10.6 Answers

10.6.1 Solution to 10.2 Example #1 (By Parts)

$$\int x\, sinh\, 3x\, dx \;=\; \frac{1}{3}\int x\, 3sinh\, 3x$$

$$L \quad I \quad D \quad I$$

$$= \frac{1}{3}\left[\, x\, cosh\, 3x \;-\; \int 1\, cosh\, 3x\, dx \,\right]$$

$$= \frac{1}{3}\left[\, x\, cosh\, 3x \;-\; \frac{1}{3}\int 3\, cosh\, 3x\, dx \,\right]$$

$$= \frac{1}{3}\left[\, x\, cosh\, 3x \;-\; \frac{1}{3}\, sinh\, 3x \,\right] + c$$

$$= \frac{x\, cosh\, 3x}{3} \;-\; \frac{sinh\, 3x}{9} + c$$

Note that if the $sinh\, 3x$ were replaced with exponentials integration by parts would still be needed on each of two pieces.

[3 marks]

10.6.2 Solution to 10.3 Example #2 (Trigonometric Odd Powers)

$$\int sinh^3 x\, dx \;=\; \int sinh\, x\, sinh^2 x\, