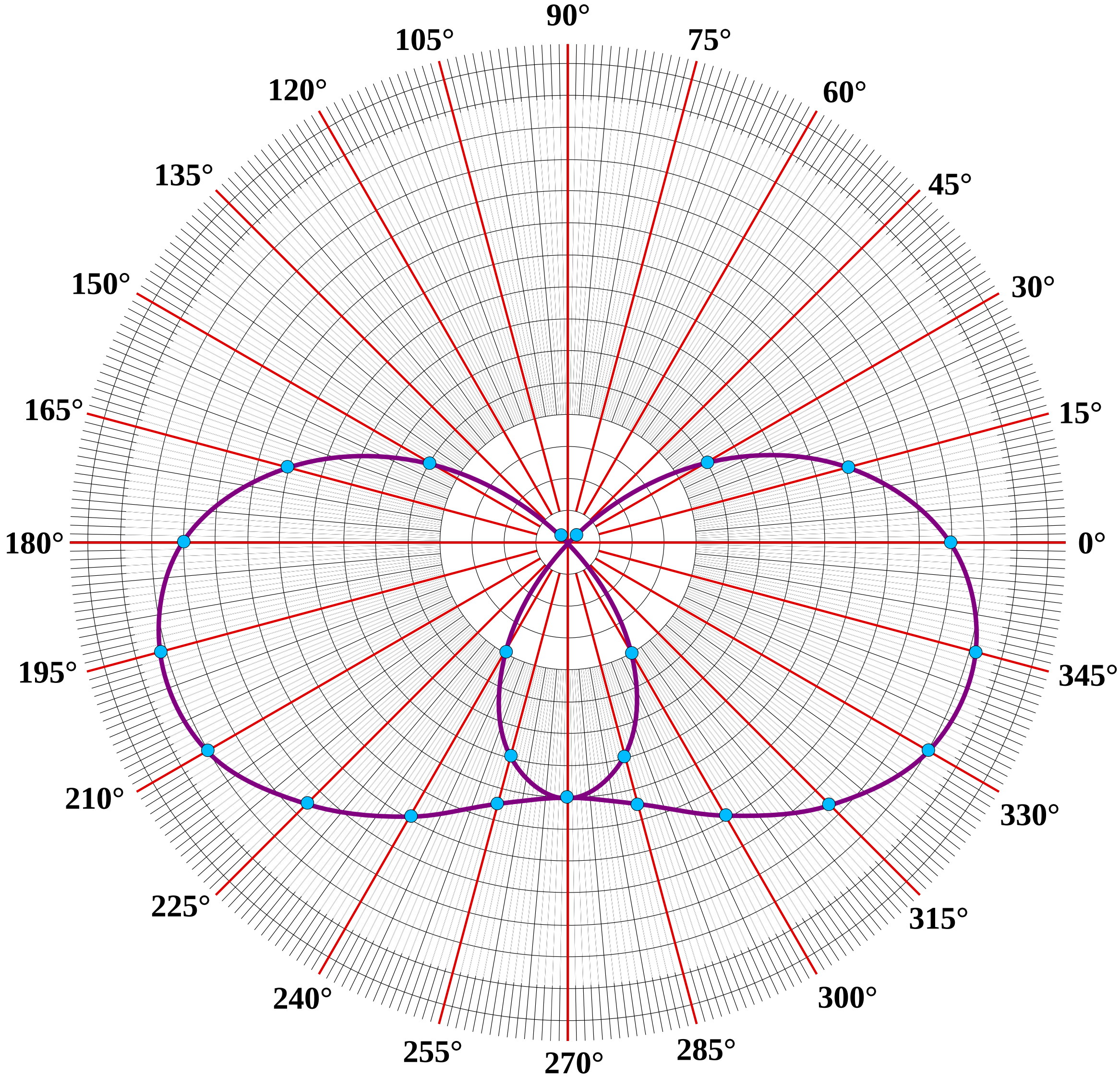
1.3 Answers to 1.2 Plotting a Polar Curve

θ (in degrees)	0	15	30	45	60	75	90
$r = 12\cos^2\theta - 8\sin\theta$	12	9.1	5	0.3	– 3.9	–6.9	– 8

θ (in degrees)	105	120	135	150	165	180
$r = 12\cos^2\theta - 8\sin\theta$	– 6.9	– 3.9	0.3	5	9.1	12

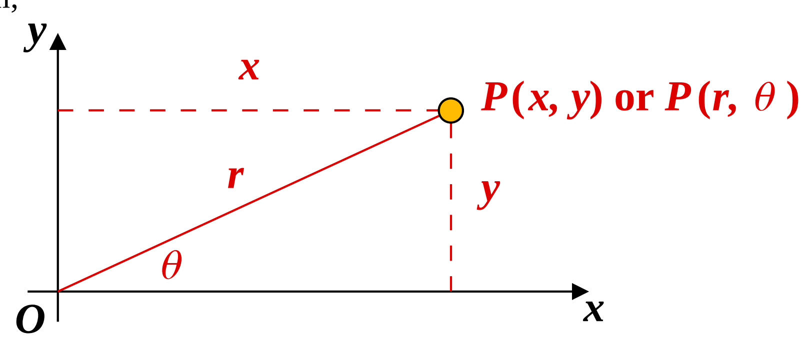
θ (in degrees)	195	210	225	240	255	270
$r = 12\cos^2\theta - 8\sin\theta$	13.3	13	11.7	9.9	8.5	8

θ (in degrees)	285	300	315	330	345	360
$r = 12\cos^2\theta - 8\sin\theta$	8.5	9.9	11.7	13	13.3	12



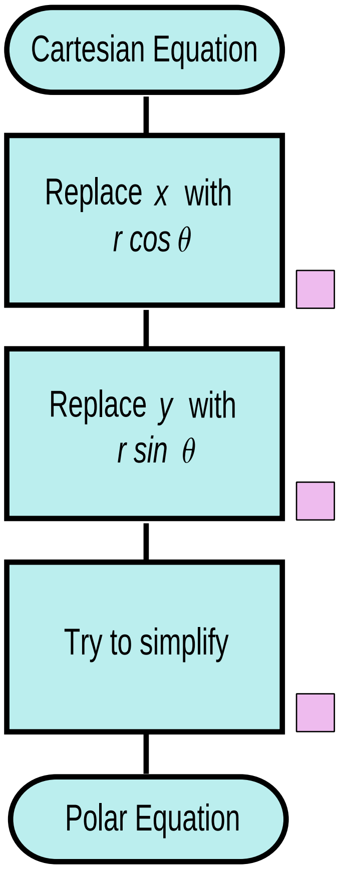
2.1 Curious and Interesting Curves

The key to moving between points and equations written in Cartesian form and the equivalent points and equations written in Polar form is the following diagram;



From the diagram, • $\cos\theta = \dfrac{x}{r}$ • $\sin\theta = \dfrac{y}{r}$ • $r^2 = x^2 + y^2$

This suggests the following strategy for manipulating the algebra of a Cartesian equation to recast it in Polar form.

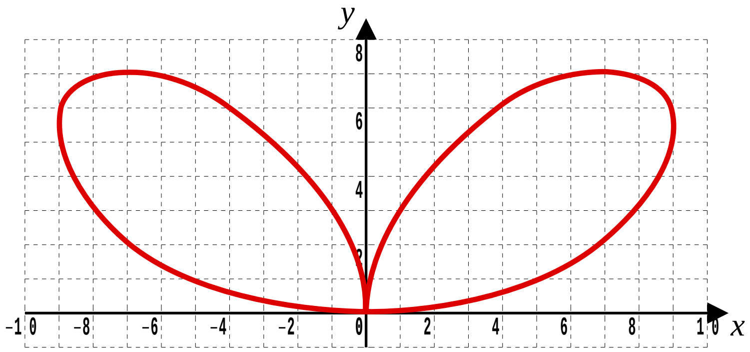


"Try to simplify" is vague; the ideal output is an equation of the form $r = f(\theta)$ or $r^2 = f(\theta)$ but that may not be possible.

2.2 Example

The famous curve below is called The Double Follium. It has Cartesian equation,

$$\left(x^2 + y^2 \right)^2 = 28\,x^2\,y$$



Find the polar equation of The Double Follium.

[4 marks]

Solution

A shortcut is to spot that the $x^2 + y^2$ piece of the equation can be replaced with r^2 alongside the more usual replacement of x with $r\cos\theta$ and y with $r\sin\theta$

$$\begin{aligned}
\left(x^2 + y^2 \right)^2 &= 28\,x^2\,y \\
\left(r^2 \right)^2 &= 28\,\left(r\cos\theta \right)^2 \left(r\sin\theta \right) \\
r^4 &= 28\,r^3\cos^2\theta\,\sin\theta \\
r &= 28\cos^2\theta\,\sin\theta
\end{aligned}$$

2.3 Exercise

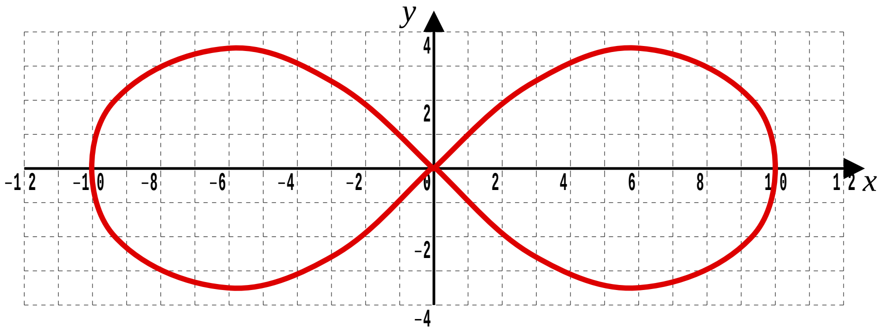
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

The curve below is called The Lemniscate of Bernoulli and has Cartesian equation

$$\left(x^2 + y^2\right)^2 = 100\left(x^2 - y^2\right)$$



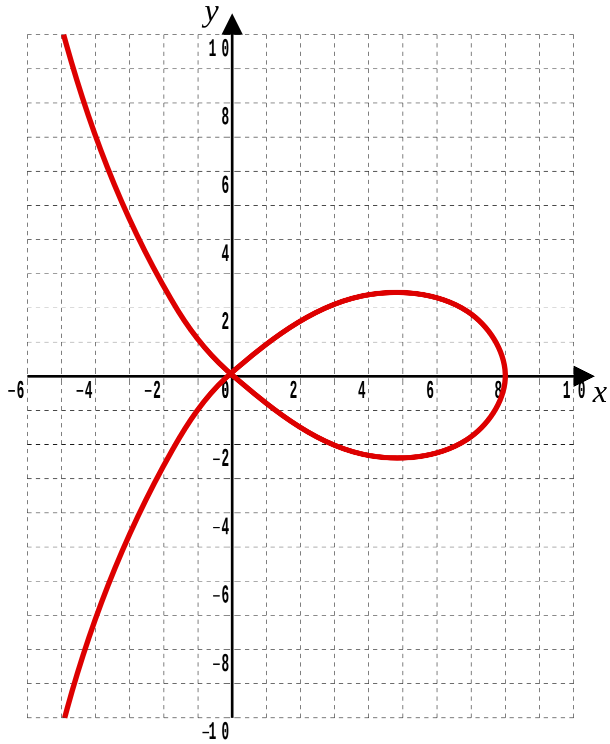
Show that the polar equation of The Lemniscate of Bernoulli can be written in the form $r^2 = 100\,cos\left(2\theta\right)$

[3 marks]

Question 2

The curve below is a Right Strophoid and has Cartesian equation

$$y^2(8 + x) = x^2(8 - x)$$



(i) Show that the Right Strophoid has polar equation $r = \dfrac{8\cos(2\theta)}{\cos\theta}$

[6 marks]

(ii) Show that the point with Cartesian coordinates $\left(-4, -4\sqrt{3}\right)$ is on the curve.

[2 marks]

(iii) By means of implicit differentiation of the Cartesian equation show that

$$\frac{dy}{dx} = \frac{16x - 3x^2 - y^2}{16y + 2xy}$$

[4 marks]

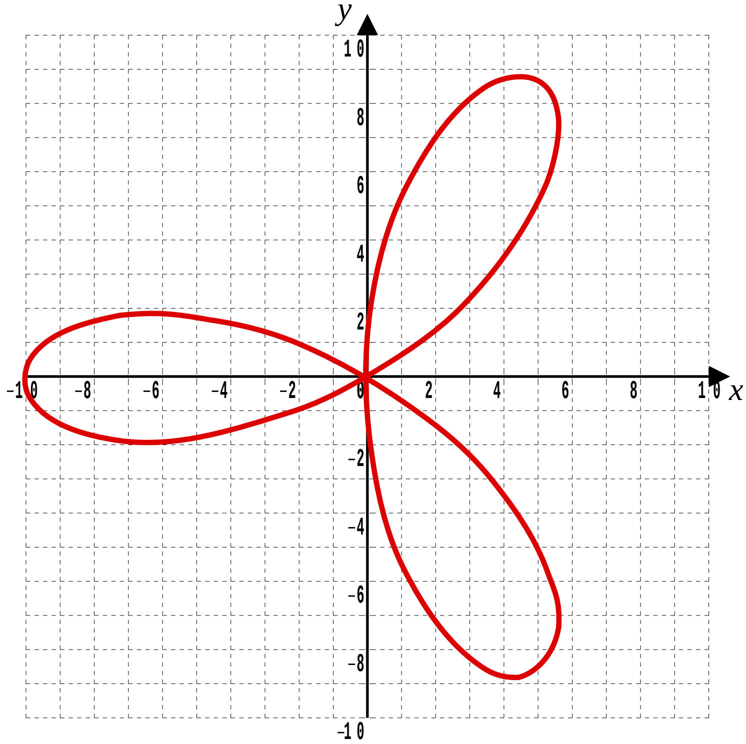
(iv) Find the equation of the tangent to the curve at the point $\left(-4, -4\sqrt{3}\right)$ in the form $y = mx + c$ where m and c are irractional constants to be found.

[3 marks]

Question 3

The curve below is a Trifolium (three-loop) and has Cartesian equation

$$\left(x^2 + y^2\right)\left(y^2 + x\left(x + 10\right)\right) = 40\,x\,y^2$$



(i) Show that the Trifolium has polar equation $r = 10\cos\theta\left(4\sin^2\theta - 1\right)$

[5 marks]

(ii) Find the value of r when $\theta = 60°$

[1 mark]

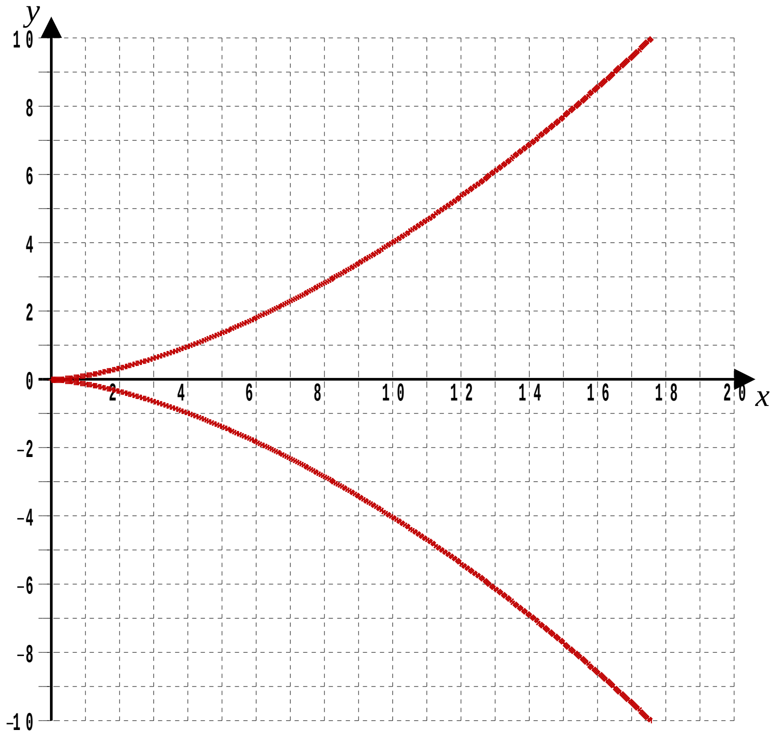
(iii) Find the value of r when $\theta = 30°$ and deduce the equation of the tangent to the curve for this value of θ

[3 marks]

Question 4

The curve below is The Cissoid of Diocles and has Cartesian equation

$$y^2 = \frac{x^3}{72 - x}$$



(i) Show that The Cissoid of Diocles has polar equation $r = 72 \tan\theta \sin\theta$

[4 marks]

(ii) Show that the point with Cartesian coordinates $\left(8,\ 2\sqrt{2}\right)$ is on the curve

[2 marks]

(iii) By means of implicit differentiation of the Cartesian equation show that

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{144y - 2xy}$$

[4 marks]

(iv) Find the equation of the tangent to the curve at the point $\left(8,\ 2\sqrt{2}\right)$ in the form $y = a\sqrt{2}\,x + b\sqrt{2}$ where a and b are rational numbers.

[3 marks]

2.4 Answers to 2.3 Exercise

Answer 1

$$\left(x^2 + y^2\right)^2 = 100\left(x^2 - y^2\right)$$
$$\left(r^2\right)^2 = 100\left(r^2 \cos^2\theta - r^2 \sin^2\theta\right)$$
$$r^4 = 100\, r^2\left(\cos^2\theta - \sin^2\theta\right)$$
$$r^2 = 100\cos(2\theta) \qquad \square$$

[3 marks]

Answer 2

(i)

$$y^2\,(8 + x) = x^2\,(8 - x)$$
$$r^2 \sin^2\theta\,(8 + r\cos\theta) = r^2\cos^2\theta\,(8 - r\cos\theta)$$
$$\sin^2\theta\,(8 + r\cos\theta) = \cos^2\theta\,(8 - r\cos\theta)$$
$$8\sin^2\theta + r\sin^2\theta\cos\theta = 8\cos^2\theta - r\cos^3\theta$$
$$r\cos^3\theta + r\sin^2\theta\cos\theta = 8\cos^2\theta - 8\sin^2\theta$$
$$r\cos\theta\left(\cos^2\theta + \sin^2\theta\right) = 8\left(\cos^2\theta - \sin^2\theta\right)$$
$$r\cos\theta = 8\cos(2\theta)$$
$$r = \frac{8\cos(2\theta)}{\cos\theta} \qquad \square$$

[6 marks]

(ii)

$$\text{LHS} = y^2(8 + x)$$
$$= \left(-4\sqrt{3}\right)^2(8 + (-4)) \qquad \text{when } \left(-4, -4\sqrt{3}\right) \text{ is inserted}$$
$$= 16 \times 3 \times 4$$
$$= 192$$

$$\text{RHS} = x^2\,(8 - x)$$
$$= (-4)^2 \times (8 - (-4)) \qquad \text{when } \left(-4, -4\sqrt{3}\right) \text{ is inserted}$$
$$= 16 \times 12$$
$$= 192$$
$$= \text{LHS}$$

$$\therefore\ \left(-4, -4\sqrt{3}\right) \text{ is on the curve} \qquad \square$$

[2 marks]

(iii)

$$y^2 (8 + x) = x^2 (8 - x)$$

$$y^2 (8 + x) = 8x^2 - x^3$$

Differentiating Implicitly

$$y^2 (1) + 2y \frac{dy}{dx} (8 + x) = 16x - 3x^2$$

$$\frac{dy}{dx} (16y + 2xy) = 16x - 3x^2 - y^2$$

$$\frac{dy}{dx} = \frac{16x - 3x^2 - y^2}{16y + 2xy} \qquad \square$$

[4 marks]

(iv)

$$\frac{dy}{dx} = \frac{16(-4) - 3(-4)^2 - \left(-4\sqrt{3}\right)^2}{16\left(-4\sqrt{3}\right) + 2(-4)\left(-4\sqrt{3}\right)}$$

$$= \frac{-64 - 48 - 48}{-64\sqrt{3} + 32\sqrt{3}}$$

$$= \frac{-160}{-32\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{5\sqrt{3}}{3}$$

$$y = mx + c$$

$$-4\sqrt{3} = \frac{5\sqrt{3}}{3} \times (-4) + c$$

$$c = -\frac{12\sqrt{3}}{3} + \frac{20\sqrt{3}}{3}$$

$$= \frac{8\sqrt{3}}{3}$$

$$\therefore \text{ Tangent is } y = \frac{5\sqrt{3}}{3} x + \frac{8\sqrt{3}}{3}$$

[3 marks]

Answer 3

(i)

$$\left(x^2 + y^2\right)\left(y^2 + x\left(x + 10\right)\right) = 40\,x\,y^2$$
$$\left(x^2 + y^2\right)\left(y^2 + x^2 + 10x\right) = 40\,x\,y^2$$
$$r^2\left(r^2 + 10\,r\cos\theta\right) = 40\,r\cos\theta\,r^2\sin^2\theta$$
$$r^3\left(r + 10\cos\theta\right) = 40\,r^3\cos\theta\sin^2\theta$$
$$r + 10\cos\theta = 40\cos\theta\sin^2\theta$$
$$r = 40\cos\theta\sin^2\theta - 10\cos\theta$$
$$r = 10\cos\theta\left(4\sin^2\theta - 1\right) \qquad \square$$

[5 marks]

(ii) When $\theta = 60°$,

$$r = 10\cos 60\left(4\sin^2 60 - 1\right)$$
$$= 10 \times \frac{1}{2}\left(4 \times \left(\frac{\sqrt{3}}{2}\right)^2 - 1\right)$$
$$= 10 \times \frac{1}{2}\left(4 \times \frac{3}{4} - 1\right)$$
$$= 10 \times \frac{1}{2} \times 2$$
$$= 10$$

This is the tip of the loop in the first quadrant

[1 mark]

(iii) When $\theta = 30°$,

$$r = 10\cos 30\left(4\sin^2 30 - 1\right)$$
$$= 10 \times \frac{\sqrt{3}}{2}\left(4 \times \left(\frac{1}{2}\right)^2 - 1\right)$$
$$= 0$$

The implication is that this is the angle of the slope as the curve makes one of its passes through the origin.

$$m = \tan\theta$$
$$= \tan 30$$
$$= \frac{\sqrt{3}}{3}$$

$\therefore$ The tangent would be $y = \dfrac{\sqrt{3}}{3}\,x$

[3 marks]

Answer 4

(i)

$$\begin{aligned}
y^2 &= \frac{x^3}{72 - x} \\
y^2 (72 - x) &= x^3 \\
r^2 \sin^2\theta \left(72 - r\cos\theta\right) &= r^3\cos^3\theta \\
\sin^2\theta \left(72 - r\cos\theta\right) &= r\cos^3\theta \\
72\sin^2\theta - r\sin^2\theta\cos\theta &= r\cos^3\theta \\
r\cos^3\theta + r\sin^2\theta\cos\theta &= 72\sin^2\theta \\
r\cos\theta\left(\cos^2\theta + \sin^2\theta\right) &= 72\sin^2\theta \\
r\cos\theta &= 72\sin^2\theta \\
r &= 72 \times \frac{\sin\theta}{\cos\theta} \times \sin\theta \\
r &= 72\tan\theta\sin\theta \qquad \square
\end{aligned}$$

[4 marks]

(ii)

$$\begin{aligned}
y^2 &= \frac{x^3}{72 - x} \\
y^2 (72 - x) &= x^3
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= y^2\left(72 - x\right) \\
&= \left(2\sqrt{2}\right)^2\left(72 - 8\right) \qquad \text{when } \left(8,\ 2\sqrt{2}\right) \text{ is inserted} \\
&= 4 \times 2 \times 64 \\
&= 512
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= x^3 \\
&= 8^3 \qquad \text{when } \left(8,\ 2\sqrt{2}\right) \text{ is inserted} \\
&= 512 \\
&= \text{LHS}
\end{aligned}$$

$\therefore \left(8,\ 2\sqrt{2}\right)$ is on the curve $\square$

[2 marks]

(iii)

$$y^2 = \frac{x^3}{72 - x}$$

$$y^2 (72 - x) = x^3$$

Differentiate Implicitly

$$y^2 (-1) + 2y \frac{dy}{dx} (72 - x) = 3x^2$$

$$\frac{dy}{dx} (144y - 2xy) = 3x^2 + y^2$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{144y - 2xy}$$

[4 marks]

(iv)

$$\frac{dy}{dx} = \frac{3(8)^2 + (2\sqrt{2})^2}{144(2\sqrt{2}) - 2(8)(2\sqrt{2})}$$

$$= \frac{192 + 8}{288\sqrt{2} - 32\sqrt{2}}$$

$$= \frac{200}{256\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{25\sqrt{2}}{64}$$

$$y = mx + c$$

$$2\sqrt{2} = \frac{25\sqrt{2}}{64} \times (8) + c$$

$$2\sqrt{2} = \frac{25\sqrt{2}}{8} + c$$

$$c = \frac{16\sqrt{2}}{8} - \frac{25\sqrt{2}}{8}$$

$$= -\frac{9\sqrt{2}}{8}$$

$$\therefore \text{ Tangent is } y = \frac{25\sqrt{2}}{64} x - \frac{9\sqrt{2}}{8}$$

[3 marks]

Lesson 3

3.1 Old Friends in New Clothes

To properly understand a polar curve, there is no substitute for going to the effort of plotting it by hand. Thought should be applied to focussing on any parts of the curve where it's less predictable, and on any transitions where r flips sign, which can be the result of $r \to \pm\infty$ or $r \to 0$.

In many situations it will be desirable to insist that r be non-negative. However, on looking back at the polar plot of the first lesson, excluding the short piece of curve where r became negative would have destroyed the symmetry and flow of the curve. Many text books, internet articles and videos on polar curves are not clear if what is being discussed applies only for $r \geqslant 0$, or for all real values of r. This is a source of muddle and confusion as is often makes no difference but sometimes does !

3.2 Exercise

Marks Available : 50

Question 1

Here is a polar equation which is to be graphed.

$$r(\theta) \;=\; \frac{8}{2\cos\theta \;+\; \sin\theta}$$

(i) Explain why $r(90)$ and $r(270)$ yield the same point on the graph

[2 marks]

(ii) Complete the following table.
Notice the table is "intelligent" with a focus on key features.

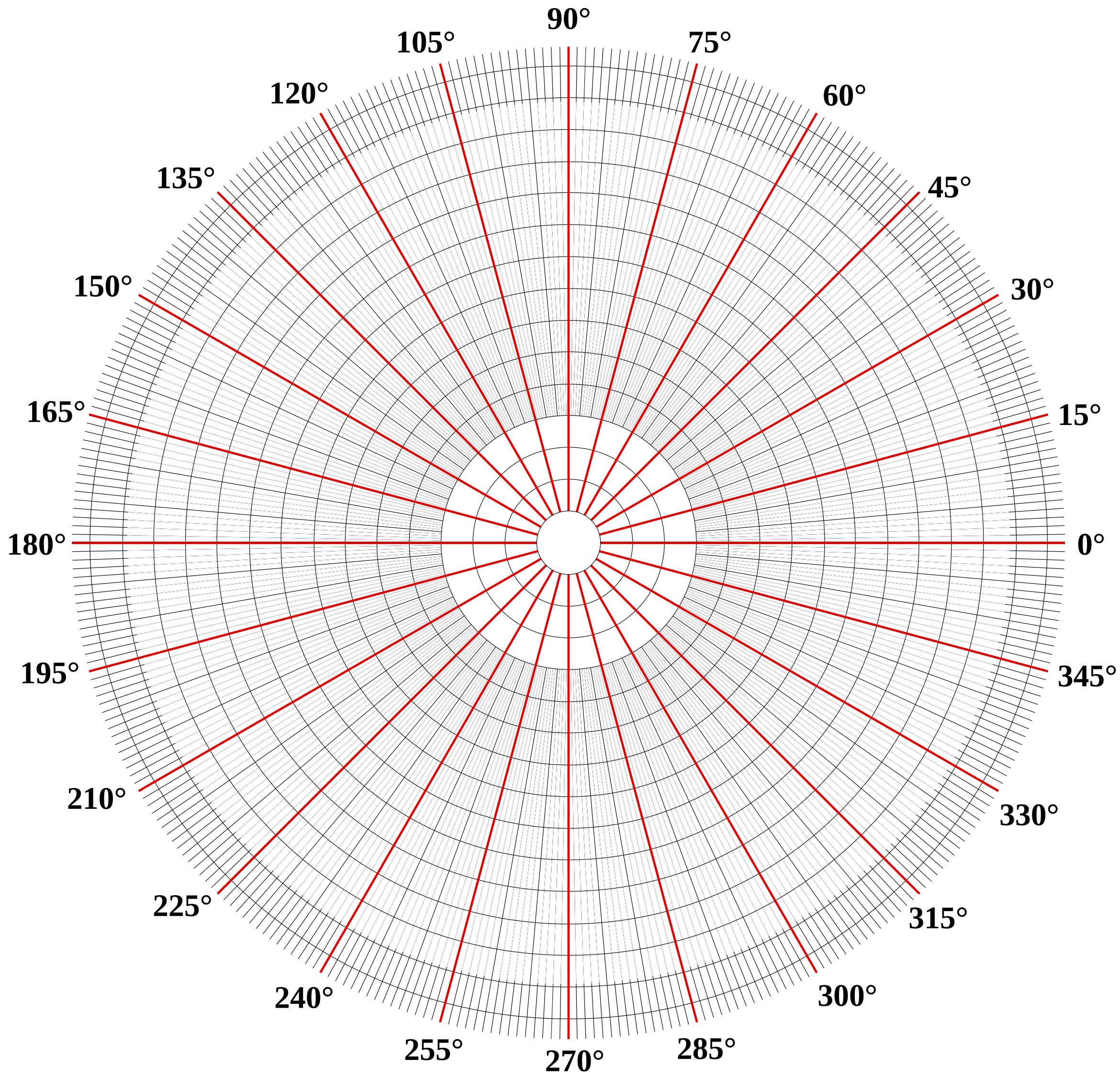
Work to 1 decimal place (and with your calculator in degrees mode !)

θ (in degrees)	0	45	90	100	[redacted]	135	165
$r = \dfrac{8}{2\cos\theta + \sin\theta}$					[redacted]		

θ (in degrees)	180	225	270	280	[redacted]	315	345
$r = \dfrac{8}{2\cos\theta + \sin\theta}$					[redacted]		

[4 marks]

(iii) Plot the polar coordinates from the table onto the graph below,



[4 marks]

(iv) This is an example of a polar curve of the form

$$r = \frac{c}{a\cos\theta + b\sin\theta} \quad \text{where } a,\ b \text{ and } c \text{ are constants}$$

"Reverse Engineer" this general equation to obtain the Cartesian equivalent.

[4 marks]

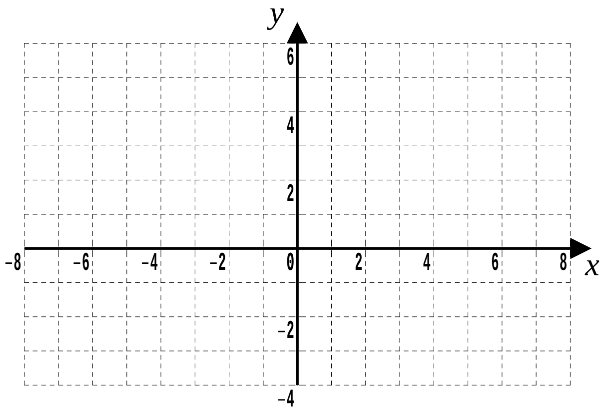
Question 2

(i) For the polar equation $r(\theta) = \dfrac{\theta}{90}$ complete the following table,

θ (in degrees)	90	180	270	360	450	540
$r = \dfrac{\theta}{90}$						

[2 marks]

(ii) Plot the graph of $r(\theta) = \dfrac{\theta}{90}$



[3 marks]

Question 3

Here is a polar equation which is to be graphed.

$$r(\theta) = 6\cos\theta + 8\sin\theta$$

(i) Explain why $r(0)$ and $r(180)$ yield the same point on the graph

[2 marks]

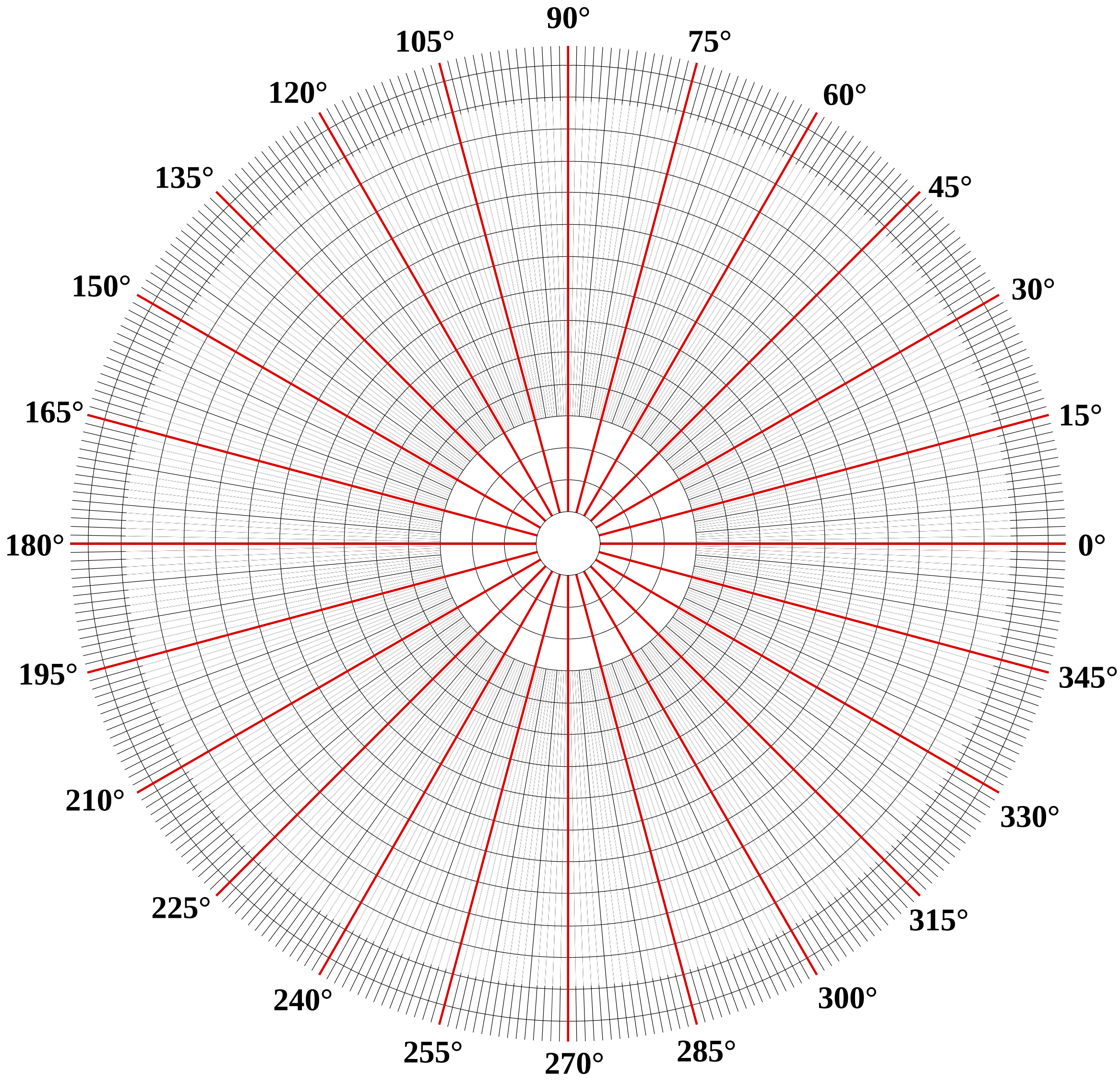
(ii) Complete the following table.
Notice the table is "intelligent" with a focus on key features.
(Plot extra points if you need to)
Work to 1 decimal place.

θ (in degrees)	0	10	20	30	45	75	90
$r = 6\cos\theta + 8\sin\theta$							

θ (in degrees)	100	110	120	140	180	210	270
$r = 6\cos\theta + 8\sin\theta$							

[5 marks]

(iii) Plot the polar coordinates from the table onto the graph below,



[5 marks]

(iv) This is an example of a polar curve of the form

$r = 2a\cos\theta + 2b\sin\theta$ where a and b are constants

"Reverse Engineer" this general equation to obtain the Cartesian equivalent.

[5 marks]

Question 4

(i) Complete the following table for $r(\theta) = \dfrac{4}{1 - \sin\theta}$

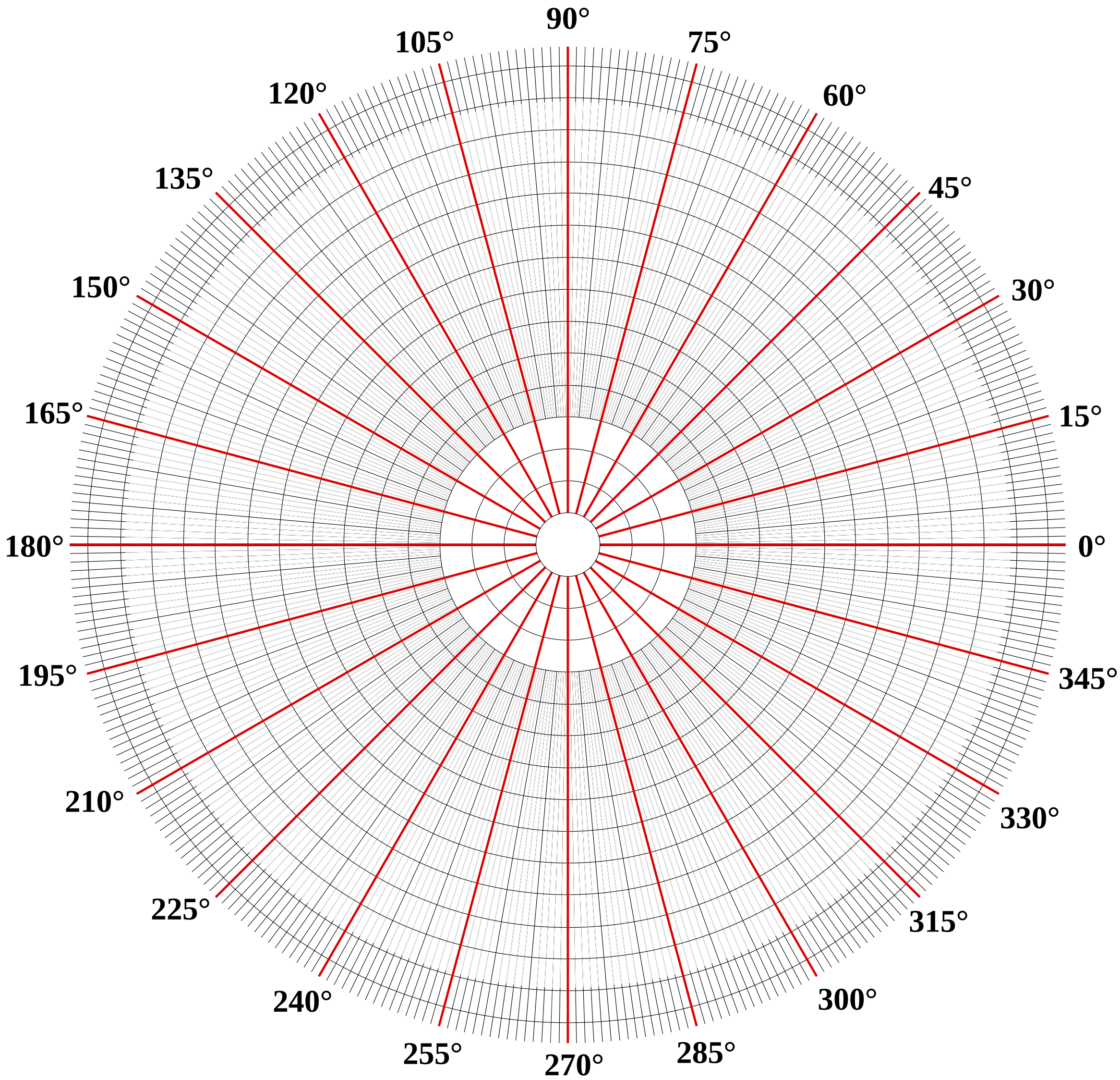
Work to 1 decimal place.

θ (in degrees)	0	15	30	45	[redacted]	135
$r = \dfrac{4}{1 - \sin\theta}$					[redacted]	

θ (in degrees)	150	165	180	225	270	315
$r = \dfrac{4}{1 - \sin\theta}$						

[4 marks]

(ii) Plot the polar coordinates from the table onto the graph below,



[4 marks]

(iii) This is an example of a polar curve of the form

$$r = \frac{p}{1 - \sin\theta} \qquad \text{where } p \text{ is a constant}$$

"Reverse Engineer" this general equation to obtain the Cartesian equivalent.

[6 marks]

3.5 Answers

3.5.1 Solution to 3.2 Exercise

Answer 1

(i)

$$r(90) = \frac{8}{2\,cos\,90 + sin\,90}$$

$= 8$ yielding the polar coordinate $(8,\ 90°)$

$$r(270) = \frac{8}{2\,cos\,270 + sin\,270}$$

$= -8$ yielding the polar coordinate $(-8,\ 270°)$

However, $(8,\ 90°)$ and $(-8,\ 270°)$ are the same point □

[2 marks]

(ii)

θ (in degrees)	0	45	90	100	[redacted]	135	165
$r = \frac{8}{2\,cos\,\theta + sin\,\theta}$	4	3.8	8	12.5	[redacted]	−11.3	− 4.8

θ (in degrees)	180	225	270	280	[redacted]	315	345
$r = \frac{8}{2\,cos\,\theta + sin\,\theta}$	− 4	− 3.8	− 8	− 12.5	[redacted]	11.3	4.8

[4 marks]

(iii) See the graph on the following page
The graph looks suspiciously like a straight line which the next part of the question will prove is indeed the case.

[4 marks]

(iv)

$$r = \frac{c}{a\,cos\,\theta + b\,sin\,\theta}$$

$$r\,(a\,cos\,\theta + b\,sin\,\theta) = c$$

$$a\,r\,cos\,\theta + b\,r\,sin\,\theta = c$$

$$ax + by = c$$

The Cartesian equation of a straight line !

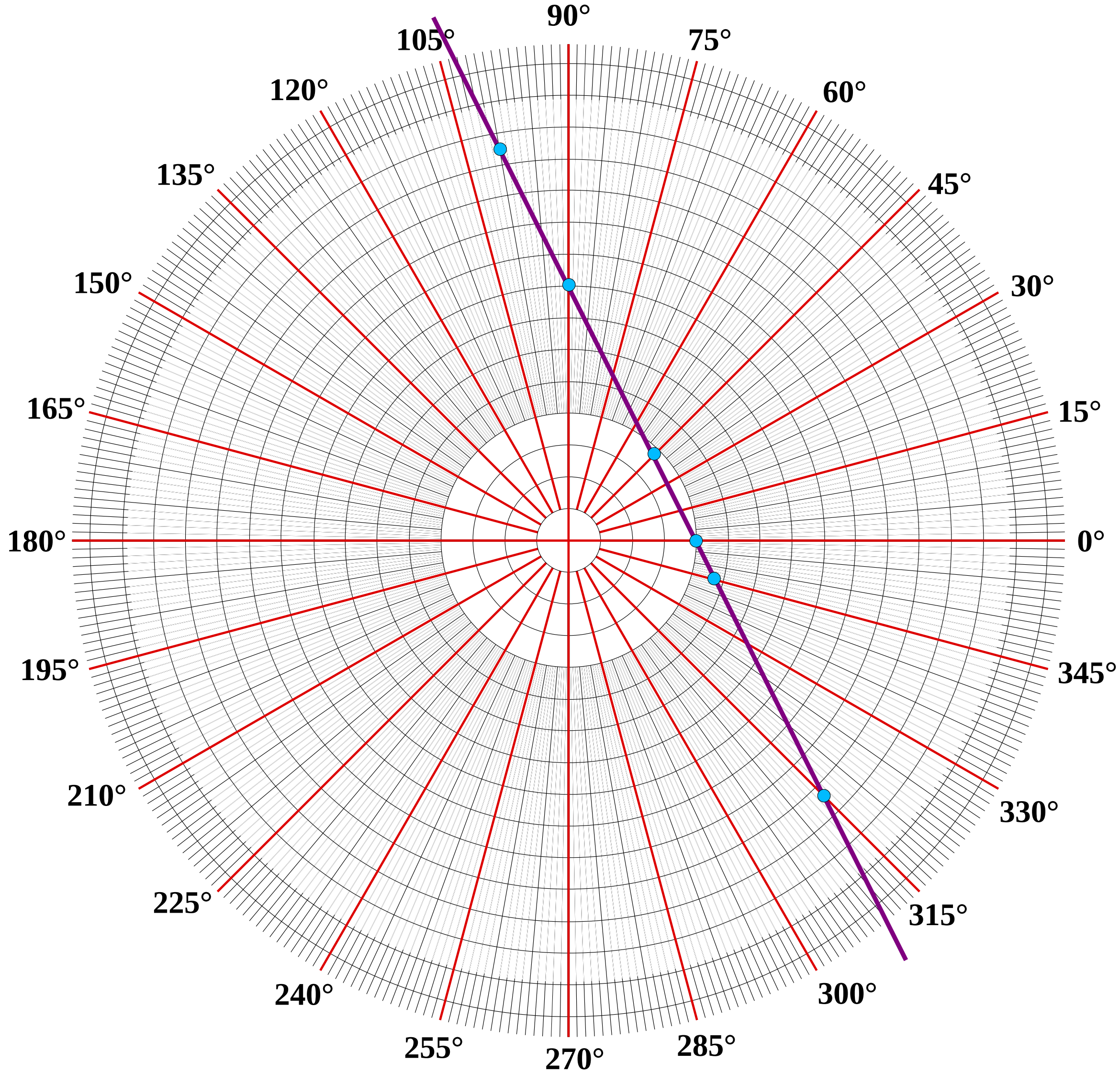
Dividing through by b

$$\frac{a}{b}x + y = \frac{c}{b}$$

$$y = -\frac{a}{b}x + \frac{c}{b}$$

$$y = mx + k \quad \text{with } m = -\frac{a}{b},\ \ k = \frac{c}{b}$$ □

[4 marks]



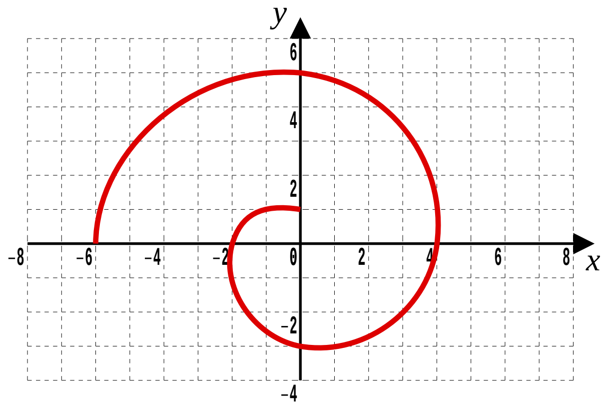
Answer 2

(i)

θ (in degrees)	90	180	270	360	450	540
$r = \dfrac{\theta}{90}$	1	2	3	4	5	6

[2 marks]

(ii)



[3 marks]

Answer 3

(i) $r(0) = 6\,cos\,0 + 8\,sin\,0$

$= 6$ yielding the polar coordinate $(6,\ 0°)$

$r(180) = 6\,cos\,180 + 8\,sin\,180$

$= -6$ yielding the polar coordinate $(-6,\ 270°)$

However, $(6,\ 0°)$ and $(-6,\ 180°)$ are the same point ☐

[2 marks]

(ii)

θ (in degrees)	0	10	20	30	45	75	90
$r = 6\,cos\,\theta + 8\,sin\,\theta$	6	7.3	8.3	9.2	9.9	9.2	8

θ (in degrees)	100	110	120	140	180	210	270
$r = 6\,cos\,\theta + 8\,sin\,\theta$	6.8	5.5	3.9	0.5	– 6	– 9.2	– 8

[5 marks]

(iii) See the graph on the following page.
The graph looks suspiciously like a circle which the next part of the question will prove is indeed the case.

(iv) r is the variable in the polar equation
R is the constant radius of a given circle

$$r = 2a\,cos\,\theta + 2b\,sin\,\theta$$

Multiply both sides by r

$$r^2 = 2a\,r\,cos\,\theta + 2b\,r\,sin\,\theta$$

$$x^2 + y^2 = 2a\,x + 2b\,y$$

$$x^2 - 2ax + y^2 - 2ay = 0$$

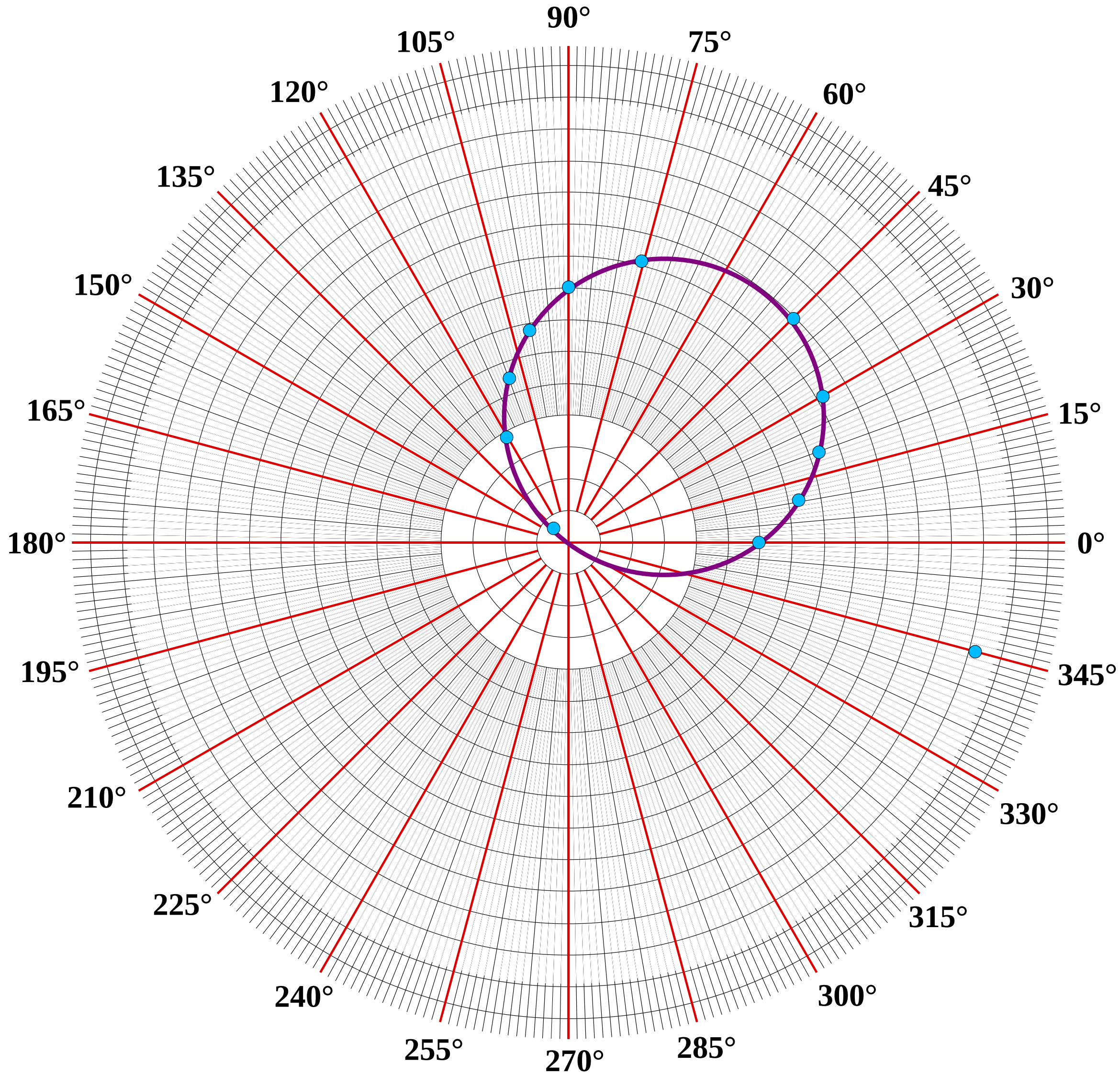
$$(x - a)^2 - a^2 + (y - b)^2 - b^2 = 0$$

$$(x - a)^2 + (y - b)^2 = a^2 + b^2$$

$$(x - a)^2 + (y - b)^2 = R^2$$

which is the Cartesian equation of a circle, centre $(a,\ b)$, Radius R

[5 marks]



[5 marks]

Answer 4

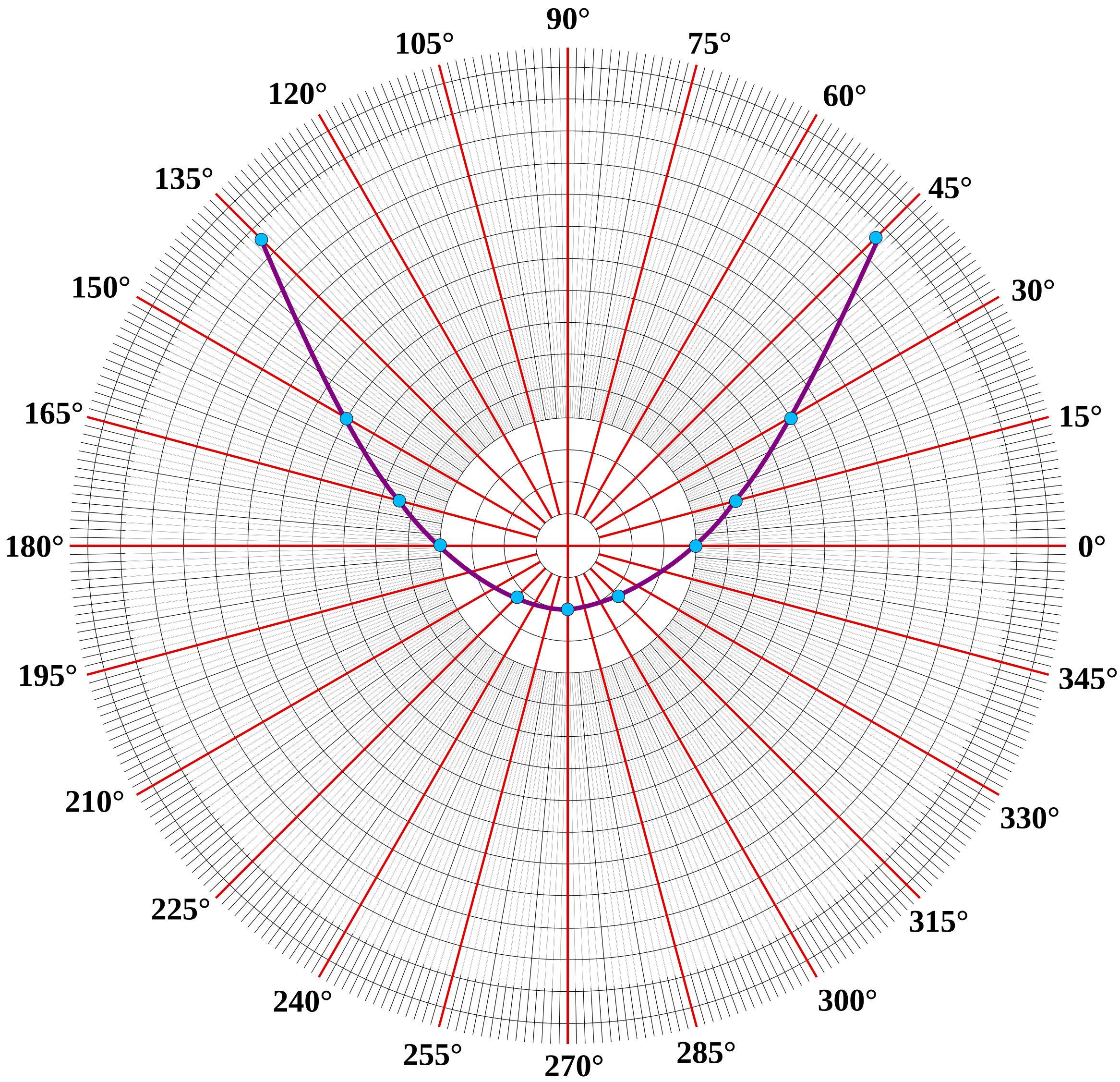
(i)

θ (in degrees)	0	15	30	45	[redacted]	135
$r = \dfrac{4}{1 - \sin\theta}$	4	5.4	8	13.7		13.7

θ (in degrees)	150	165	180	225	270	315
$r = \dfrac{4}{1 - \sin\theta}$	8	5.4	4	2.3	2	2.3

[4 marks]

(ii)



[4 marks]

This looks suspiciously like a parabola.
The next part of the question will prove that this is indeed the case !

(iii)

$$r = \frac{p}{1 - \sin\theta}$$

$$r\left(1 - \sin\theta\right) = p$$

$$r - r\sin\theta = p$$

$$r - y = p$$

$$r = p + y$$

square both sides

$$r^2 = (p + y)^2$$

$$x^2 + y^2 = p^2 + 2py + y^2$$

$$x^2 = p^2 + 2py$$

$$2py = x^2 - p^2$$

$$y = \frac{1}{2p}x^2 - \frac{p}{2}$$

Which is the form of a parabola

[6 marks]

4.1 Key Polar Curves I

In much the same way that it is useful to know the graph that goes with a few Cartesian equations, it's useful to know the graph that goes with a few Polar equations. The interest is in trying to get "a lot from a little", and so the desire is to find simple Polar equations that yield graphs that would need formidable algebra to be described in Cartesian form. In this lesson the focus is on graphs that are familiar, but recast in polar form.

4.1 The Line

(i) Generalised Lines

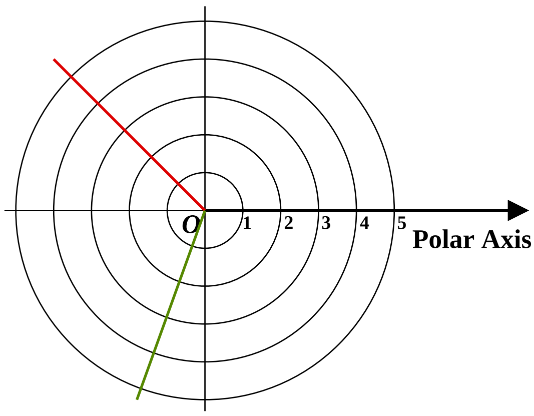
: Cartesian $y = mx + c$: $ax + by = c$

: Polar $r = \dfrac{c}{\sin\theta - m\cos\theta}$: $r = \dfrac{c}{a\cos\theta + b\sin\theta}$

(ii) Half Lines through the pole at an angle of ϕ to the polar axis

: Cartesian $y = \tan(\phi)\, x$ (Whole line)

: Polar $\theta = \phi$ $r \geqslant 0$ (Half line)



Half lines $\theta = 135°$ (red) and $\theta = 250°$ (green)

(iii) Vertical Lines

: Cartesian $x = a$

: Polar $r = a\sec\theta$

(iv) Horizontal Lines

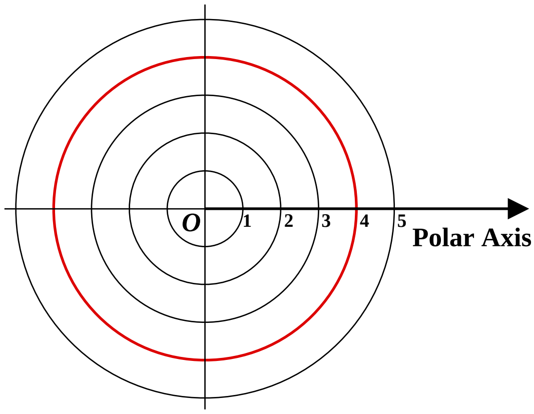
: Cartesian $y = b$

: Polar $r = b\csc\theta$

4.2 The Circle

(i) Centre origin, radius R : Cartesian $x^2 + y^2 = R^2$

: Polar : $r = R$



A circle centre (0, 0) and radius 4 has polar equation, $r = 4$ (red)

(ii) Centre (a, 0), radius a ($\therefore$ passing through the pole)

: Cartesian $(x - a)^2 + y^2 = r^2$ with $r = a$

: Polar $r = 2a \cos\theta$

(iii) Centre (0, b), radius b ($\therefore$ passing through the pole)

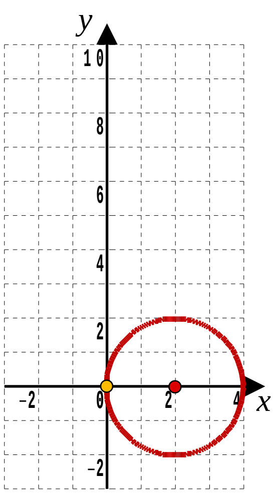
: Cartesian $x^2 + (y - b)^2 = r^2$ with $r = b$

: Polar $r = 2b \sin\theta$

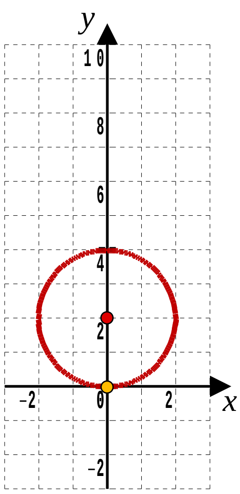
(iv) Centre (a, b), radius r ($\therefore$ passing through the pole)

: Cartesian $(x - a)^2 + (y - b)^2 = r^2$ with $r^2 = a^2 + b^2$

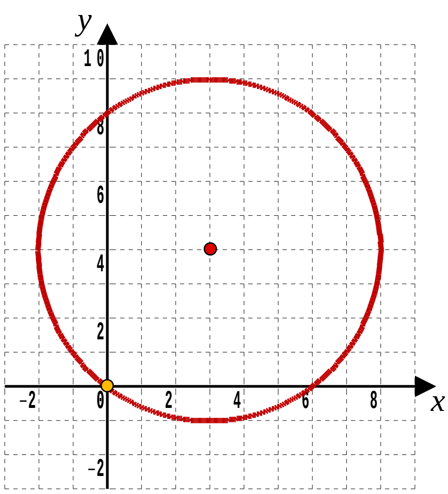
: Polar $r = 2a \cos\theta + 2b \sin\theta$



$r = 4 \cos\theta$



$r = 4 \sin\theta$



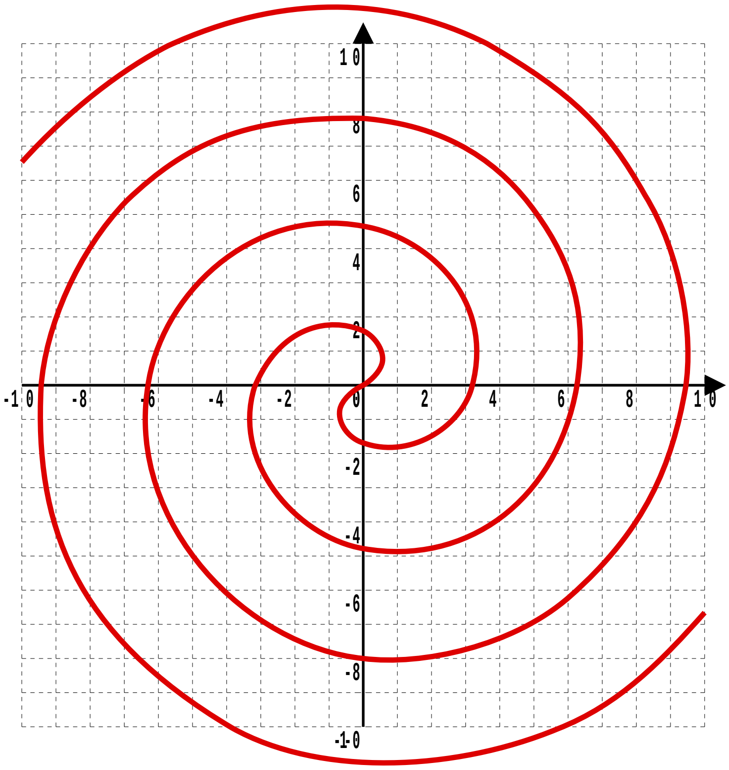
$r = 6 \cos\theta + 8 \sin\theta$

4.3 The Spiral

(i) The Archimedean Spiral

: Cartesian $y = x\,tan\left(\sqrt{x^2 + y^2}\right)$

: Polar $r = \theta, \quad r \geqslant 0$



The Double Archimedean Spiral $r = \pm\theta$ with θ in radians

The triskelion, a symbol dating back over 6000 years of three interlocking Archimedean spirals, often associated with Ireland, where it adorns many graves

(ii) The Equiangular Spiral (also called Logarithmic Spiral, Bernoulli's Spiral)

: Cartesian $y = x\,tan\left(ln\left(\sqrt{x^2 + y^2}\right)\right)$

: Polar $r = e^{\theta}$

3.4 Exercise

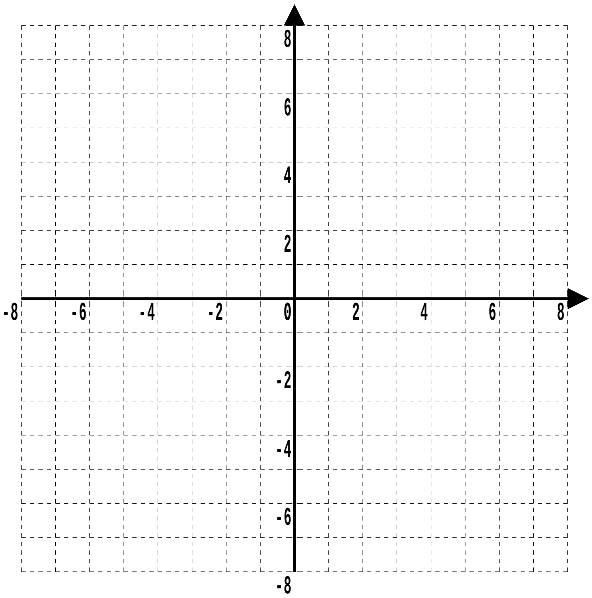
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(i) On the graph, sketch the polar equations,

$$r = 6 \qquad \text{and} \qquad r = 3\sqrt{3}\ sec\ \theta$$



[2 marks]

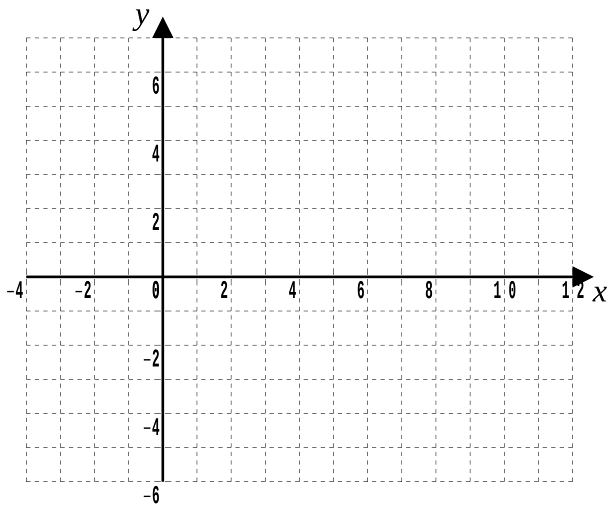
(ii) Use algebra to find the polar coordinates of the points of intersection of the two graphs sketched in part (i)

[4 marks]

Question 2

(i) On the graph, sketch the polar equations,

$$r_1 = 6 \sin \theta_1 \quad \text{and} \quad r_2 = 6\sqrt{3} \cos \theta_2$$



[2 marks]

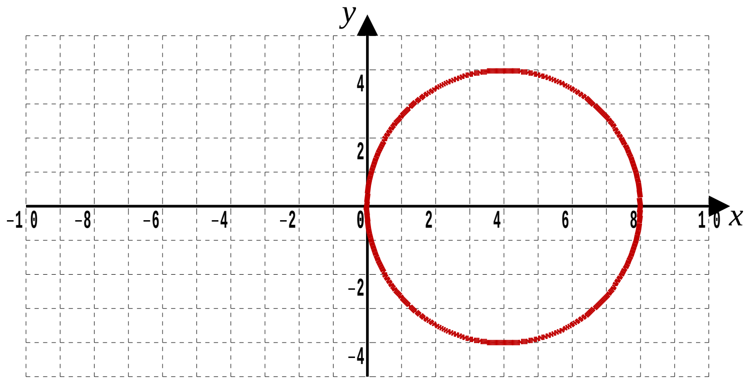
(ii) Use algebra to find the polar coordinates of the point of intersection of the two graphs sketched in part (i)

Is the pole when $r_1 = r_2 = 0$ an additional point of intersection ?

[4 marks]

Question 3
On the graph is plotted the polar equation,

$$r \;=\; 8\,cos\,\theta$$



(i) To the graph add the points with the following polar coordinates;

$A\left(4\sqrt{2},\; 45°\right)$, $B\left(8,\; 0°\right)$, $C\left(4\sqrt{2},\; 315°\right)$

[**3 marks]**

(iii) Add the circle $r = 4$ to the graph.

[1 mark]

(iv) Use algebra to find both points of intersection of the two graphs.
Give your answer in polar coordinates.

[4 marks]

Question 4

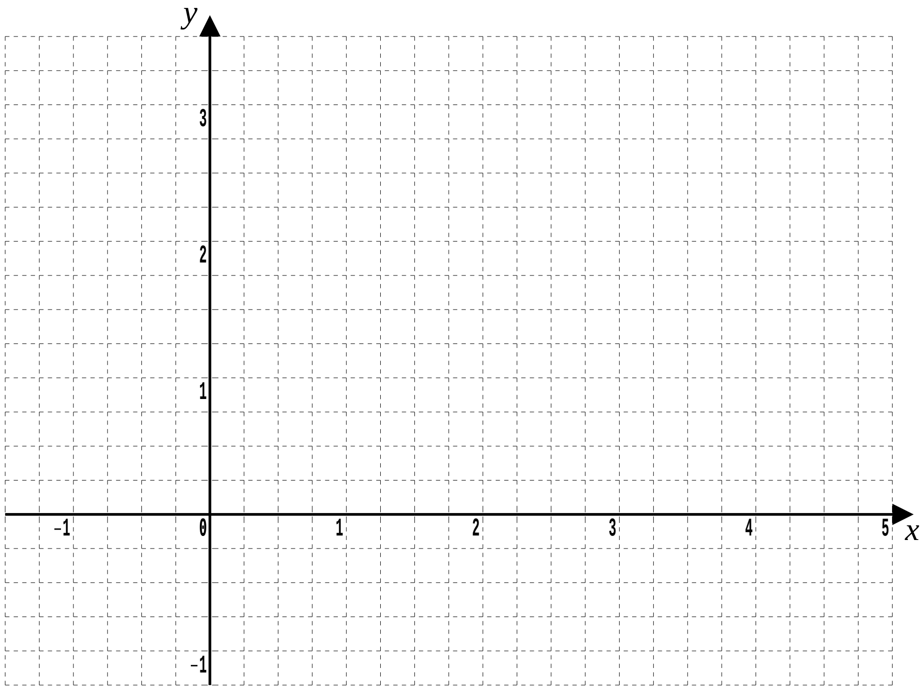
Consider the curve with polar equation $r = 4\,cos\left(\theta - 30°\right)$

(i) Prove that this equation can be written as,

$$r = 2\sqrt{3}\ cos\,\theta + 2\,sin\,\theta$$

[2 marks]

(ii) Hence, sketch the curve $r = 4\,cos\left(\theta - 30°\right)$



[3 marks]

(iii) Write down the Cartesian equation of the curve.

[2 marks]

(iv) What transformation would result from replacing θ in the polar equation equation $r = 4\,cos\,\theta$ with $\left(\theta - 30°\right)$?

[2 marks]

Question 5

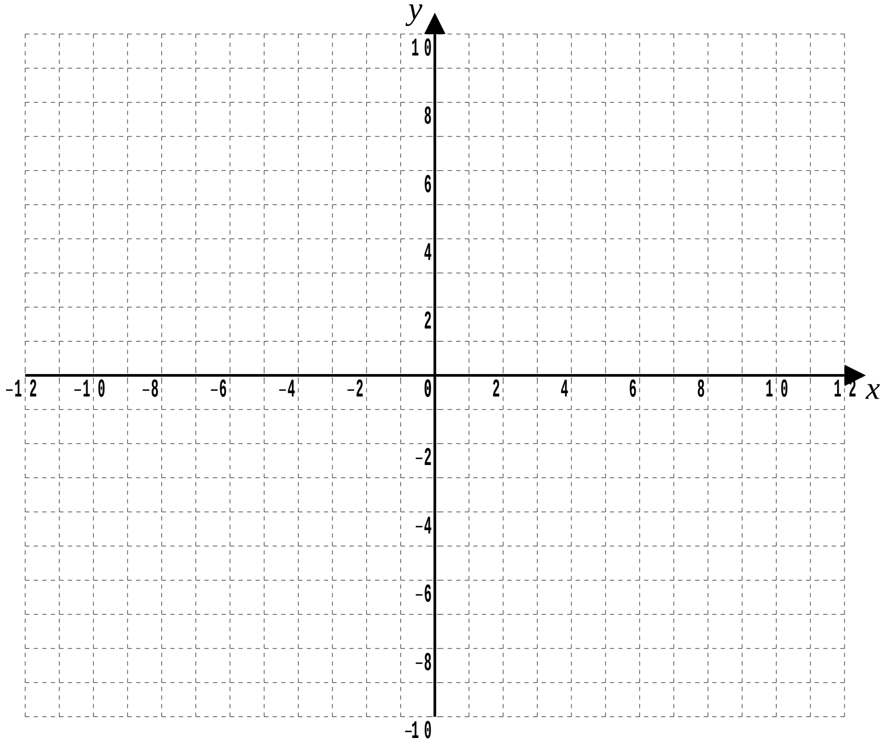
Consider the polar equation $r = 5\,sec\left(\theta - 60°\right)$

(i) Prove that this equation can be written as,

$$r = \frac{10}{cos\,\theta + \sqrt{3}\,sin\,\theta}$$

[2 marks]

(ii) Hence, sketch the graph of $r = 5\,sec\left(\theta - 60°\right)$



[3 marks]

(iii) Write down the Cartesian equation of the curve.

[2 marks]

(iv) What transformation would result from replacing θ in the polar equation equation $r = 5\,sec\,\theta$ with $\left(\theta - 60°\right)$?

[2 marks]

Question 6
Starting with the Cartesian equation of a circle, centre (5, 12), radius 13, and using the relations

- $x = r \cos\theta$
- $y = r \sin\theta$
- $r^2 = x^2 + y^2$

prove using algebra that the polar equation of the circle will be

$$r = 10\cos\theta + 24\sin\theta$$

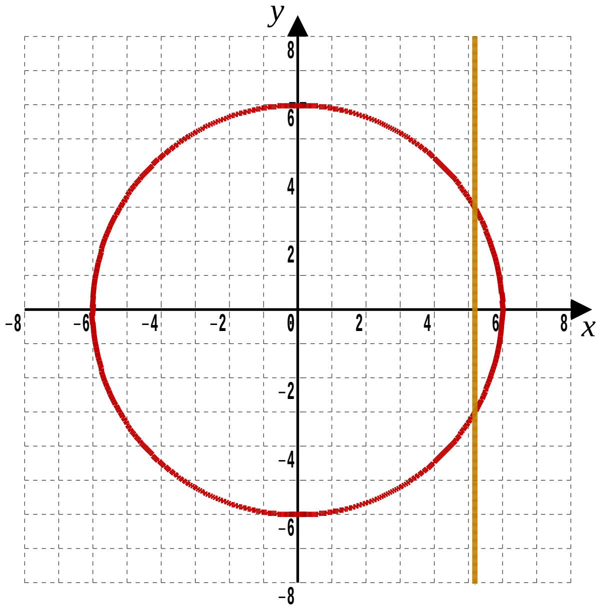
[6 marks]

Question 7
Starting with the Cartesian equation of the Archimedian spiral, prove that its polar equation is $r = \theta$

[3 marks]

Answer 1

(i)



$r = 6$ is a circle, centre origin, radius 6

$r = 3\sqrt{3}\ sec\ \theta$ is the vertical line $x = 3\sqrt{3}$

(ii) At points of intersection,

$$3\sqrt{3}\ sec\ \theta = 6$$

$$\frac{1}{cos\ \theta} = \frac{6}{3\sqrt{3}}$$

$$cos\ \theta = \frac{\sqrt{3}}{2}$$

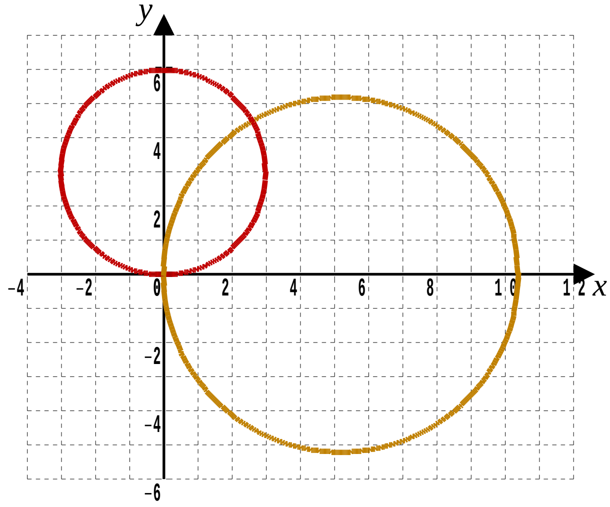
$$\theta = 30° \text{ or } 330°$$

So the polar coordinates of intersection are $(6,\ 30°)$, $(6.\ 330°)$

[4 marks]

Answer 2

(i)



$r_1 = 6 \sin \theta_1$ is a circle, centre $(0,\ 3)$ radius 3

$r_2 = 6\sqrt{3} \cos \theta_2$ is a circle, centre $(3\sqrt{3},\ 0)$ and radius $3\sqrt{3}$

(ii) At the point of intersection,

$$6 \sin \theta = 6\sqrt{3} \cos \theta$$

$$\frac{\sin \theta}{\cos \theta} = \sqrt{3}$$

$$\tan \theta = \sqrt{3}$$

$$\theta = 60°,\ 240°$$

When $\theta = 60^0$, $r_1 = 6 \sin(60)$

$= 3\sqrt{3} \quad \therefore (3\sqrt{3},\ 60°)$

When $\theta = 240°$, $r_1 = 6 \sin(240)$

$= -3\sqrt{3} \quad \therefore (-3\sqrt{3},\ 240°)$

But $(-3\sqrt{3},\ 240°) = (3\sqrt{3},\ 60°)$ so this is the same point twice.

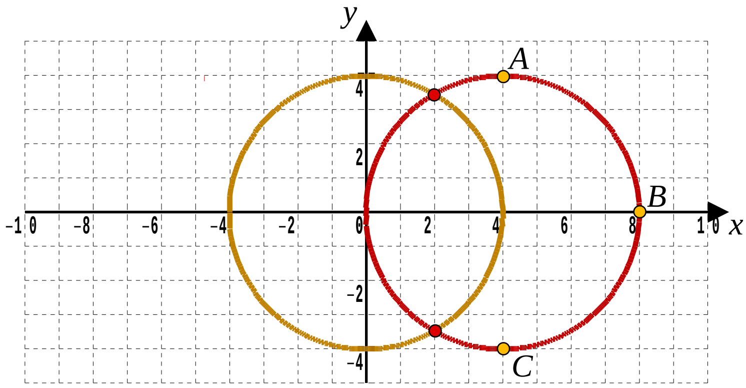
However, there is another point of intersection at the pole !

Note: At the pole the graphs intersect, not because $\theta_1 = \theta_2$ but because $r_1 = r_2 = 0$; Always check if the pole is a point of intersection !

The pole can be written as the polar coordinate $(0,\ \phi°)$ where ϕ is any angle at all !

Answer 3

(i), **(ii)** and **(iii)**



[3, 3, 1 marks)

(iv)

$$4 = 8 \cos \theta$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = 60°, \; 300°$$

The polar coordinates of intersection are;

$$(4, \; 60°), \; (4, \; 300°)$$

[4 marks]

Answer 4

(i)

$$
\begin{aligned}
r &= 4\cos(\theta - 30^\circ) \\
&= 4\cos\theta\cos 30 + 4\sin\theta\sin 30 \\
&= 4\cos\theta\,\frac{\sqrt{3}}{2} + 4\sin\theta \times \frac{1}{2} \\
&= 2\sqrt{3}\cos\theta + 2\sin\theta \qquad \square
\end{aligned}
$$

[2 marks]

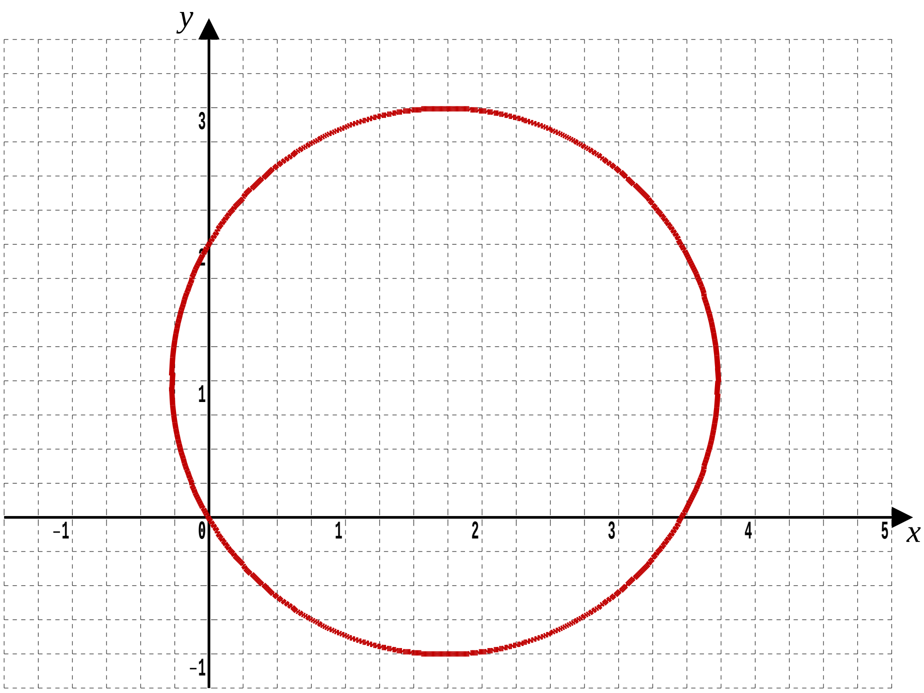
(ii) It is known that a circle,

Centre $(a,\ b)$, radius r ($\therefore$ passing through the pole)

: Cartesian $(x - a)^2 + (y - b)^2 = r^2$ with $r^2 = a^2 + b^2$

: Polar $r = 2a\cos\theta + 2b\sin\theta$

Thus $r = 2\sqrt{3}\cos\theta + 2\sin\theta$ has centre $(\sqrt{3},\ 1)$, radius 2



[3 marks]

(iii) $(x - \sqrt{3})^2 + (y - 1)^2 = 2^2$

[2 marks]

(iv) Rotation about the pole of 30° (anticlockwise)

[2 marks]

Answer 5

(i)

$$r = 5\,sec\,(\,\theta - 60°\,)$$

$$= \frac{5}{cos\;(\,\theta - 60\,)}$$

$$= \frac{5}{cos\,\theta\, cos\,60 + sin\,\theta\, sin\,60}$$

$$= \frac{5}{cos\,\theta \times \left(\frac{1}{2}\right) + sin\,\theta \times \left(\frac{\sqrt{3}}{2}\right)} \times \frac{2}{2}$$

$$= \frac{10}{cos\,\theta + \sqrt{3}\;sin\,\theta} \qquad \square$$

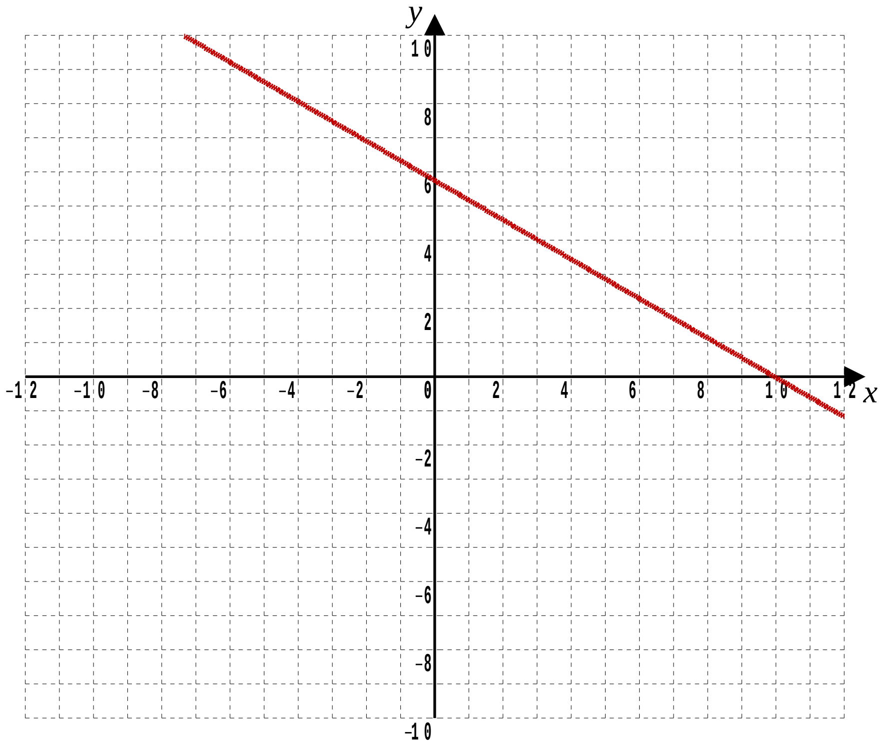
[2 marks]

(ii) It is known that for lines,

: Cartesian $y = mx + c$: $ax + by = c$

: Polar $r = \dfrac{c}{sin\,\theta - m\,cos\,\theta}$: $r = \dfrac{c}{a\,cos\,\theta + b\,sin\,\theta}$

Thus $r = \dfrac{10}{cos\,\theta + \sqrt{3}\;sin\,\theta}$ is the line $x + \sqrt{3}\,y = 10$



[3 marks]

(iii) $y = -\dfrac{1}{\sqrt{3}}\,x + \dfrac{10}{\sqrt{3}}$

[2 marks]

(iv) Rotation about the pole of 60° (anticlockwise)

[2 marks]

Answer 6

$$(x - 5)^2 + (y - 12)^2 = 169$$

$$x^2 - 10x + 25 + y^2 - 24y + 144 = 169$$

$$x^2 - 10x + y^2 - 24y = 0$$

$$x^2 + y^2 = 10x + 24y$$

$$r^2 = 10\, r \cos\theta + 24\, r \sin\theta$$

$$r = 10 \cos\theta + 24 \sin\theta \qquad \square$$

[6 marks]

Answer 7

$$y = x \tan\left(\sqrt{x^2 + y^2} \right)$$

$$y = x \tan\left(\sqrt{r^2} \right)$$

$$r \sin\theta = r \cos\theta \tan(r)$$

$$\sin\theta = \cos\theta \tan(r)$$

$$\frac{\sin\theta}{\cos\theta} = \tan(r)$$

$$\tan\theta = \tan(r)$$

$$\theta = \pm r$$

If r is restricted to being positive

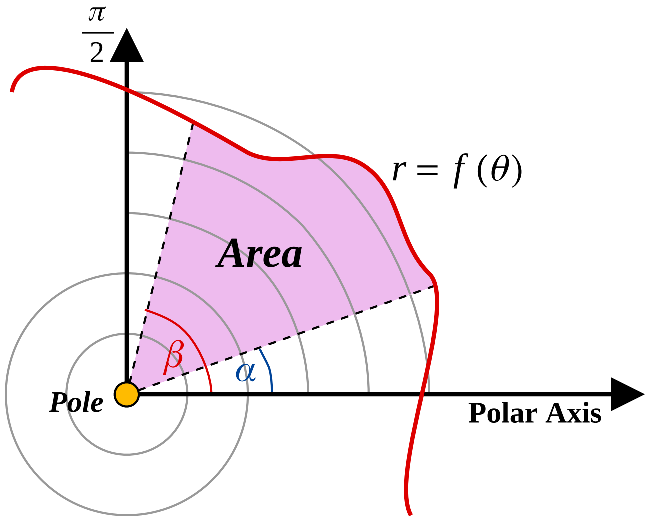
$$\theta = r \qquad \square$$

[3 marks]

5.1 Areas and Polar Curves

Integration can be used to find the area bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$ radiating out from the pole to the curve. A diagram to keep firmly in mind and the formula to apply are given below;

Polar Area Formula



$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Notice the importance of the pole in the areas being found, and also the similarity between the Polar Area Formula and the formula for the Area of a Sector.
Here is a reminder of that previous formula from the Year 2 course;

Arc Length and Sector Area

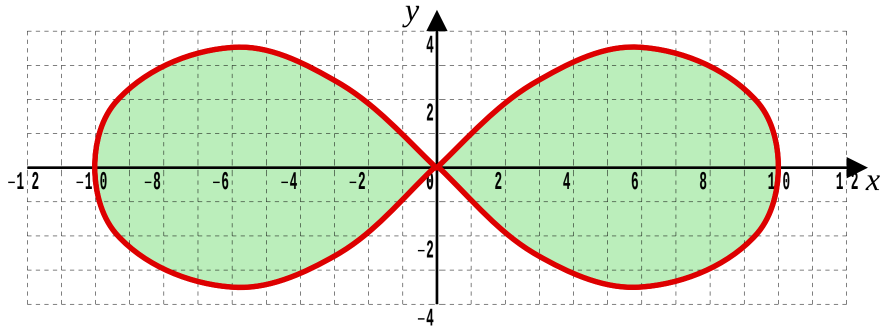
$$Arc\ length = r\,\theta^c$$

$$Sector\ area = \frac{1}{2} r^2 \theta^c$$

Given that integration is a summation process, it should not come as a surprise that what is being summed by the Polar Area Formula is an infinite number of infinitely thin sectors, all radiating out from the pole.
Hence the similarity of the Polar Area Formula with the Sector Area formula.
It goes without saying that angles α and β need to be in radians.

5.2 Example

Previously† it was shown that the famous curve graphed below, The Lemniscate of Bernoulli, had the polar equation $r^2 \; = \; 100\,cos\,(2\theta)$



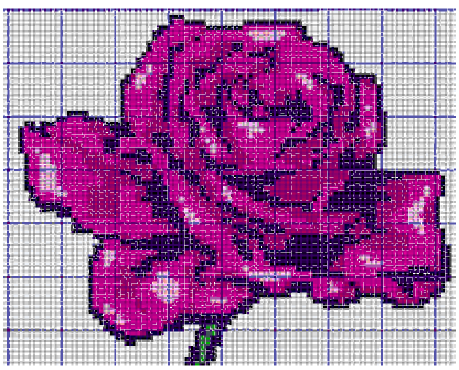
What is the total area of the region enclosed by the two loops ?

Teaching Video: http://www.NumberWonder.co.uk/v9101/5.mp4

[6 marks]

5.3 The Rose Graphs

The Lemniscate of Bernoulli is an example of a "Rose Graph" with two petals. Also called "flower graphs", they increasingly resemble a flower as the multiple of θ is increased and more and more petals occur.
Another rose graphs is explored in the exercise (Question 4)



† In Lesson 2, Exercise 2.3, Question 1

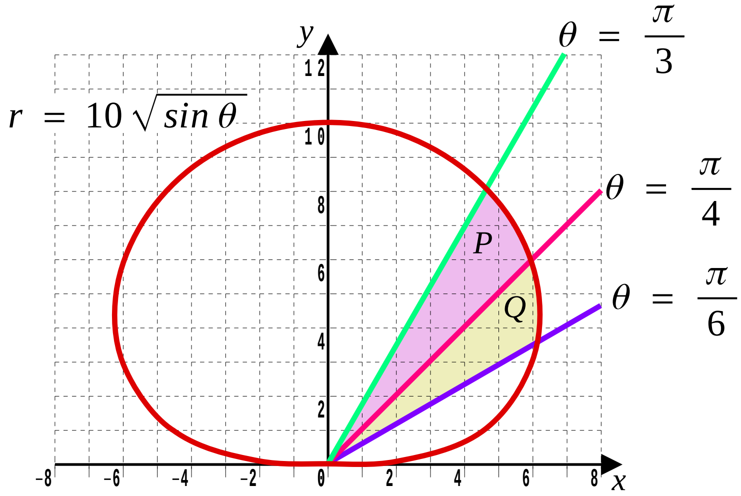
5.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 70

Question 1

The graph is of the curve with polar equation $r = 10\sqrt{\sin\theta}$ and the three half-lines with polar equations, $\theta = \dfrac{\pi}{6}$, $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{3}$



(i) Calculate the exact value of the area of P by means of the integration;

$$Area\ P = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(10\sqrt{\sin\theta}\right)^2 d\theta$$

[4 marks]

(ii) In a similar manner, show that,

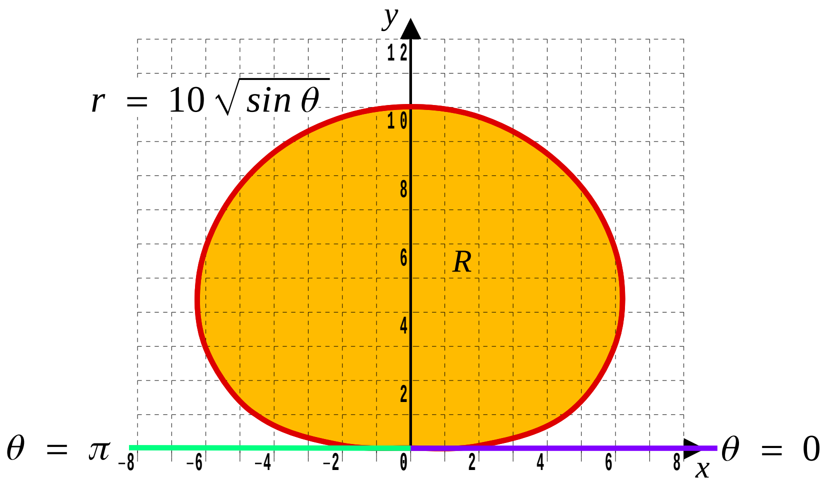
$$Area\ Q = 25\left(\sqrt{3} - \sqrt{2}\right)$$

[4 marks]

(iii) Explain why the domain for θ is $0 \leqslant \theta \leqslant \pi$ rather than $0 \leqslant \theta \leqslant 2\pi$

[2 marks]

(iv)



Keeping in mind the domain for θ identified in part (iii), find the exact total area within the loop, marked R in the above graph.

[3 marks]

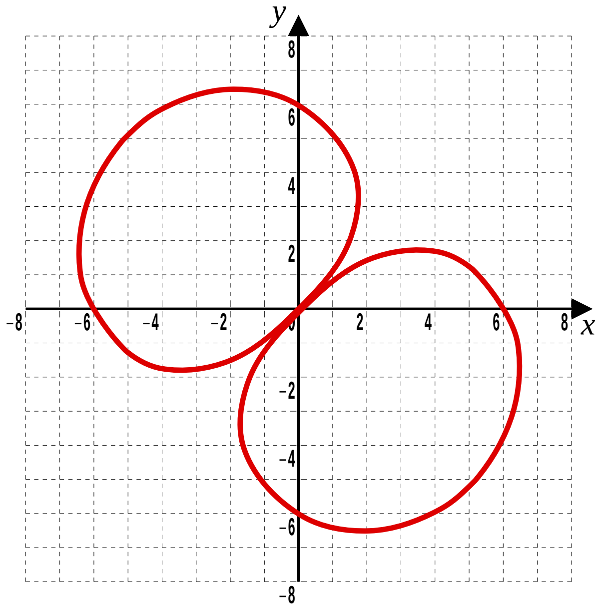
(v) Show that the Cartesian equation for the curve is,

$$\left(x^2 + y^2\right)^3 = 10000\,y^2, \qquad y \geqslant 0$$

[3 marks]

Question 2

The graph is of the curve with polar equation $r^2 = 36\,(\sin\theta - \cos\theta)$



(i) Show that if $I = \dfrac{1}{2}\displaystyle\int_0^{2\pi} r^2\,d\theta$ then $I = 0$

[4 marks]

Stay Positive !

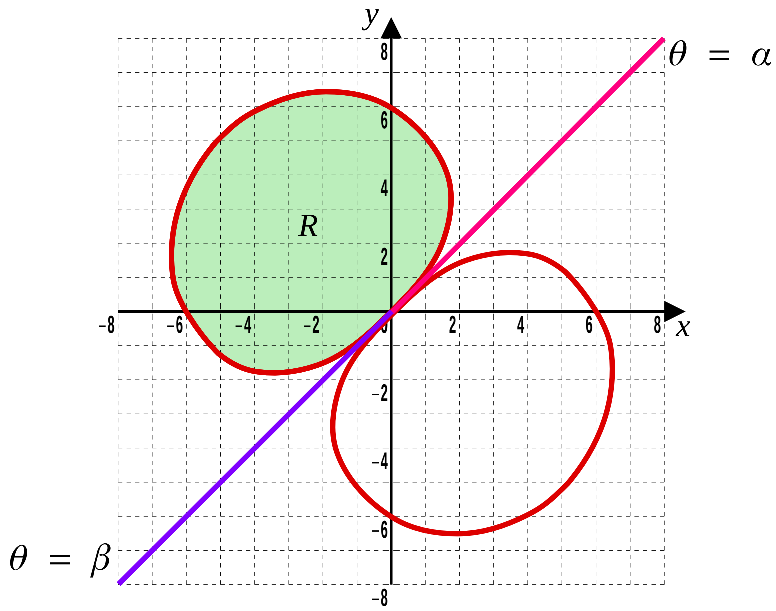
The reason for the failure of I to be the total area within the two loops is because the upper left loop corresponds to r being positive yielding a positive area, and the loop lower right corresponds to r being negative, yielding a negative area. Notice that, unlike with Cartesian integration, the quadrant the curve is in is not what determines whether or not the integral is positive or negative; from the polar point of view there is no x-axis for a curve to be below and thus negative.

The A-Level course recommends that limits should be chosen carefully such that $r \geqslant 0$ for all values within the domain of the integral.

(ii) Let α and β be the solutions to the equation $r^2 = 0$ with $\alpha < \beta$.
Find the value of α and the value of β for $0 \leqslant \alpha < 2\pi,\ 0 \leqslant \beta < 2\pi$

[3 marks]

(iii) Use your part (ii) values to find the exact area on the upper left loop, R.



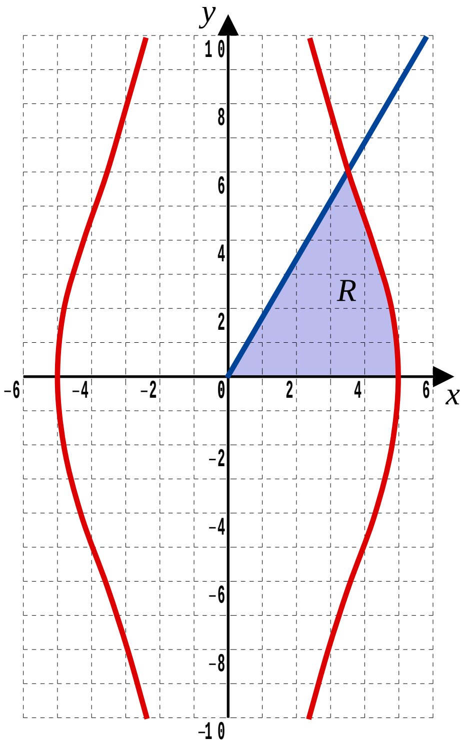
[4 marks]

(iv) Hence state the total exact area enclosed within the two loops

[1 mark]

Question 3

The graph is of the polar curve, $r^2 = 25\sec\theta$ and the half-line $\theta = \dfrac{\pi}{3}$



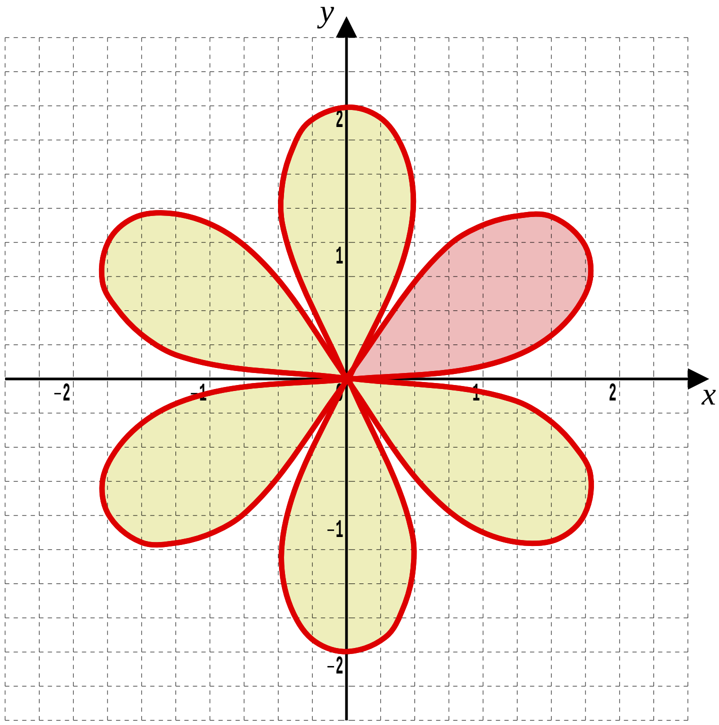
(i) Find the exact value of the area of the region R, shown shaded.

[7 marks]

(ii) Find a Cartesian equation for the curve $r^2 = 25\sec\theta$

[4 marks]

Question 4



(i) By first finding the area of one petal of the polar curve $r^2 = 4\,sin\,(3\theta)$ find the total area enclosed by the six petals of this rose.

[6 marks]

(ii) Prove that,

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

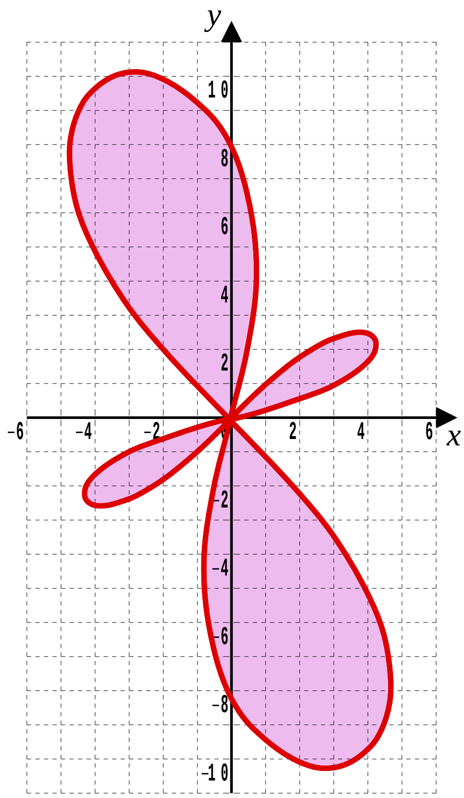
[3 marks]

(iii) Hence, or otherwise, find a Cartesian equation for the rose curve.

[4 marks]

Question 5

The graph is of the polar equation $r^2 = 64(\sin(4\theta) - \cos(2\theta))$



(i) Find the area enclosed by the four loops.

(ii) Find a Cartesian equation for the curve.

[18 marks]

5.4 Answers

5.4.1 Solution to 5.2 Example (Teaching Video)

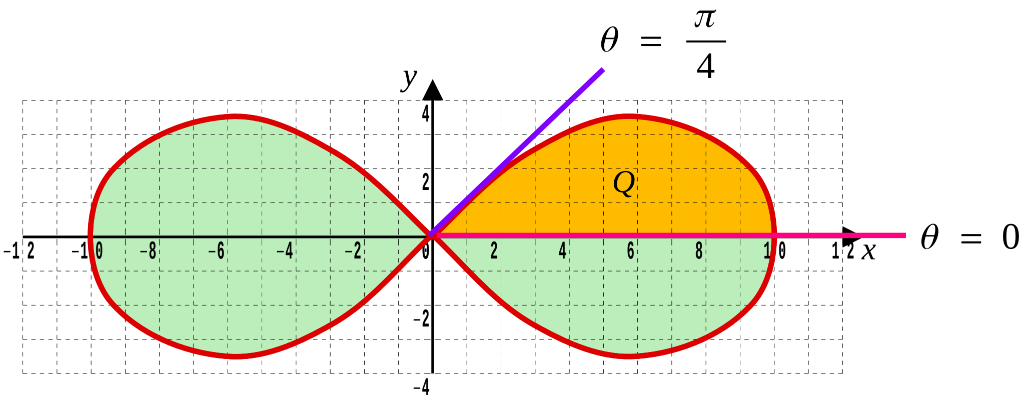
Integrations tend to be less work when the lower limit is zero, so a good tactic to get started on this problem is to work out where on the curve the point corresponding to $\theta = 0$ is.

As we have the graph, this is easy; simply look along the polar axis.
As there is only one point on the x-axis (not at the origin) $\theta = 0$ is $(\ 10,\ 0°\ )$ which is easily confirmed by putting $\theta = 0$ into the polar curve's equation.

With, along with the graph, the equation of the curve in mind,

$$r^2 = 100 \cos(2\theta)$$

it's reasonably obvious that the curve proceeds anticlockwise until at an angle of $\frac{\pi}{4}$ radians it's passing through the pole.



So one way of getting the required area is to do the following integration,

$$\begin{aligned}
Area &= 4 \times Area\ Q \\
&= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{4}} 100 \cos(2\theta)\ d\theta \\
&= 200 \int_0^{\frac{\pi}{4}} \cos(2\theta)\ d\theta \\
&= 100 \int_0^{\frac{\pi}{4}} 2 \cos(2\theta)\ d\theta \\
&= 100\ \Big[\ \sin(2\theta)\ \Big]_0^{\frac{\pi}{4}} \\
&= 100 \left[\sin\left(\frac{\pi}{2}\right) - \sin(0) \right] \\
&= 100
\end{aligned}$$

[6 marks]

5.4.2 Solutions to 5.3 Exercise

Answer 1

(i)

$$Area\ P = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(10\sqrt{sin\ \theta}\ \right)^2 d\theta$$

$$= 50 \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} sin\ \theta\ d\theta$$

$$= 50 \left[-cos\ \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= 50 \left[-cos\left(\frac{\pi}{3}\right) + cos\left(\frac{\pi}{4}\right) \right]$$

$$= 50 \left[-\frac{1}{2} + \frac{\sqrt{2}}{2} \right]$$

$$= 25\left(\sqrt{2} - 1 \right)$$

[4 marks]

(ii)

$$Area\ Q = \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(10\sqrt{sin\ \theta}\ \right)^2 d\theta$$

$$= 50 \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} sin\ \theta\ d\theta$$

$$= 50 \left[-cos\ \theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= 50 \left[-cos\left(\frac{\pi}{4}\right) + cos\left(\frac{\pi}{6}\right) \right]$$

$$= 50 \left[-\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \right]$$

$$= 25\left(\sqrt{3} - \sqrt{2} \right)$$

(iii) Over the interval $\pi < \theta < 2\pi$, $sin\ \theta$ is negative, so if that interval were part of the domain, the taking of a square root of a negative number would be an issue.

Had the equation been $r^2 = 100\ sin\ \theta$ the graph would have had a second petal that was a mirror reflection of the first in the polar axis.

[2 marks]

(iv)

$$Area\ R = \frac{1}{2}\int_0^{\pi}\left(10\sqrt{sin\ \theta}\right)^2 d\theta$$
$$= 50\int_0^{\pi} sin\ \theta\ d\theta$$
$$= 50\left[-cos\ \theta\right]_0^{\pi}$$
$$= 50\left[-cos\ \pi + cos\ 0\right]$$
$$= 50\left[-(-1) + 1\right]$$
$$= 100$$

[3 marks]

(v)

$$r = 10\sqrt{sin\ \theta}$$

Square both sides

(This will make the second 'ghost' loop visible)

$$r^2 = 100\ sin\ \theta$$

Multiply both sides by r

$$r^3 = 100\ r\ sin\ \theta$$
$$r^3 = 100\ y$$

Square both sides

$$\left(r^3\right)^2 = 10000\ y^2$$
$$\left(r^2\right)^3 = 10000\ y^2$$
$$\left(x^2 + y^2\right)^3 = 10000\ y^2$$

Add in a restriction to stop the 'ghost' loop being visible

$$\left(x^2 + y^2\right)^3 = 10000\ y^2, \qquad y \geqslant 0$$

[3 marks]

Answer 2

(i)
$$I = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} 36\ (\sin\theta - \cos\theta)\ d\theta$$
$$= 18\ \Big[-\cos\theta - \sin\theta\ \Big]_0^{2\pi}$$
$$= 18\left[-\cos(2\pi) - \sin(2\pi) + \cos 0 + \sin 0\right]$$
$$= 18\left[\ -1 - 0 + 1 + 0\right]$$
$$= 0 \qquad \square$$

[4 marks]

(ii)
$$r^2 = 0$$
$$36\ (\sin\theta - \cos\theta) = 0$$
$$\sin\theta - \cos\theta = 0$$
$$\sin\theta = \cos\theta$$
$$\frac{\sin\theta}{\cos\theta} = 1$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4},\ \frac{5\pi}{4}$$
$$\therefore\ \alpha = \frac{\pi}{4},\ \beta = \frac{5\pi}{4}$$

[3 marks]

(iii)
$$Area\ R = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 36\ (\sin\theta - \cos\theta)\ d\theta$$
$$= 18\Big[-\cos\theta - \sin\theta\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$
$$= 18\left[-\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)\ \right]$$
$$= 18\left[\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\ \right]$$
$$= 36\sqrt{2}$$

[4 marks]

(iv) Total Area $= 72\sqrt{2}$

[1 mark]

Answer 3

(i)

$$Area\ R = \frac{1}{2}\int_0^{\frac{\pi}{3}} 25\, sec\ \theta\ d\theta$$

$$= \frac{25}{2}\int_0^{\frac{\pi}{3}} sec\ \theta\ d\theta$$

$$= \frac{25}{2}\left[\ ln\left| sec\ \theta + tan\ \theta \right|\ \right]_0^{\frac{\pi}{3}}$$

$$= \frac{25}{2}\left[ln\left| \frac{1}{cos\left(\frac{\pi}{3}\right)} + tan\left(\frac{\pi}{3}\right)\right| - ln\left| \frac{1}{cos\ 0} + tan\ 0 \right| \right]$$

$$= \frac{25}{2}\left[\ ln\left| 2 + \sqrt{3}\ \right| - ln\left| 1 + 0 \right|\ \right]$$

$$= \frac{25}{2} ln\left(2 + \sqrt{3}\ \right)$$

[7 marks]

(ii)

$$r^2 = 25\, sec\ \theta$$

$$r^2 = \frac{25}{cos\ \theta}$$

$$r\ r\ cos\ \theta = 25$$

$$r\ x = 25$$

Square both sides

$$r^2\ x^2 = 625$$

$$(\ x^2 + y^2\ )\ x^2 = 625$$

[4 marks]

Answer 4

(i) For the petal highlighted in a peach colour,

$$Area = \frac{1}{2} \int_0^{\frac{\pi}{3}} 4\,sin(3\theta)\ d\theta$$

$$= \frac{2}{3} \int_0^{\frac{\pi}{3}} 3\,sin(3\theta)\ d\theta \qquad \text{Setting up a "chain rule backwards"}$$

$$= \frac{2}{3}\left[-\,cos(3\theta)\ \right]_0^{\frac{\pi}{3}}$$

$$= \frac{2}{3}\left[-\,cos\,\pi\ +\ cos\,0\right]$$

$$= \frac{2}{3}\left[-\ (-\,1)\ +\ 1\right]$$

$$= \frac{4}{3}$$

$\therefore$ Total area of all six petals is 8

[6 marks]

(ii)

$$\begin{aligned}
\text{LHS} &= sin\,3\theta \\
&= sin\ (\,2\theta\ +\ \theta\,) \\
&= sin\,2\theta\ cos\,\theta\ +\ cos\,2\theta\ sin\,\theta \\
&= 2\,sin\,\theta\ cos\,\theta\ cos\,\theta\ +\ \left(cos^2\,\theta\ -\ sin^2\,\theta\right) sin\,\theta \\
&= 2\,sin\,\theta\ cos^2\,\theta\ +\ cos^2\,\theta\ sin\,\theta\ -\ sin^3\,\theta \\
&= 3\,sin\,\theta\ cos^2\,\theta\ -\ sin^3\,\theta \\
&= 3\,sin\,\theta\ \left(1\ -\ sin^2\,\theta\right)\ -\ sin^3\,\theta \\
&= 3\,sin\,\theta\ -\ 3\,sin^3\,\theta\ -\ sin^3\,\theta \\
&= 3\,sin\,\theta\ -\ 4\,sin^3\,\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

(iii)

$$r^2 = 4\,sin\,(3\theta)$$

$$r^2 = 4\left(3\,sin\,\theta\ -\ 4\,sin^3\,\theta\right)$$

Multiply both sides by r^3

$$r^5 = 12\,r^2\,r\,sin\,\theta\ -\ 16\,r^3\,sin^3\,\theta$$

$$\left(x^2\ +\ y^2\right)^{\frac{5}{2}} = 12\left(x^2\ +\ y^2\right)y\ -\ 16\,y^3$$

$$\left(x^2\ +\ y^2\right)^{\frac{5}{2}} = 12\,x^2\,y\ -\ 4\,y^3$$

[4 marks]

Answer 5

(i) To get the limit lines,

$$\begin{aligned} r^2 &= 0 \\ 64\,(\,\sin(4\theta) - \cos(2\theta)\,) &= 0 \\ \sin(4\theta) - \cos(2\theta) &= 0 \\ 2\sin(2\theta)\cos(2\theta) - \cos(2\theta) &= 0 \\ \cos(2\theta)(2\sin(2\theta) - 1\,) &= 0 \end{aligned}$$

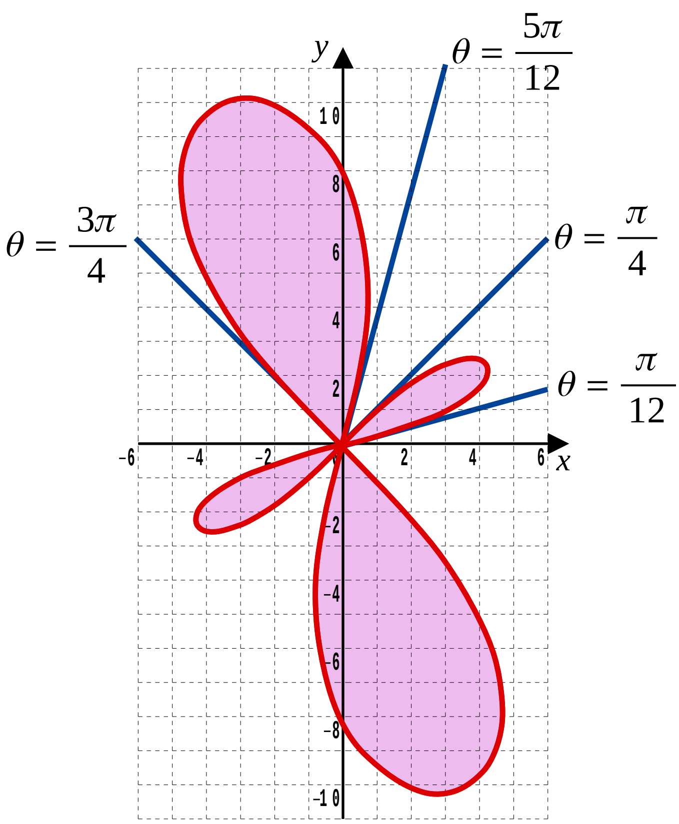
$$\text{Either } \cos(2\theta) = 0 \text{ or } \sin(2\theta) = \frac{1}{2}$$

$$\text{For } \cos(2\theta) = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\text{For } \sin(2\theta) = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$



So the areas sought are,

$$\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} r^2\, d\theta \quad \text{and} \quad \frac{1}{2}\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} r^2\, d\theta$$

Area of loop wholly in 1st Quadrant:

$$Area = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} r^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 64\left(sin(4\theta) - cos(2\theta) \right) d\theta$$

$$= 32\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} sin(4\theta) - cos(2\theta)\ d\theta$$

$$= 8\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 4\,sin(4\theta)\ d\theta - 16\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 2\,cos(2\theta)\ d\theta$$

$$= 8\left[- cos(4\theta)\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}} - 16\left[sin(2\theta)\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$$

$$= 8\left[- cos\,\pi + cos\left(\frac{\pi}{3}\right)\right] - 16\left[sin\left(\frac{\pi}{2}\right) - sin\left(\frac{\pi}{6}\right)\right]$$

$$= 8\left[-(-1) + \frac{1}{2}\right] - 16\left[1 - \frac{1}{2}\right]$$

$$= 8 \times \frac{3}{2} - 16 \times \frac{1}{2}$$

$$= 12 - 8$$

$$= 4$$

For the larger loop, mostly in the 2nd Quadrant

$$Area = \frac{1}{2}\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} r^2\, d\theta$$

$$= 32\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} sin(4\theta) - cos(2\theta)\ d\theta$$

$$= 8\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} 4\,sin(4\theta)\ d\theta - 16\int_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} 2\,cos(2\theta)\ d\theta$$

$$= 8\left[- cos(4\theta)\right]_{\frac{5\pi}{12}}^{\frac{3\pi}{4}} - 16\left[sin(2\theta)\right]_{\frac{5\pi}{12}}^{\frac{3\pi}{4}}$$

$$= 8\left[- cos\,3\pi + cos\left(\frac{5\pi}{3}\right)\right] - 16\left[sin\left(\frac{3\pi}{2}\right) - sin\left(\frac{5\pi}{6}\right)\right]$$

$$= 8\left[-(-1) + \frac{1}{2}\right] - 16\left[-1 - \frac{1}{2}\right]$$

$$= 36$$

$$\therefore \text{ Total Area} = 2(4 + 36)$$

$$= 80$$

[12 marks]

(ii)

$$\begin{aligned} r^2 &= 64 (\sin(4\theta) - \cos(2\theta)) \\ &= 64 (2 \sin(2\theta) \cos(2\theta) - \cos(2\theta)) \\ &= 64 \cos(2\theta) (2 \sin(2\theta) - 1) \\ &= 64 \left(\cos^2\theta - \sin^2\theta \right) (4 \sin\theta \cos\theta - 1) \\ r^6 &= 64 \left(r^2 \cos^2\theta - r^2 \sin^2\theta \right) \left(4 r \sin\theta \, r \cos\theta - r^2 \right) \\ \left(x^2 + y^2 \right)^3 &= 64 \left(x^2 - y^2 \right) \left(4yx - x^2 - y^2 \right) \end{aligned}$$

[6 marks]

6.1 Exam : Bring It ON

Polar equations, by their predominantly circular nature, are awash with trigonometric functions. In consequence, the resulting integrations will often involve squares of those functions especially when finding sector areas.

That's because of the r^2 in the formula $Area = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta$

Rather than memorising lots of results the prevailing wisdom is that it is better to focus on a few that are key, from which others are easily obtained when required.

6.2 Cosine

How to find $\int cos^2 \theta \, d\theta$

The two key results to use are;

$$cos^2 \theta + sin^2 \theta = 1$$
$$cos^2 \theta - sin^2 \theta = cos(2\theta)$$

Combine by addition;

$$2\,cos^2 \theta = 1 + cos(2\theta)$$

$$\int cos^2 \theta \, d\theta = \frac{1}{2} \int 1 + cos(2\theta) \, d\theta$$

6.3 Sine

How to find $\int sin^2 \theta \, d\theta$

The two key results to use are;

$$cos^2 \theta + sin^2 \theta = 1$$
$$cos^2 \theta - sin^2 \theta = cos(2\theta)$$

Combine by subtraction;

$$2\,sin^2 \theta = 1 - cos(2\theta)$$

$$\int sin^2 \theta \, d\theta = \frac{1}{2} \int 1 - cos(2\theta) \, d\theta$$

6.4 And Some...

$\int sec^2 \, d\theta = tan\, x + c$: Told this in examination formula booklet

$\int tan^2 \theta \, d\theta$: Use the identity $1 + tan^2 \theta = sec^2 \theta$

$\int csc^2 \theta \, d\theta = -cot\, \theta + c$: Told this in examination formula booklet

$\int cot^2 \theta \, d\theta$: Use the identity $cot^2 \theta + 1 = csc^2 \theta$

6.5 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 42

Question 1

Further A-Level Examination Question from June 2019, Core Paper 1, Q3 (Edexcel)

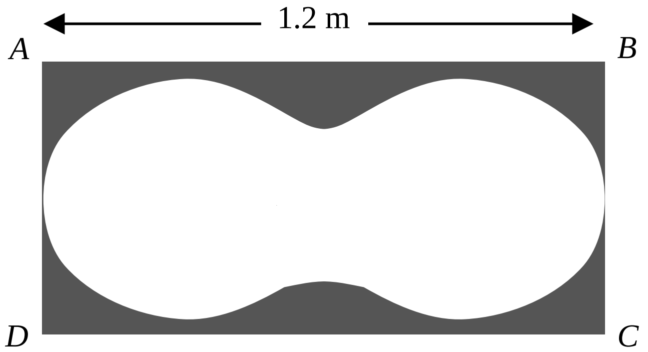


Diagram not to scale

The diagram shows the design for a table top in the shape of a rectangle *ABCD*.
The length of the table, *AB*, is 1.2 m.
The area inside the closed curve is made of glass.
The surrounding area, shown shaded, is made of wood.

The perimeter of the glass is modelled by the curve with polar equation

$$r = 0.4 + a\cos 2\theta \qquad 0 \leqslant \theta < 2\pi$$

where a is a constant.

(a) Show that $a = 0.2$

[2 marks]

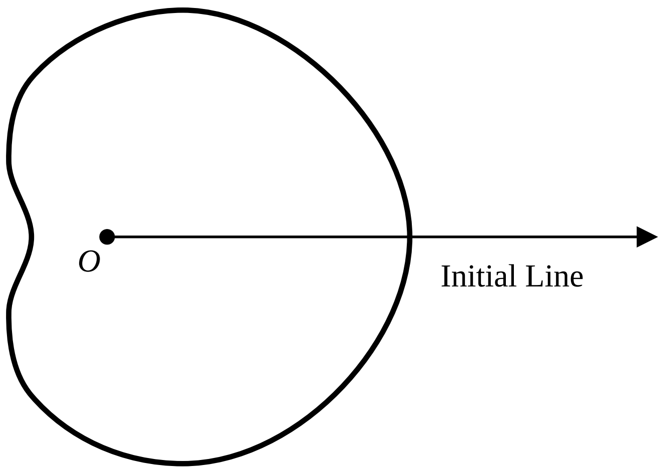
Hence, given that $AD = 60$ cm,

(b) find the area of the wooden part of the table top, giving your answer in m^2 to 3 significant figures.

[8 marks]

Question 2

Further A-Level Examination Question from June 2009, FP2, Q4 (Edexcel)



The sketch is of the curve with polar equation,

$$r = a + 3\cos\theta, \quad a > 0, \quad 0 \leqslant \theta < 2\pi$$

The area enclosed by the curve $\dfrac{107}{2}\pi$

Find the value of a

[8 marks]

Question 3

Further A-Level Examination Question from June 2015, FP2, Q1 (MEI)

(i) A curve has polar equation $r = 2a\cos\theta + 2b\sin\theta$,

where $a > 0$ and $b > 0$

Show, by considering the Cartesian equation, that the curve is a circle which passes through the origin.

Find the centre and radius of the circle in term of a and b

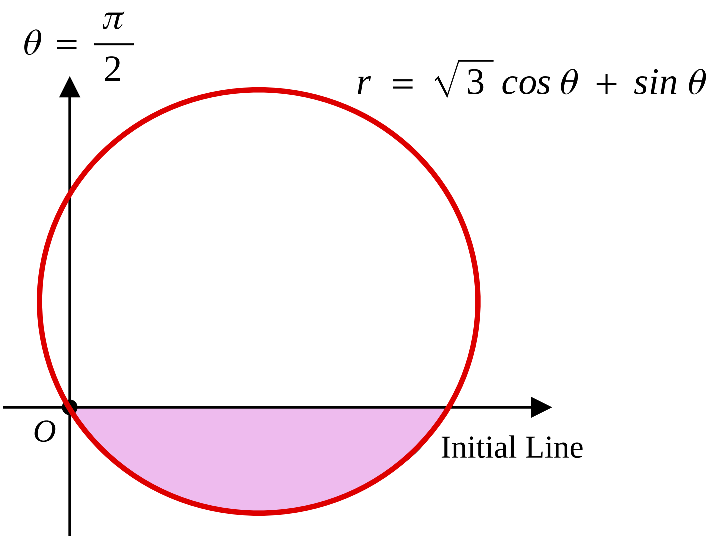
[5 marks]

(ii) For the case $a = b = 1$, use integration to show that the region bound by a minor arc of the circle and lines $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{\pi}{3}$ has area $1 + \dfrac{\pi}{3}$

[5 marks]

Question 4

Further A-Level Examination Question, Paper O, FP2 (Madas Maths)



The diagram above shows the curve with polar equation

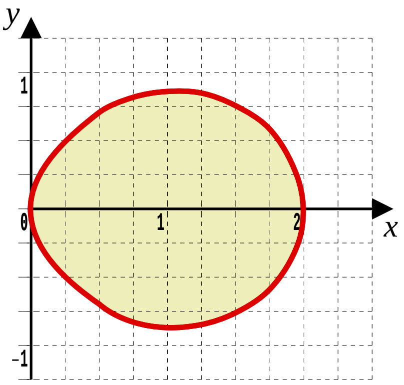
$$r = \sqrt{3}\cos\theta + \sin\theta, \quad -\frac{\pi}{3} \leqslant \theta < \frac{2\pi}{3}$$

By using a method involving integration in polar coordinates, show that the area of the shaded region is $\frac{1}{12}\left(4\pi - 3\sqrt{3}\right)$

[7 marks]

Question 5

Further A-Level Examination Question from Summer 2000, Mock P4, Q7 (Edexcel)



The curve C is given by; $\quad r = 2\cos^{\frac{3}{2}}\theta \qquad -\dfrac{\pi}{2} \leqslant \theta < \dfrac{\pi}{2}$

Find the area of the region enclosed by C

[7 marks]

Answers 1

(a) Clearly the pole is at the centre of the diagram in which case, also from the diagram, it would seem that r is 0.6 m when $\theta = 0$

$$r = 0.4 + a \cos 2\theta$$

$$0.6 = 0.4 + a \cos 0$$

$$a = 0.2 \qquad \square$$

[2 marks]

(b) The area of the table top will be,

$$\text{Area Rectangle} = 1.2 \times 0.6$$

$$= 0.72 \text{ m}^2$$

For the glass, this is a forgiving question in that there is no issue with a negative value of r as the curve would have to go through the pole to change sign which it clearly does not.

So any suitable pair of limits will do, although it'll be easier to use the polar axis with $\theta = 0$ as the lower limit,

Sensible upper limits would be $\theta = \dfrac{\pi}{2}$ or $\theta = \pi$ or $\theta = 2\pi$

$$Area = \frac{1}{2} \int (0.4 + 0.2 \cos 2\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int (0.2 (2 + \cos 2\theta))^2 \, d\theta$$

$$= \frac{1}{2} 0.2^2 \int (2 + \cos 2\theta)^2 \, d\theta$$

$$= \frac{1}{50} \int 4 + 4 \cos 2\theta + \cos^2 2\theta \, d\theta$$

$$= \frac{1}{50} \left[\int 4 \, d\theta + 2 \int 2 \cos 2\theta \, d\theta + \frac{1}{2} \int 1 + \cos 4\theta \, d\theta \right]$$

$$= \frac{1}{50} \left[4\theta + 2 \sin 2\theta + \frac{\theta}{2} + \frac{1}{8} \int 4 \cos 4\theta \, d\theta \right]$$

$$= \frac{1}{50} \left[4\theta + 2 \sin 2\theta + \frac{\theta}{2} + \frac{1}{8} \sin 4\theta \right]$$

$$= \frac{1}{400} [32\theta + 16 \sin 2\theta + 4\theta + \sin 4\theta]$$

$$= \frac{1}{400} [36\theta + 16 \sin 2\theta + \sin 4\theta]$$

I'll use limits of 0 and 2π,

$$Area = \frac{1}{400}\left[\,72\pi + 16\sin 4\pi + \sin 8\pi - 0 - 0 - 0\,\right]$$

$$= \frac{72\pi}{400}$$

$$= \frac{9\pi}{50}$$

$\therefore$ The area of the wood is $0.72 - \dfrac{9\pi}{50}$

$$= \frac{36 - 9\pi}{50}$$

$$= \frac{9\,(\,4 - \pi\,)}{50}$$

$$= 0.155\ \text{m}^2$$

[8 marks]

Answer 2

$$I = \frac{1}{2}\int r^2\, d\theta$$

$$= \frac{1}{2}\int (\,a + 3\cos\theta\,)^2\, d\theta$$

$$= \frac{1}{2}\int a^2 + 6a\cos\theta + 9\cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int a^2 + 6a\cos\theta + \frac{9}{2} + \frac{9}{4}\,2\cos 2\theta\ d\theta$$

$$= \frac{1}{2}\left[\,a^2\theta + 6a\sin\theta + \frac{9\theta}{2} + \frac{9}{4}\sin 2\theta\,\right] + c$$

With lower limit 0 and upper limit 2π this will become,

$$= \frac{1}{2}\left[\,a^2\,2\pi + 0 + 9\pi + 0 - 0 - 0 - 0 - 0\,\right]$$

$$= \frac{\pi}{2}\left(2\,a^2 + 9\right)$$

Using the given area,

$$\frac{\pi}{2}\left(2\,a^2 + 9\right) = \frac{107\pi}{2}$$

$$2\,a^2 + 9 - 107 = 0$$

$$2\,a^2 = 98$$

$$a = 7 \qquad \text{as} \quad a > 0$$

[8 marks]

Answer 3

(i)

$$r = 2a \cos\theta + 2b \sin\theta$$

Multiply through by r

$$r^2 = 2a\, r \cos\theta + 2b\, r \sin\theta$$

$$x^2 + y^2 = 2ax + 2by$$

$$x^2 - 2ax + y^2 - 2by = 0$$

$$(x - a)^2 - a^2 + (y - b)^2 - b^2 = 0$$

$$(x - a)^2 + (y - b)^2 = a^2 + b^2$$

Which is recognized as a circle, centre $(a,\ b)$ and radius $\sqrt{a^2 + b^2}$

[5 marks]

(ii)

$$R = \frac{1}{2}\int (2\cos\theta + 2\sin\theta)^2\, d\theta \quad \text{as} \quad a = b = 1$$

$$= 2\int (\cos\theta + \sin\theta)^2\, d\theta \qquad \text{as} \quad a = b = 1$$

$$= 2\int \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\, d\theta$$

$$= 2\int 1 + \sin 2\theta\, d\theta$$

$$= 2\theta - \cos 2\theta + c$$

With the lower limit of $\frac{\pi}{6}$ and an upper limit of $\frac{\pi}{3}$ this becomes,

$$= 2\left(\frac{\pi}{3}\right) - \cos\left(\frac{2\pi}{3}\right) - 2\left(\frac{\pi}{6}\right) + \cos\left(\frac{2\pi}{6}\right)$$

$$= \frac{2\pi}{3} + \frac{1}{2} - \frac{\pi}{3} + \frac{1}{2}$$

$$= 1 + \frac{\pi}{3} \qquad \square$$

[5 marks]

Answer 4

A circle with centre (a, b), radius r that passes through the pole has equation

: Cartesian $(x - a)^2 + (y - b)^2 = r^2$ with $r^2 = a^2 + b^2$

: Polar $r = 2a\cos\theta + 2b\sin\theta$

$\therefore\ r = \sqrt{3}\cos\theta + \sin\theta$ has centre $\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$ and radius 2

When $r = 0$,
$$\begin{aligned}
\sqrt{3}\cos\theta + \sin\theta &= 0\\
\sin\theta &= -\sqrt{3}\cos\theta\\
\frac{\sin\theta}{\cos\theta} &= -\sqrt{3}\\
\tan\theta &= -\sqrt{3}\\
\theta &= -\frac{\pi}{3},\ \frac{2\pi}{3}
\end{aligned}$$

Which the question pretty much told us anyway !

$$\begin{aligned}
I &= \frac{1}{2}\int\left(\sqrt{3}\cos\theta + \sin\theta\right)^2 d\theta\\
&= \frac{1}{2}\int\left(3\cos^2\theta + 2\sqrt{3}\cos\theta\sin\theta + \sin^2\theta\right) d\theta\\
&= \frac{1}{2}\int\left(2\cos^2\theta + \sqrt{3}\sin 2\theta + 1\right) d\theta\\
&= \frac{1}{2}\int\left(1 + \cos 2\theta + \sqrt{3}\sin 2\theta + 1\right) d\theta\\
&= \frac{1}{2}\int\left(2 + \cos 2\theta + \sqrt{3}\sin 2\theta\right) d\theta\\
&= \int 1\, d\theta + \frac{1}{4}\int 2\cos 2\theta\, d\theta + \frac{\sqrt{3}}{4}\int 2\sin 2\theta\, d\theta\\
&= \theta + \frac{1}{4}\sin 2\theta - \frac{\sqrt{3}}{4}\cos 2\theta + c
\end{aligned}$$

With lower limit $-\dfrac{\pi}{3}$ and upper limit 0 this becomes,

$$\begin{aligned}
&= 0 + \frac{1}{4}\sin 0 - \frac{\sqrt{3}}{4}\cos 0 - \left(-\frac{\pi}{3}\right) - \frac{1}{4}\sin\left(-\frac{2\pi}{3}\right) + \frac{\sqrt{3}}{4}\cos\left(-\frac{2\pi}{3}\right)\\
&= -\frac{\sqrt{3}}{4} + \frac{\pi}{3} + \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8}\\
&= \frac{1}{12}\left(4\pi - 3\sqrt{3}\right) \quad \square
\end{aligned}$$

[7 marks]

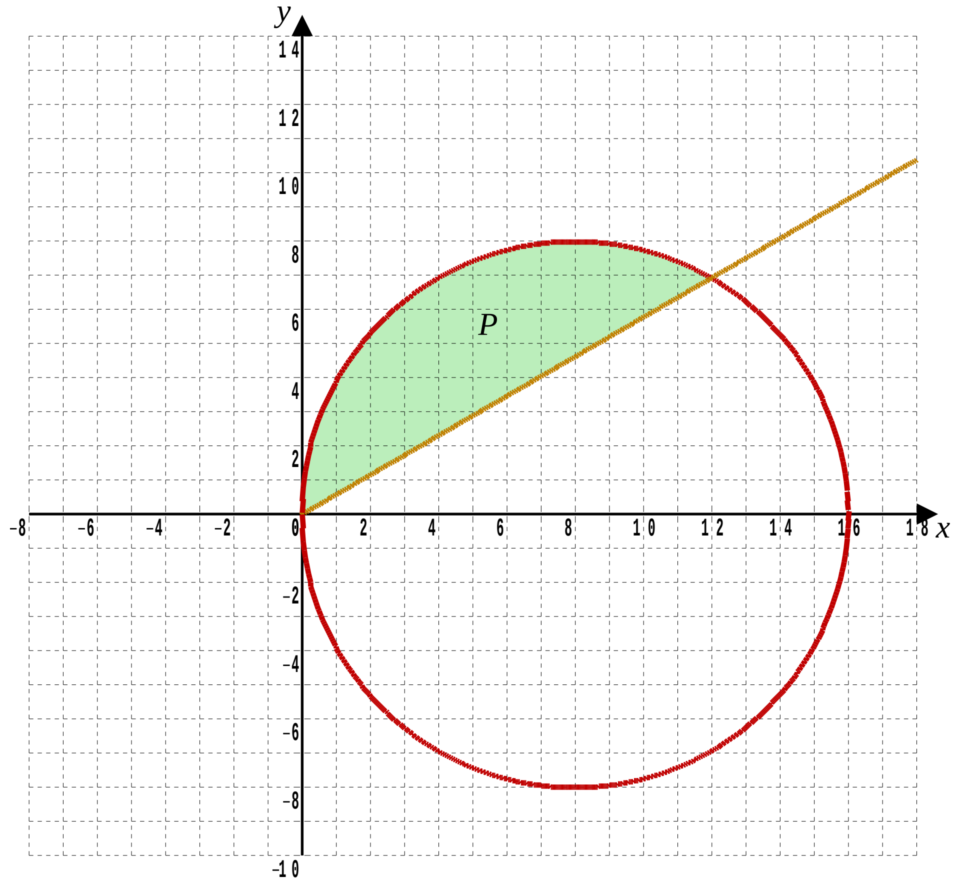
Answer 5

$$
\begin{aligned}
Area &= \frac{1}{2} \int r^2 \, d\theta \\
&= 2 \times \frac{1}{2} \int_0^{\frac{\pi}{2}} 4 \cos^3 \theta \, d\theta \\
&= \int_0^{\frac{\pi}{2}} 4 \cos \theta \cos^2 \theta \, d\theta \\
&= 4 \int_0^{\frac{\pi}{2}} \cos \theta \, (1 - \sin^2 \theta) \, d\theta \\
&= 4 \int_0^{\frac{\pi}{2}} \cos \theta \, d\theta - 4 \int_0^{\frac{\pi}{2}} (\cos \theta) (\sin \theta)^2 \, d\theta \\
&= 4 \left[\sin \theta \right]_0^{\frac{\pi}{2}} - 4 \left[\frac{\sin^3 \theta}{3} \right]_0^{\frac{\pi}{2}} \\
&= 4 \left[\sin\left(\frac{\pi}{2}\right) - \sin 0 \right] - \frac{4}{3} \left[\sin^3\left(\frac{\pi}{2}\right) - \sin^3 0 \right] \\
&= 4 - \frac{4}{3} \\
&= \frac{8}{3}
\end{aligned}
$$

[7 marks]

7.1 Area Problems, Taken Further

Here is a recap of the type of problem tackled in the previous lesson,



The segment shaded is in the circle $r = 16\cos\theta$ and bounded by $\theta = \dfrac{\pi}{6}$

It's area is given by,

$$P = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 16^2 \cos^2\theta \, d\theta$$

$$= \frac{16^2}{2} \times \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 + 2\cos 2\theta) \, d\theta$$

$$= 32\,[\,2\theta + \sin 2\theta\,]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 32\left[\,\pi + \sin\pi - \frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\,\right]$$

$$= 32\left[\,\pi + 0 - \frac{\pi}{3} - \frac{\sqrt{3}}{2}\,\right]$$

$$= \frac{64\pi}{3} - 16\sqrt{3} \qquad \text{(About 39.3)}$$

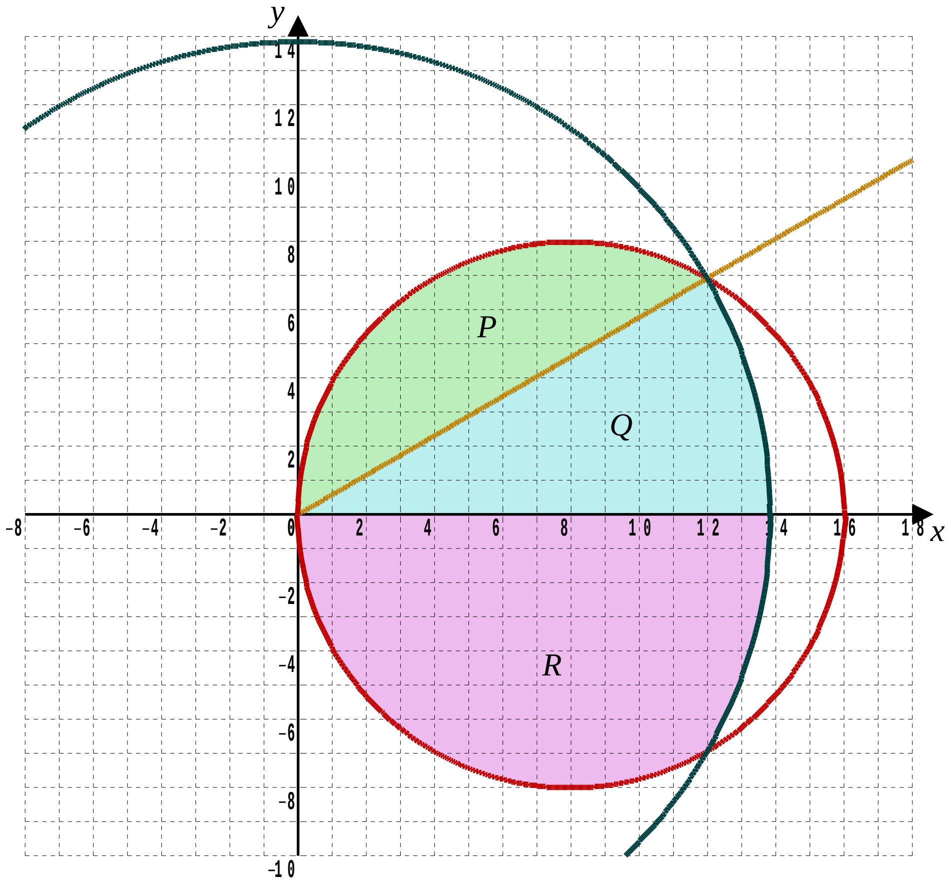
With this in mind, a related problem is considered next.

7.2 Area of Overlap

Find the area of the overlap between the two circles with polar equations,

$$r = 16\cos\theta \text{ and } r = \sqrt{192}$$

This area is the total areas of the green, blue and red shaded regions below,



To find where the circles intersect,

$$16\cos\theta = \sqrt{192} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \pm\frac{\pi}{6}$$

Thus the gold line $\theta = \frac{\pi}{6}$ is added to the graph.

Half of the area sought is the green area, P, plus the blue area, Q.
But the green area, P, is the same as that worked out in the previous problem.

$$Q = \frac{1}{2}\int_0^{\frac{\pi}{6}} 192\, d\theta$$
$$= 16\pi$$

$$\therefore \text{Area of overlap} = 2 \times (P + Q)$$
$$= 2\left(\frac{64\pi}{3} - 16\sqrt{3} + 16\pi\right)$$
$$= \frac{224\pi}{3} - 32\sqrt{3} \quad \text{(About 179)}$$

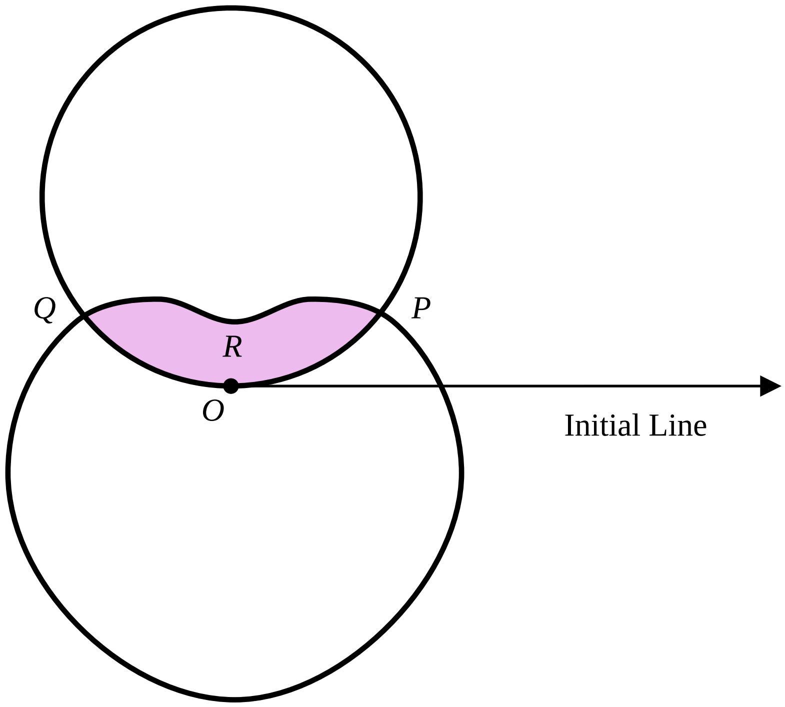
7.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 21

Question 1

Further A-Level Examination Question from June 2018, FP2, Q8 (Edexcel)



Th sketch is of the curves with polar equations

$$r = 2\sin\theta \qquad 0 \leqslant \theta \leqslant \pi$$

$$r = 1.5 - \sin\theta \qquad 0 \leqslant \theta \leqslant 2\pi$$

The curves intersect at the points P and Q

(a) Find the polar coordinates of the point P and the polar coordinates of the point Q

[3 marks]

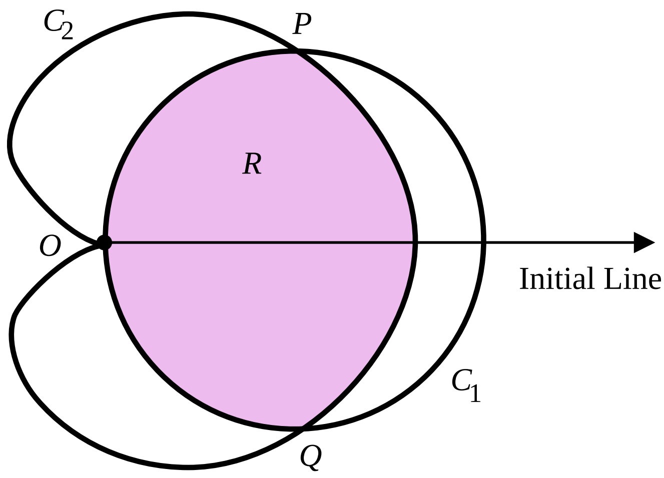
The region R, shown shaded, is enclosed by the two curves.

(b) Find the exact area of R giving your answer in the form $p\pi + q\sqrt{3}$, where p and q are rational numbers to be found.

[8 marks]

Question 2

Further A-Level Examination Question from June 2016, FP2, Q8 (Edexcel)



The sketch is of the curve C_1 with equation

$$r = 7\cos\theta \qquad -\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$$

and the curve C_2 with equation

$$r = 3(1 + \cos\theta) \qquad -\pi < \theta \leqslant \pi$$

The curves C_1 and C_2 both pass through the pole and intersect at the point P and at the point Q

(a) Find the polar coordinates of P and the polar coordinates of Q

[3 marks]

The regions enclosed by the curve C_1 and the curve C_2 overlap, and the common region is shaded in the sketch.

(b) Find the area of R

[7 marks]

7.4 Answers to 7.3 Exercise

Answer 1

(a) When the curves intersect,

$$2 \sin\theta = 1.5 - \sin\theta$$

$$3 \sin\theta = 1.5$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \ \frac{5\pi}{6}$$

With the diagram as a guide, $P\left(1, \frac{\pi}{6}\right)$, $Q\left(1, \frac{5\pi}{6}\right)$

[3 marks]

(b)

$$Area = 2 \times \frac{1}{2} \int_0^{\frac{\pi}{6}} (2 \sin\theta)^2 \, d\theta + 2 \times \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{3}{2} - \sin\theta \right)^2 d\theta$$

$$= 4 \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{9}{4} - 3 \sin\theta + \sin^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 2 - 2\cos 2\theta \, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{9}{4} - 3 \sin\theta + \frac{2}{4} - \frac{1}{4} \times 2 \cos 2\theta \, d\theta$$

$$= \left[2\theta - \sin 2\theta \right]_0^{\frac{\pi}{6}} + \left[\frac{11\theta}{4} + 3\cos\theta - \frac{1}{4} \sin 2\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{11\pi}{8} - \frac{11\pi}{24} - \frac{3\sqrt{3}}{2} + \frac{1}{4} \times \frac{\sqrt{3}}{2}$$

$$= \frac{5}{4}\pi - \frac{15}{8}\sqrt{3}$$

[8 marks]

Answer 2

(a) When the poles intersect,

$$7 \cos\theta = 3(1 + \cos\theta)$$

$$4 \cos\theta = 3$$

$$\cos\theta = \frac{3}{4}$$

$$\theta = \pm 0.7227...$$

With the diagram as a guide, $P\left(\frac{21}{4},\ 0.7227...\right),\ Q\left(\frac{21}{4}, -0.7227...\right)$

[3 marks]

(b) Let $\phi = \arccos\left(\frac{3}{4}\right) \Rightarrow \cos\phi = \frac{3}{4},\ \sin\phi = \frac{\sqrt{7}}{4}$

$$Area = 2 \times \frac{1}{2}\int_0^{\phi} 3^2(1 + \cos\theta)^2\ d\theta + 2 \times \frac{1}{2}\int_{\phi}^{\frac{\pi}{2}} 7^2 \cos^2\theta\ d\theta$$

$$= 9\int_0^{\phi} 1 + 2\cos\theta + \cos^2\theta\ d\theta + 49\int_{\phi}^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

$$= 9\int_0^{\phi} 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{4} \times 2\cos 2\theta\ d\theta + 49\int_{\phi}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{4} \times 2\cos 2\theta\ d\theta$$

$$= 9\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\phi} + 49\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_{\phi}^{\frac{\pi}{2}}$$

$$= \frac{27\phi}{2} + 18\sin\phi + \frac{9}{4}\sin 2\phi + \frac{49\pi}{4} - \frac{49\phi}{2} - \frac{49}{4}\sin 2\phi$$

$$= \frac{49\pi}{4} - 11\phi + 18\sin\phi - 10\sin 2\phi$$

$$= \frac{49\pi}{4} - 11\phi + 18\sin\phi - 20\sin\phi\cos\phi$$

$$= \frac{49\pi}{4} - 11\arccos\left(\frac{3}{4}\right) + 18 \times \frac{\sqrt{7}}{4} - 20 \times \frac{\sqrt{7}}{4} \times \frac{3}{4}$$

$$= \frac{49\pi}{4} - 11\arccos\left(\frac{3}{4}\right) + \frac{3\sqrt{7}}{4} \qquad * \text{ The exact answer } *$$

$$= 32.5187....$$

[7 marks]

8.1 Tangents

A curve in polar form is typically given as, $r = f(\theta)$

Using the relationships that $x = r\cos\theta$ and $y = r\sin\theta$ such a polar curve can immediately be recast as a parametric curve in Cartesian form,

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta$$

All of our existing knowledge about parametric curves can then be applied.
For example, the gradient of the curve can be found using the chain rule written in the form,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

Of particular use is the following immediate consequence;

Parallel and Perpendicular Tangents Theorem

- To find a tangent parallel to the polar axis, solve $\frac{dy}{d\theta} = 0$
- To find a tangent perpendicular to the polar axis, solve $\frac{dx}{d\theta} = 0$

It may help to think of of $\frac{dy}{d\theta} = 0$ as "no change in height" (as θ changes)

and, similarly, $\frac{dx}{d\theta} = 0$ as "only change in height" (as θ changes) where there's a fudge in which the polar axis is being viewed as a horizontal axis.

Note that *the polar axis* is also often called *the initial line.*

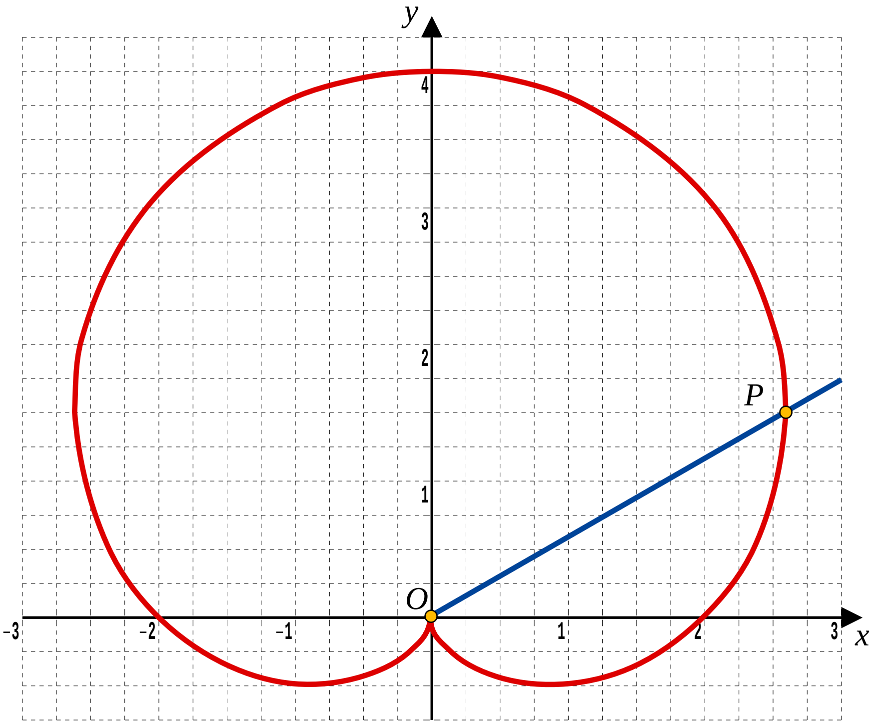
8.2 Polar Curve Sketching

When asked to sketch a polar curve that's not immediately recognised, working out points where there is a tangent parallel or perpendicular to the polar axis may helpful especially if it's quick and easy to do so.
Another short cuts is to spotting if the curve is an odd or an even function.

Points where $\frac{dr}{d\theta} = 0$ correspond to a "stationary point" possibly indicating the tip of a loop where the distance from the pole has stopped increasing and is about to start decreasing (or vice versa).

8.3 Example

The graph shows the polar equation $r = 2(1 + \sin\theta)$, $0 \leqslant \theta < 2\pi$, and a half-line that meets the curve at the pole, O, and at the point P



The tangent to the curve at P is perpendicular to the polar axis.
The polar coordinates of P are (R, ϕ). Determine the exact values of R and ϕ

[6 marks]

The Solution :

$$x = r\cos\theta$$

$$= 2(1 + \sin\theta)\cos\theta \qquad \text{As } r = 2(1 + \sin\theta)$$

$$= 2\cos\theta + 2\sin\theta\cos\theta$$

$$= 2\cos\theta + \sin 2\theta \qquad \text{Trigonometric Identity}$$

$$\frac{dx}{d\theta} = -2\sin\theta + 2\cos 2\theta$$

But $\dfrac{dx}{d\theta} = 0$ for a tangent to be perpendicular to the polar axis

$$\therefore \cos 2\theta - \sin\theta = 0 \qquad \text{After dividing through by 2}$$

$$1 - 2\sin^2\theta - \sin\theta = 0 \qquad \text{Trigonometric Identity}$$

$$2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin + 1) = 0 \Rightarrow P \text{ corresponds to } \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

Substitution this back into $r = 2(1 + \sin\theta)$ gives $P\left(3, \dfrac{\pi}{6}\right)$

[6 marks]

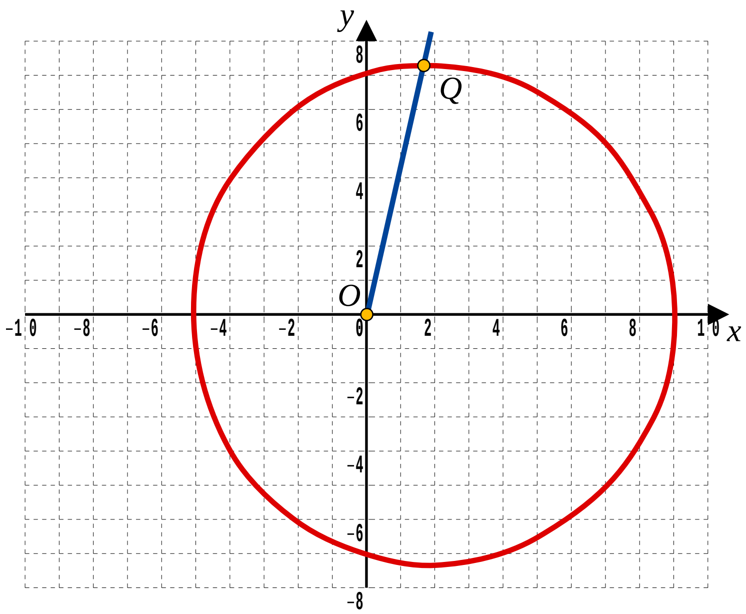
8.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 21

Question 1

The graph shows the polar equation $r = 7 + 2\cos\theta, \quad 0 \leqslant \theta < 2\pi$, and a half-line that meets the curve at the pole, O, and at the point Q.



The tangent to the curve at Q is parallel to the polar axis.

The polar coordinates of Q are (R, ϕ)

(i) Show that $\cos\phi = \dfrac{1}{4}$

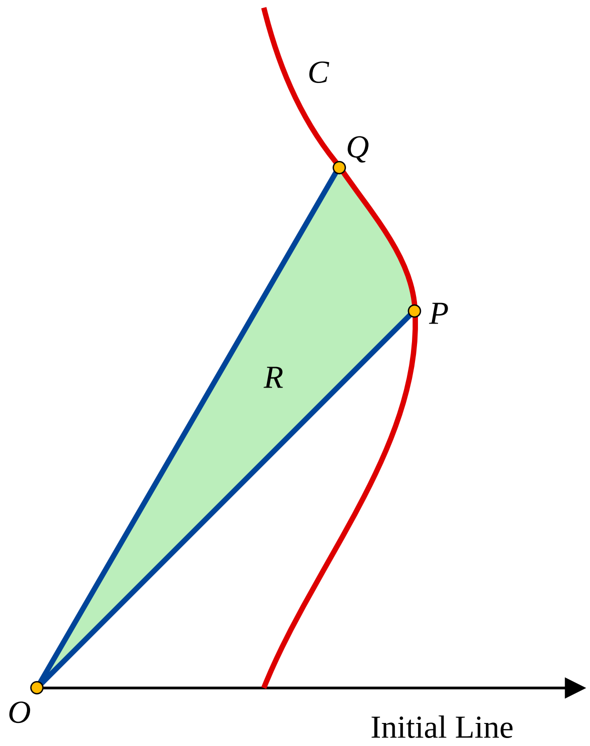
[5 marks]

(ii) Find the exact value of R

[1 mark]

Question 2

Further A-Level Examination Question from June 2014, FP2, Q8 (Edexcel)



The sketch is of part of the curve C with polar equation

$$r = 1 + \tan\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The tangent to the curve C at the point P is perpendicular to the initial line

(a) Find the polar coordinates of the point P

[5 marks]

The point Q lies on the curve C, where $\theta = \dfrac{\pi}{3}$

The shaded region R is bounded by OP, OQ and the curve C, as shown in the sketch.

(b) Find the exact area of R, giving your answer in the form

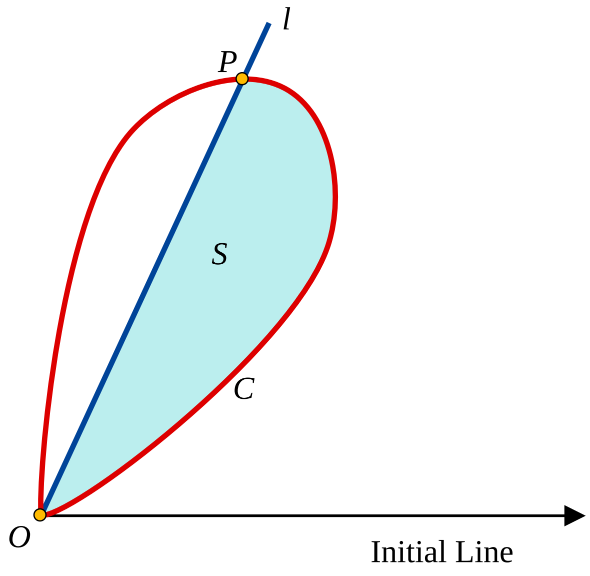
$$\frac{1}{2}\left(\ln p + \sqrt{q} + r \right)$$

where p, q and r are integers to be found

[7 marks]

Question 3

Further A-Level Examination Question from June 2013, FP2, Q8 (Edexcel)



The diagram shows a curve C with polar equation $r = a \sin 2\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

and a half-line l. The half-line l meets C at the pole O and at the point P.

The tangent to C at P is parallel to the initial line.

The polar coordinates of P are (R, ϕ)

(a) Show that $\cos \phi = \frac{1}{\sqrt{3}}$

[6 marks]

(b) Find the exact value of R

[2 marks]

The region S, shown shaded, is bounded by C and l.

(c) Use calculus to show that the exact area of S is

$$\frac{1}{36}\,a^2\left(9\,arccos\left(\frac{1}{\sqrt{3}}\right)+\sqrt{2}\right)$$

[7 marks]

8.5 Answers to Exercise 8.4

Answer 1

(i)

$$
\begin{aligned}
y &= r \sin\theta \\
&= (7 + 2\cos\theta)\sin\theta \\
&= 7\sin\theta + 2\sin\theta\cos\theta \\
&= 7\sin\theta + \sin 2\theta
\end{aligned}
$$

$$\frac{dy}{d\theta} = 7\cos\theta + 2\cos 2\theta$$

But $\frac{dy}{d\theta} = 0$ for a tangent to be parallel to the polar axis

$$
\begin{aligned}
\therefore\ 7\cos\theta + 2\cos 2\theta &= 0 \\
7\cos\theta + 2(\cos^2\theta - \sin^2\theta) &= 0 \\
7\cos\theta + 2(2\cos^2\theta - 1) &= 0 \\
4\cos^2\theta + 7\cos\theta - 2 &= 0 \\
(4\cos\theta - 1)(\cos\theta + 2) &= 0
\end{aligned}
$$

Either $\cos\theta = \frac{1}{4}$ or $\cos\theta = -2$

As $\cos\theta = -2$ has no solutions, the only solution is $\cos\theta = \frac{1}{4}$ □

[5 marks]

(ii)

$$
\begin{aligned}
r &= 7 + 2\cos\theta \\
&= 7 + 2 \times \frac{1}{4} \\
&= 7.5
\end{aligned}
$$

[1 mark]

Answer 2

(a)

$$x = r\cos\theta$$
$$= (1 + \tan\theta)\cos\theta$$
$$= \cos\theta + \sin\theta$$

$$\frac{dx}{d\theta} = -\sin\theta + \cos\theta$$

But $\frac{dx}{d\theta} = 0$ for a tangent to be perpendicular to the polar axis

$$\therefore\ -\sin\theta + \cos\theta = 0$$
$$\sin\theta = \cos\theta$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4} \Rightarrow r = 2 \Rightarrow P\left(2,\ \frac{\pi}{4}\right)$$

[5 marks]

(b) Without worrying about the limits to begin with...

$$Area\ R = \frac{1}{2}\int (1 + \tan\theta)^2\ d\theta$$
$$= \frac{1}{2}\int 1 + 2\tan\theta + \tan^2\theta\ d\theta$$
$$= \frac{1}{2}\int 1 + 2\tan\theta + \sec^2\theta - 1$$
$$= \int \tan\theta\ d\theta + \frac{1}{2}\int \sec^2\theta\ d\theta$$
$$= \ln|\sec\theta| + \frac{1}{2}\tan\theta + c$$

With a lower limit of $\frac{\pi}{4}$ and an upper limit of $\frac{\pi}{3}$ this becomes,

$$Area\ R = \ln\left|\frac{1}{\cos\left(\frac{\pi}{3}\right)}\right| + \frac{1}{2}\tan\left(\frac{\pi}{3}\right) - \ln\left|\frac{1}{\cos\left(\frac{\pi}{4}\right)}\right| - \frac{1}{2}\tan\left(\frac{\pi}{4}\right)$$
$$= \ln 2 + \frac{\sqrt{3}}{2} - \ln\sqrt{2} - \frac{1}{2}$$
$$= \ln 2 - \frac{1}{2}\ln 2 + \frac{1}{2}\sqrt{3} - \frac{1}{2}$$
$$= \frac{1}{2}\left(\ln 2 + \sqrt{3} - 1\right)$$

That is, $p = 2,\ q = 3$ and $r = -1$

[7 marks]

Answer 3

(a)

$$y = r \sin\theta$$
$$= a\ (\sin 2\theta \sin\theta)$$

$$\frac{dy}{d\theta} = a\ (\sin 2\theta \cos\theta + 2\cos 2\theta \sin\theta)$$
$$= a\left(2\sin\theta \cos^2\theta + 2\left(\cos^2\theta - \sin^2\theta \right) \sin\theta \right)$$
$$= a\left(2\sin\theta \cos^2\theta + 2\sin\theta \cos^2\theta - 2\sin^3\theta \right)$$
$$= a\left(4\sin\theta \cos^2\theta - 2\sin^3\theta \right)$$
$$= 2a\sin\theta \left(2\cos^2\theta - \sin^2\theta \right)$$
$$= 2a\sin\theta \left(2\cos^2\theta - (1 - \cos^2\theta \right)$$
$$= 2a\sin\theta \left(3\cos^2\theta - 1 \right)$$

But $\frac{dy}{d\theta} = 0$ for a tangent to be parallel to the polar axis

$$\therefore\ 2a\sin\theta \left(3\cos^2\theta - 1 \right) = 0$$

Either $\sin\theta = 0$ or $3\cos^2\theta - 1 = 0$

The $\sin\theta = 0$ result shows that the initial angle of the loop at the origin is 0

For $3\cos^2\theta - 1 = 0$

$$\cos^2\theta = \frac{1}{3}$$
$$\cos\theta = \pm\frac{1}{\sqrt{3}}$$

The $\cos\theta = -\frac{1}{3}$ result corresponds to a loop not shown in the diagram that is for values of θ outside of the domain specified.

For the angle of interest, where $\theta = \phi$, $\cos\phi = \frac{1}{\sqrt{3}}$ □

[6 marks]

(b) $r = a\sin 2\theta$ with $\theta = \phi$, $\cos\phi = \frac{1}{\sqrt{3}}$, $\sin\phi = \frac{\sqrt{2}\sqrt{3}}{3}$

$$r = 2a\sin\phi\cos\phi$$
$$= 2a\sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2} \times \frac{1}{\sqrt{3}}$$
$$= \frac{2\sqrt{2}}{3}a$$

[2 marks]

(c) Preparation...

$$\cos^2 2\theta + \sin^2 2\theta = 1$$

Subtract $\quad \cos^2 2\theta - \sin^2 2\theta = \cos 4\theta$

$$2 \sin^2 \theta = 1 - \cos 4\theta$$

$$\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 4\theta$$

$$
\begin{aligned}
Area\ S &= \frac{1}{2} \int_0^{\phi} a^2 \sin^2 2\theta \, d\theta \\
&= \frac{a^2}{4} \int_0^{\phi} 1 - \cos 4\theta \, d\theta \\
&= \frac{a^2}{16} \int_0^{\phi} 4 - 4 \cos 4\theta \, d\theta \\
&= \frac{a^2}{16} \left[4\theta - \sin 4\theta \right]_0^{\phi} \\
&= \frac{a^2 \phi}{4} - \frac{a^2 \sin 4\phi}{16} \\
&= \frac{a^2 \phi}{4} - \frac{a^2\, 2 \sin 2\phi \cos 2\phi}{16} \\
&= \frac{a^2 \phi}{4} - \frac{a^2\, 4 \sin \phi \cos \phi \left(\cos^2 \phi - \sin^2 \phi \right)}{16} \\
&= \frac{a^2 \phi}{4} - \frac{a^2}{4} \times \frac{\sqrt{2}\sqrt{3}}{3} \times \frac{1}{\sqrt{3}} \left(\frac{1}{3} - \frac{2}{3} \right) \\
&= \frac{a^2 \phi}{4} - \frac{a^2}{4} \times \frac{\sqrt{2}}{3} \left(- \frac{1}{3} \right) \\
&= \frac{1}{36} a^2 \left(9\phi + \sqrt{2} \right) \\
&= \frac{1}{36} a^2 \left(9 \arccos\left(\frac{1}{\sqrt{3}} + \sqrt{2} \right) \right)
\end{aligned}
$$

[7 marks]

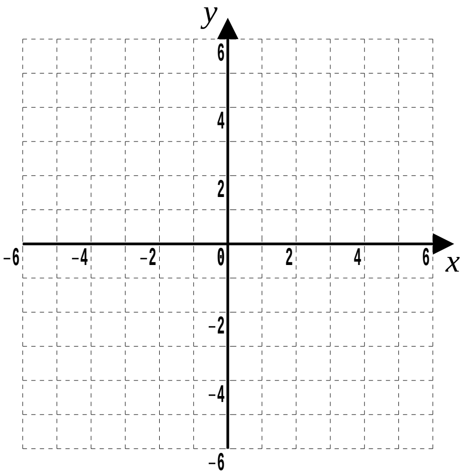
Lesson 9

Further A-Level Pure Mathematics, Core 2
Polar Coordinates

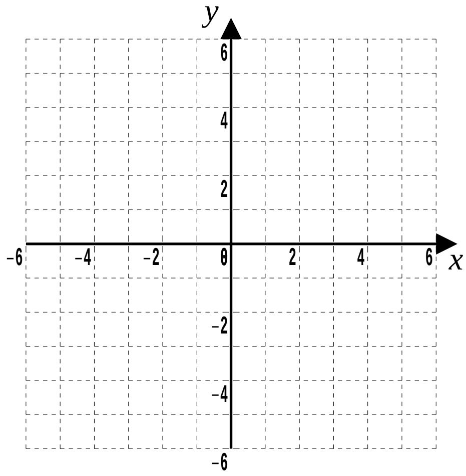
9.1 Revision

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1
On each grid, sketch the graph of the polar equation given,

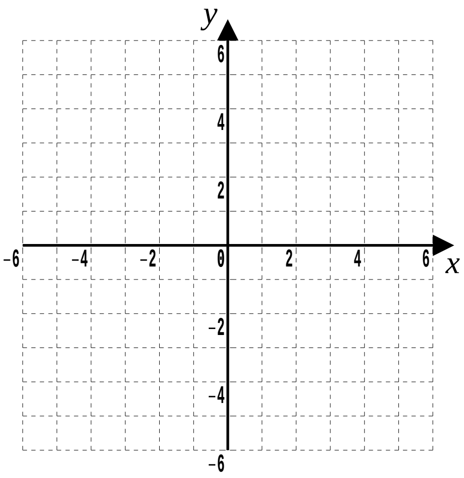


(i) $r = 3\,csc\,\theta$

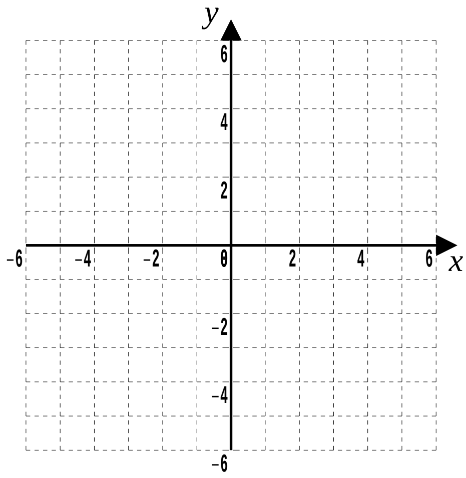


(ii) $r = 3\cos\theta$

[1, 1 marks]



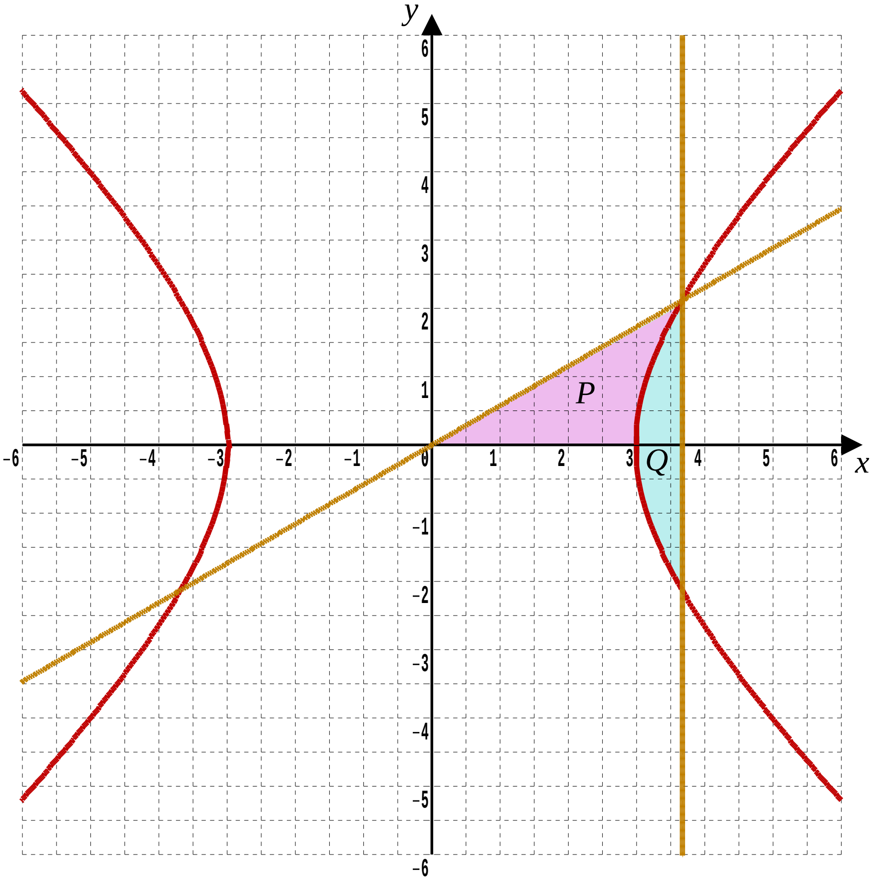
(iii) $r = \dfrac{2}{\cos\theta + 2\sin\theta}$



(iv) $r = 4\sqrt{2}\,\sin(\theta - 45°)$

[1, 1 marks]

Question 2



The curve in the graph is of the hyperbola with polar equation $r^2 = 9\,sec\,2\theta$

Also shown is the vertical line with polar equation $r = \dfrac{3\sqrt{6}}{2}\,sec\,\theta$

(i) Show that the curve and vertical line intersect when $\theta = \pm\dfrac{\pi}{6}$

[4 marks]

(ii) The sloping straight line has polar equation $\theta = \dfrac{\pi}{6}$

Show that the area shaded and marked P is $\dfrac{9}{4}\, ln\left(2 + \sqrt{3} \right)$ by evaluating the following integral,

$$\text{Area } P = \frac{1}{2}\int_0^{\frac{\pi}{6}} 9\, sec\, 2\theta \, d\theta$$

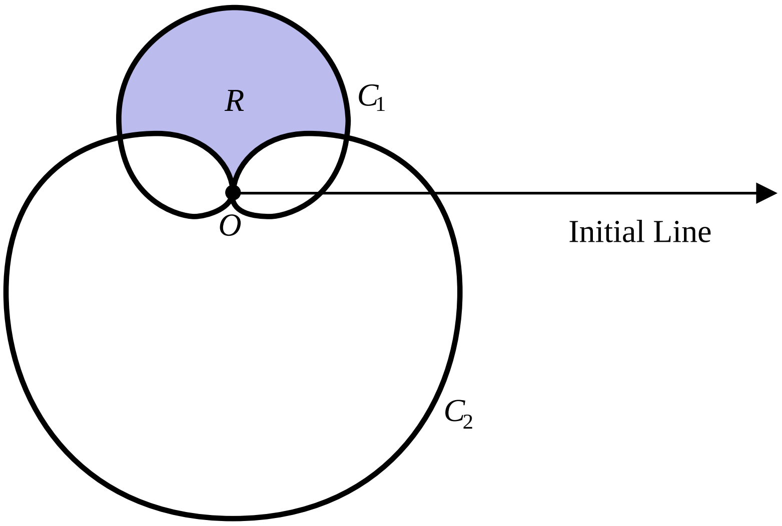
[5 marks]

(iii) Hence, or otherwise, find the exact area between the hyperbola and the vertical line which is shaded blue and marked Q.

[4 marks]

Question 3

Further A-Level Examination Question from October 2020, FP1, Q3 (Edexcel)



The sketch is of two curves C_1 and C_2 with polar equations,

$$C_1 \; : \; r = (1 + \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

$$C_2 \; : \; r = 3(1 - \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

The region R lies inside C_1 and outside C_2 and is shown shaded in the sketch.

Show the area of R is $p\sqrt{3} - q\pi$ where p and q are integers to be determined.

[9 marks]

Question 4

Further A-Level Examination Question from November 2021, FP2, Q6 (Edexcel)

The curve C has polar equation $r = a(p + 2\cos\theta)$, $\quad 0 \leqslant \theta < 2\pi$
where a and p are positive constants and $p > 2$. There are exactly four points on C where the tangent is perpendicular to the initial line.

(a) Show that the range of possible values for p is $2 < p < 4$

[5 marks]

(b) Sketch the curve with equation $r = a(3 + 2\cos\theta)$, $0 \leqslant \theta < 2\pi$
where $a > 0$

[1 mark]

John digs a hole in his garden in order to make a pond.
The pond has a uniform horizontal cross section that is modelled by the curve $r = 20(3 + 2\cos\theta)$, $0 \leqslant \theta < 2\pi$, with r measured in centimetres.
The depth of the pond is 90 centimetres.
Water flows through a hosepipe into the pond at a rate of 50 litres per minute.
Given that the pond is initially empty,

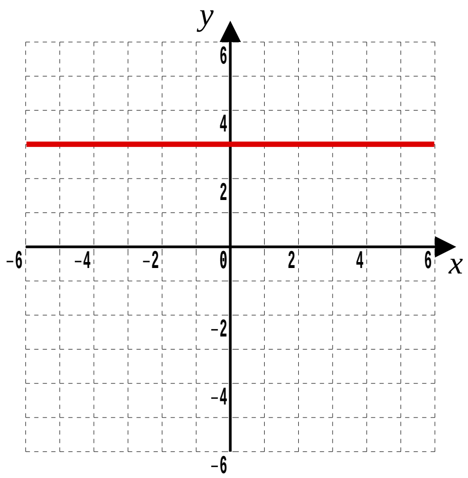
(c) determine how long it will take to completely fill the pond with water using the hosepipe, according to the model.
Give your answer to the nearest minute.

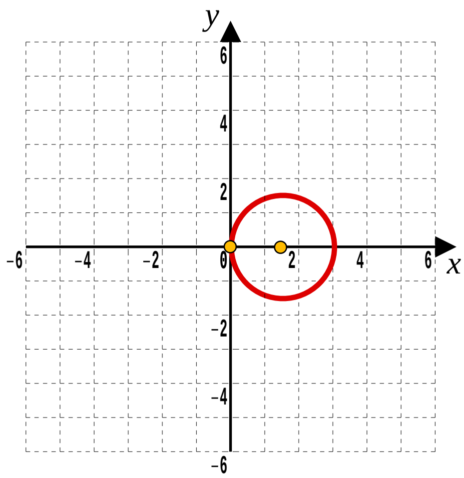
[7 marks]

(d) State a limitation of the model.

[1 mark]

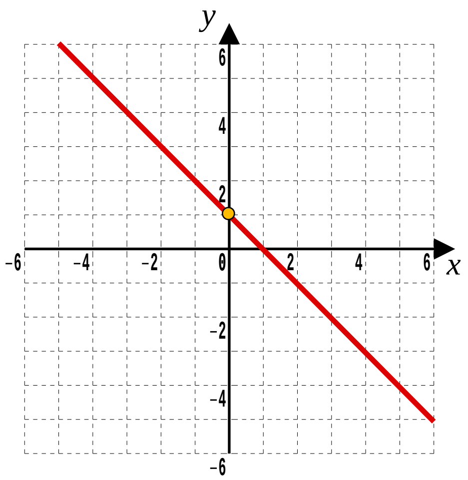
Answer 1

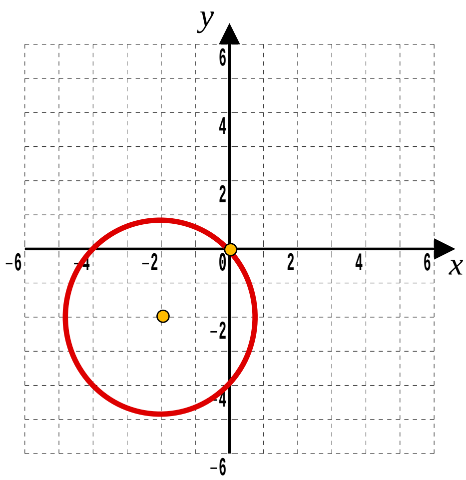




(i) $r = 3\,csc\,\theta$ **(ii)** $r = 3\cos\theta$

[2, 2 marks]





(iii) $r = \dfrac{2}{\cos\theta + 2\sin\theta}$ **(iv)** $r = 4\sqrt{2}\,\sin\,(\theta - 45°)$

[2, 2 marks]

Answer 2

(i) The curve and the line will intersect when,

$$\left(\frac{3\sqrt{6}}{2} \sec\theta \right)^2 = 9 \sec 2\theta$$

$$\frac{9 \times 6}{4 \cos^2 x} = \frac{9}{\cos 2\theta}$$

$$\frac{\cos 2\theta}{\cos^2\theta} = \frac{2}{3}$$

$$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta} = \frac{2}{3}$$

$$1 - \tan^2\theta = \frac{2}{3}$$

$$\tan^2\theta = \frac{1}{3}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = \pm\frac{\pi}{6}$$

[4 marks]

(ii) Area P $= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{6}} 9 \sec 2\theta \, d\theta$

$= \dfrac{9}{4}\displaystyle\int_0^{\frac{\pi}{6}} 2 \sec 2\theta \, d\theta$ (Setting up a chain rule backwards)

$= \dfrac{9}{4}\left[\ln\left| \sec 2\theta + \tan 2\theta \right| \right]_0^{\frac{\pi}{6}}$ (from formulae booklet)

$$= \frac{9}{4}\left[\ln\left| \sec\left(\frac{2\pi}{6}\right) + \tan\left(\frac{2\pi}{6}\right) \right| - \ln\left| \sec 0 + \tan 0 \right| \right]$$

$$= \frac{9}{4}\left[\ln\left| \frac{1}{\cos\left(\frac{\pi}{3}\right)} + \tan\left(\frac{\pi}{3}\right) \right| - \ln\left| \frac{1}{\cos 0} + \tan 0 \right| \right]$$

$$= \frac{9}{4}\left[\ln\left| 2 + \sqrt{3} \right| - \ln 1 \right]$$

$= \dfrac{9}{4}\ln\left(2 + \sqrt{3} \right)$ which is about 2.963...

[5 marks]

(iii) When $\theta = \dfrac{\pi}{6}$, $r = \dfrac{3\sqrt{6}}{2} sec\left(\dfrac{\pi}{6}\right)$ (using equation of line)

$$= \frac{3\sqrt{2}\sqrt{3}}{2} \times \frac{2}{\sqrt{3}}$$

$$= 3\sqrt{2} \quad \text{(hypotenuse of triangle)}$$

$$\text{Area } \Delta = \frac{1}{2} a\, b\, sin\, C$$

$$= \frac{1}{2} \times \frac{3\sqrt{2}\sqrt{3}}{2} \times 3\sqrt{2} \times sin\left(\frac{\pi}{6}\right)$$

$$= \frac{9\sqrt{3}}{4}$$

$$\text{Area } Q = 2 \times (\text{ Area of right angled triangle } - \text{ Area } P\,)$$

$$= 2\left(\frac{9\sqrt{3}}{4} - \frac{9}{4} ln\left(2 + \sqrt{3}\right)\right)$$

$$= \frac{9}{2}\left(\sqrt{3} - ln\left(2 + \sqrt{3}\right)\right) \quad \text{about } 1.8679...$$

[4 marks]

Answer 3

To get the angles associated with the points of intersection,

$$\begin{aligned}(1 + \sin\theta) &= 3(1 - \sin\theta)\\ 4\sin\theta &= 2\\ \sin\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}\end{aligned}$$

The same limits work on both curves,

$$\begin{aligned}
&= 2 \times \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \sin\theta)^2 \; - 9(1 - \sin\theta)^2 \; d\theta\\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + 2\sin\theta + \sin^2\theta - 9 + 18\sin\theta - 9\sin^2\theta\\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 20\sin\theta - 8 - 8\sin^2\theta \, d\theta\\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 20\sin\theta - 8 - 8\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta\\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 20\sin\theta - 12 + 2(2\cos 2\theta)\; d\theta\\
&= \left[-20\cos\theta - 12\theta + 2\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\\
&= [0 - 6\pi + 0] - \left[-10\sqrt{3} - 2\pi + \sqrt{3}\right]\\
&= 9\sqrt{3} - 4\pi
\end{aligned}$$

That is, $p = 9$ and $q = 4$

[9 marks]

Answer 4

(a) We require there to be four solutions to $\dfrac{dx}{d\theta} = 0$

$$x = r\cos\theta$$
$$= a(p + 2\cos\theta)\cos\theta$$
$$= ap\cos\theta + 2a\cos^2\theta$$
$$\frac{dx}{d\theta} = -ap\sin\theta - 4a\cos\theta\sin\theta$$

$$ap\sin\theta + 4a\cos\theta\sin\theta = 0$$

Divide through by a as we know $a \neq 0$

$$p\sin\theta + 4\sin\theta\cos\theta = 0$$
$$\sin\theta\,(p + 4\cos\theta) = 0$$

Either $\sin\theta = 0$ giving two solutions, $\theta = \{0, \pi\}$

or $(p + 4\cos\theta) = 0$

As there are four solutions $(p + 4\cos\theta) = 0$ must yield two more solutions

$$\text{so } \cos\theta = -\frac{p}{4}$$

Keeping in mind that p is a positive constant greater than 2 and

$$-1 \leqslant \cos\theta \leqslant 1$$

gives $2 < p < 4$ □

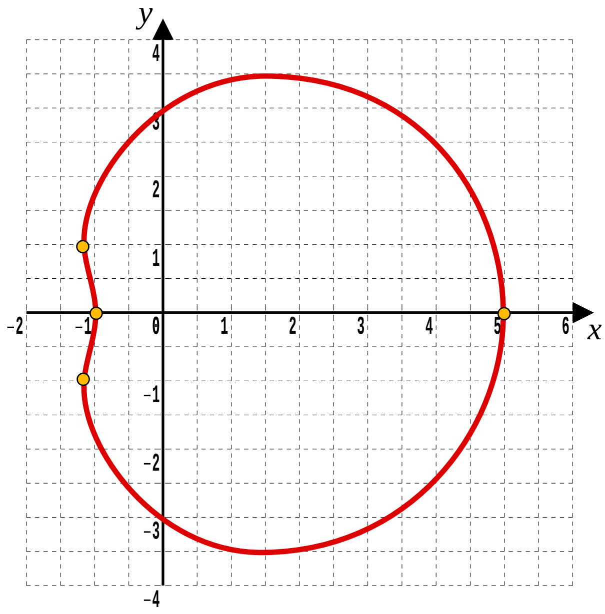
[5 marks]

Graph Plotting :

Out of interest, to plot the limaçon $r = 3 + 2\cos\theta$ on a graph plotter you may need to convert it into Cartesian form (the a is an enlargement scale factor);

$$r = 3 + 2\cos\theta$$
$$r^2 = 3r + 2r\cos\theta$$
$$x^2 + y^2 = 3\sqrt{x^2 + y^2} + 2x$$
$$x^2 + y^2 - 2x = 3\sqrt{x^2 + y^2}$$
$$\left(x^2 + y^2 - 2x\right)^2 = 9\left(x^2 + y^2\right)$$

(b)



Part (a) has provided hints on how to draw this limaçon.
Two of the four perpendicular tangents are when when $\theta = 0$ or π which is on the x-axis. Putting those values of θ into the curve's equation gives the Cartesian points (5, 0) and (− 1, 0).
With $p = 3$, the other two perpendicular tangents occur when,

$$\cos\theta = -\frac{3}{4} \text{ from which } \sin\theta = \pm\frac{\sqrt{7}}{4}$$

The associated points are given by,

$$(x, y) = (r\cos\theta,\ r\sin\theta)$$

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= (3 + 2\cos\theta)\cos\theta & &= (3 + 2\cos\theta)\sin\theta \\ &= \left(3 - \frac{2\times 3}{4}\right)\left(-\frac{3}{4}\right) & &= \left(3 - \frac{2\times 3}{4}\right)\left(\pm\frac{\sqrt{7}}{4}\right) \\ &= -\frac{9}{8} & &= \pm\frac{3\sqrt{7}}{8} \\ &\quad(-1.125) & &\quad(\text{about } \pm 0.992\ldots) \end{aligned}$$

All four of these points are plotted above and as there are no other points with a perpendicular gradient, the simplest curve through those four points is what is required.

[1 mark]

(c) Realising the shape has mirror symmetry in the x-axis suggests using limits of 0 and π and doubling the area found from that.

$$Area = 2 \times \frac{1}{2} \int_0^{\pi} (20 (3 + 2 \cos\theta))^2 \, d\theta$$

$$= \int_0^{\pi} 400 \left(9 + 12 \cos\theta + 4 \cos^2\theta \right) d\theta$$

$$= 400 \int_0^{\pi} 9 + 12 \cos\theta + 4 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta$$

$$= 400 \int_0^{\pi} 11 + 12 \cos\theta + 2 \cos 2\theta \, d\theta$$

$$= 400 \left[11\theta + 12 \sin\theta + \sin 2\theta \right]_0^{\pi}$$

$$= 400 \left[11\pi + 0 + 0 - 0 - 0 - 0 \right]$$

$$= 4400\pi \text{ cm}^2$$

So the volume is $4400\pi \times 90 = 396000\pi$ cm^3

$$\text{Time} = \frac{396000\pi}{50000}$$

$= 25$ minutes (as requested, to the nearest minute)

[7 marks]

(d) In practice, John may not want the pond to be filled to the brim and even if he did, it would need to have a perfectly built with, for example, a perfectly horizontal brim.
Lots of other answers are possible and acceptable.
Anything sensible gets the mark.

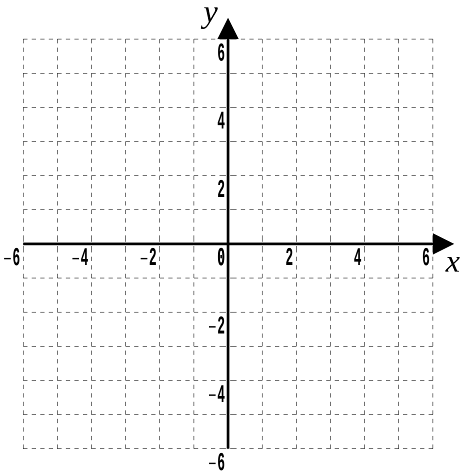
[1 mark]

10.1 Test

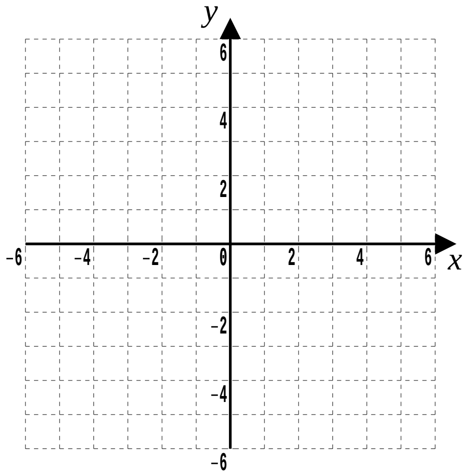
Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

On each grid, sketch the graph of the polar equation given,

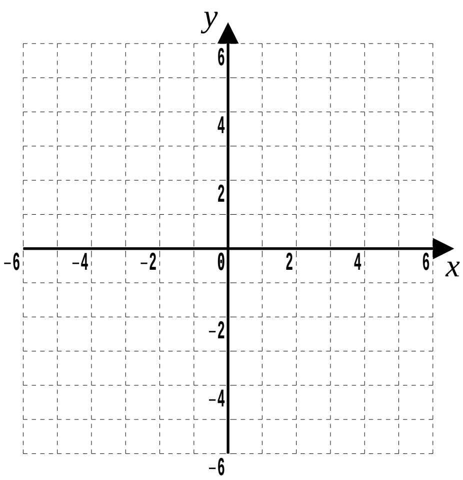


(i) $r = 6 \cos \theta$

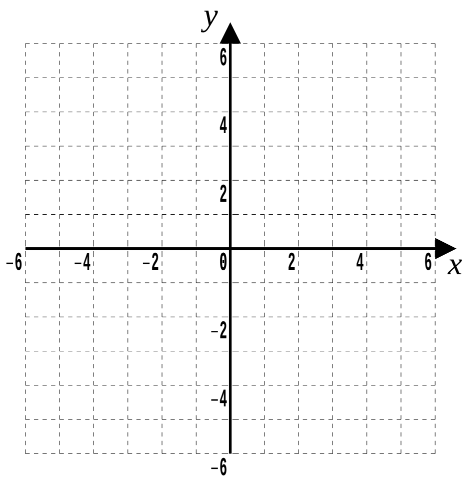


(ii) $r = \sec \theta$

[2, 2 marks]



(iii) $r = \dfrac{1}{2 \cos \theta + \sin \theta}$



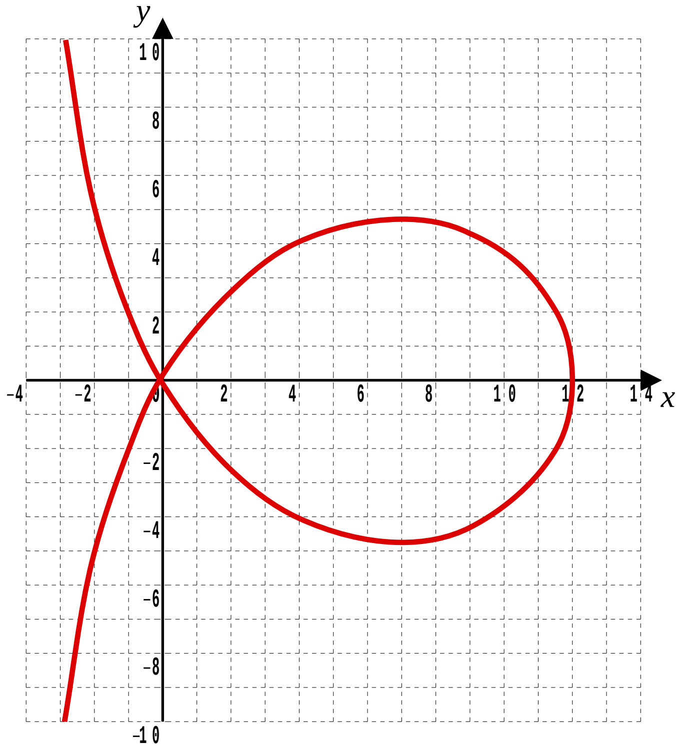
(iv) $r = 4\sqrt{2} \cos (\theta - 45°)$

[2, 2 marks]

Question 2

The curve below is The Trisectrix of Maclaurin and has Cartesian equation

$$y^2(4+x) = x^2(12-x)$$



(i) Show that The Trisectrix of Maclaurin has polar equation

$$r = \frac{4(2\cos(2\theta)+1)}{\cos\theta}$$

[4 marks]

(ii) Show that the point with Cartesian coordinates (4, 4) is on the curve.

[2 marks]

(iii) By means of implicit differentiation of the Cartesian equation show that

$$\frac{dy}{dx} = \frac{24x - 3x^2 - y^2}{8y + 2xy}$$

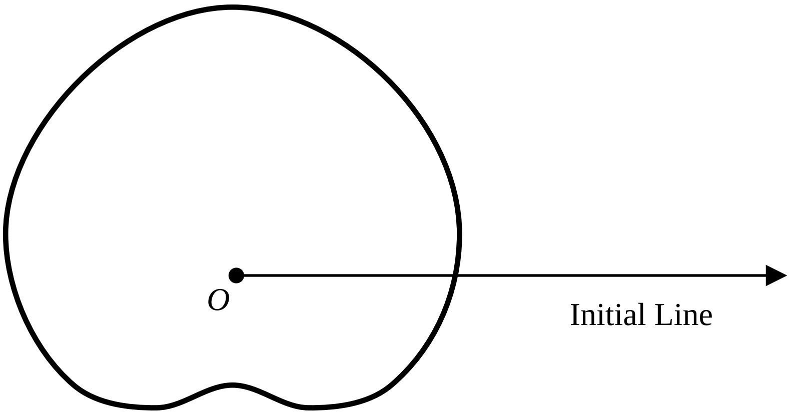
[4 marks]

(iv) Find the equation of the tangent to the curve at the point (4, 4) in the form $ax + by + c = 0$ where a, b and c are integer constants.

[3 marks]

Question 3

Further A-Level Examination Question from June 2017, FP2, Q6 (Edexcel)



The sketch is of a curve with polar equation,

$$r = 6 + a \sin \theta$$

where $0 < a < 6$ and $0 \leqslant \theta < 2\pi$

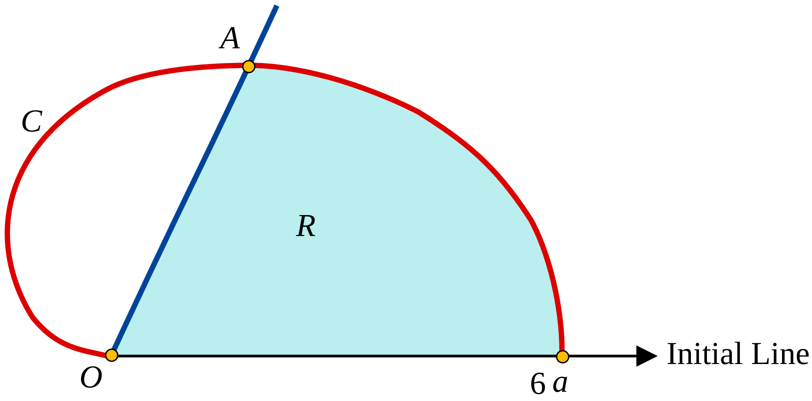
The are enclosed by the curve is $\dfrac{97\pi}{2}$

Find the value of the constant a

[8 marks]

Question 4

Further A-Level Examination Question from June 2015, FP2, Q6 (Edexcel)



The curve C has polar equation,

$$r = 3a(1 + \cos\theta), \qquad 0 \leqslant \theta < \pi$$

The tangent to C at the point A is parallel to the initial line.

(a) Find the polar coordinates of A

[6 marks]

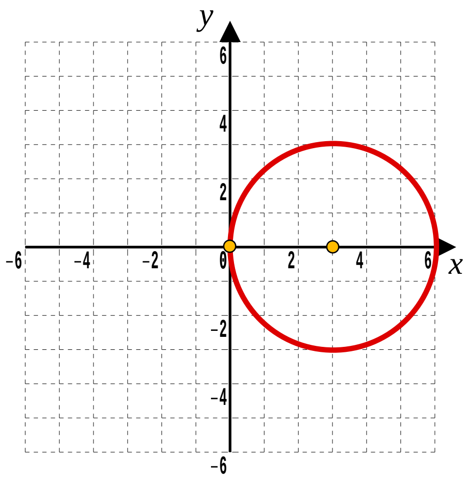
The finite region R, shown shaded, is bounded by the curve C, the initial line and the line OA

(b) Use calculus to find the area of the shaded region R, giving your answer in the form $a^2 (p\pi + q\sqrt{3})$, where p and q are rational numbers

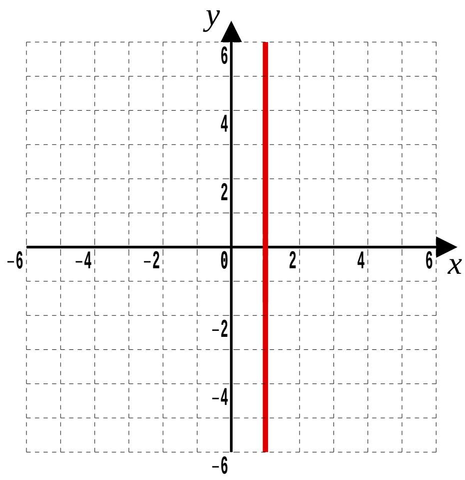
[5 marks]

10.2 Answers to 10.1 Test

Answer 1

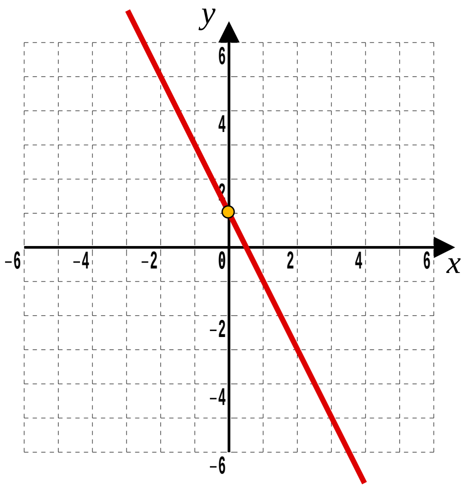


(i) $r = 6 \cos \theta$

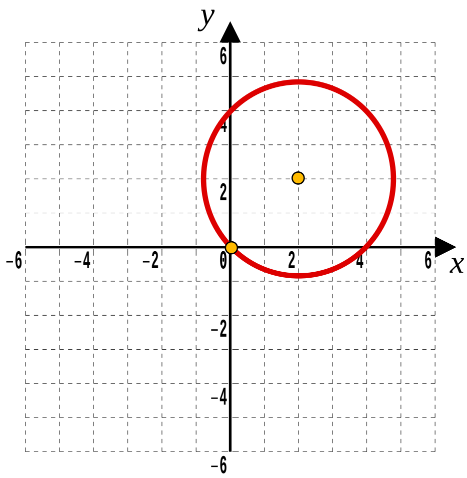


(ii) $r = \sec \theta$

[2, 2 marks]



(iii) $r = \dfrac{1}{2 \cos \theta + \sin \theta}$



(iv) $r = 4\sqrt{2} \cos (\theta - 45°)$

[2, 2 marks]

Answer 2

(i)

$$y^2 \ (4 + x) = x^2 (12 - x)$$

$$r^2 \sin^2 \theta \ (4 + r \cos \theta) = r^2 \cos^2 \theta \ (12 - r \cos \theta)$$

$$\sin^2 \theta \ (4 + r \cos \theta) = \cos^2 \theta \ (12 - r \cos \theta)$$

$$4 \sin^2 \theta + r \sin^2 \theta \cos \theta = 12 \cos^2 \theta - r \cos^3 \theta$$

$$r \cos^3 \theta + r \sin^2 \theta \cos \theta = 12 \cos^2 \theta - 4 \sin^2 \theta$$

$$r \cos \theta \left(\cos^2 \theta + \sin^2 \theta \right) = 4 \left(3 \cos^2 \theta - \sin^2 \theta \right)$$

$$r \cos \theta = 4 \left(\cos^2 \theta + \sin^2 \theta + 2 \cos^2 \theta - 2 \sin^2 \theta \right)$$

$$r \cos \theta = 4 (1 + 2 \cos (2\theta))$$

$$r = \frac{4 \ (2 \cos (2\theta) + 1)}{\cos \theta} \qquad \square$$

[4 marks]

(ii)

$$\text{LHS} = y^2 \left(4 + x \right)$$

$$= 4^2 \left(4 + 4 \right) \qquad \text{when } \left(4, \ 4 \right) \text{ is inserted}$$

$$= 16 \times 8$$

$$= 128$$

$$\text{RHS} = x^2 \ (12 - x)$$

$$= 4^2 \times \left(12 - 4 \right) \qquad \text{when } \left(4, \ 4 \right) \text{ is inserted}$$

$$= 16 \times 8$$

$$= 128$$

$$= \text{LHS}$$

$\therefore \ \left(4, \ 4 \right)$ is on the curve $\square$

[2 marks]

(iii)

$$y^2 \left(4 + x \right) = x^2 \ (12 - x)$$

$$y^2 \left(4 + x \right) = 12 x^2 - x^3$$

Differentiating Implicitly with respect to x

$$y^2 (1) + 2y \frac{dy}{dx} \left(4 + x \right) = 24x - 3 x^2$$

$$\frac{dy}{dx} \left(8y + 2xy \right) = 24x - 3 x^2 - y^2$$

$$\frac{dy}{dx} = \frac{24x - 3 x^2 - y^2}{8y + 2xy} \qquad \square$$

[4 marks]

(iv)

$$\frac{dy}{dx} = \frac{24(4) - 3(4)^2 - (4)^2}{8(4) + 2(4)(4)}$$

$$= \frac{96 - 48 - 16}{32 + 32}$$

$$= \frac{32}{64}$$

$$= \frac{1}{2}$$

$$y = mx + c$$

$$4 = \frac{1}{2} \times (4) + c$$

$$c = 2$$

$\therefore$ Tangent is $y = \dfrac{1}{2}x + 2$, that is, $x - 2y + 4 = 0$

[3 marks]

Answer 3

$$I = \frac{1}{2}\int r^2 \, d\theta$$

$$= \frac{1}{2}\int (6 + a \sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int 36 + 12a \sin\theta + a^2 \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int 36 + 12a \sin\theta + \frac{a^2}{2} - \frac{a^2}{4} 2\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[36\theta - 12a \cos\theta + \frac{a^2\theta}{2} - \frac{a^2}{4}\sin 2\theta \right] + c$$

With lower limit 0 and upper limit 2π this will become,

$$= \frac{1}{2}\left[72\pi - 12a + a^2\pi - 0 - 0 + 12a - 0 + 0 \right]$$

$$= \frac{\pi}{2}\left(72 - a^2\right)$$

Using the given area,

$$\frac{\pi}{2}\left(72 - a^2\right) = \frac{97\pi}{2}$$

$$a^2 + 97 - 72 = 0$$

$$a^2 = 25$$

$$a = 5 \qquad \text{as} \quad 0 < a < 6$$

[8 marks]

Answer 4

(a)

$$\begin{aligned} y &= r \sin\theta \\ &= 3a\,(1 + \cos\theta)\,\sin\theta \\ &= 3a \sin\theta + 3a \sin\theta \cos\theta \\ &= 3a \sin\theta + 3a\,\frac{1}{2}\,\sin 2\theta \end{aligned}$$

$$\begin{aligned} \frac{dy}{d\theta} &= 3a\cos\theta + 3a\cos 2\theta \\ &= 3a\left(\cos\theta + \cos^2\theta - \sin^2\theta \right) \\ &= 3a\left(\cos\theta + \cos^2\theta - 1 + \cos^2\theta \right) \\ &= 3a\left(2\cos^2\theta + \cos\theta - 1 \right) \\ &= 3a\left(2\cos\theta - 1 \right)\left(\cos\theta + 1 \right) \end{aligned}$$

But $\frac{dy}{d\theta} = 0$ for a tangent to be parallel to the polar axis

Either $\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

The $\cos\theta = -1$ result shows that the angle of the curve at the origin is π^c

For $\cos\theta = \frac{1}{2}$

$$\theta = \frac{\pi}{3} \Rightarrow A\left(\frac{9a}{2}, \frac{\pi}{3} \right)$$

[6 marks]

(b) Ignoring limits for the time being....

$$I = \frac{1}{2} \int r^2 \, d\theta$$

$$= \frac{1}{2} \int (3a)^2 \; (1 + \cos\theta)^2 \; d\theta$$

$$= \frac{9a^2}{2} \int 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{9a^2}{2} \int 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{4} 2\cos 2\theta \, d\theta$$

$$= \frac{9a^2}{2} \left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta \right]$$

$$= \frac{9a^2}{8} [6\theta + 8\sin\theta + \sin 2\theta] + c$$

With lower limit 0 and upper limit $\frac{\pi}{3}$ this will become,

$$= \frac{9a^2}{8} \left[2\pi + 4\sqrt{3} + \frac{\sqrt{3}}{2} - 0 - 0 - 0 \right]$$

$$= \frac{9a^2}{16} [4\pi + 8\sqrt{3} + \sqrt{3}]$$

$$= \frac{9a^2}{16} [4\pi + 9\sqrt{3}]$$

$$= a^2 \left(\frac{9}{4}\pi + \frac{81}{16}\sqrt{3} \right)$$

That is, $p = \frac{9}{4}$ and $q = \frac{81}{16}$

[5 marks]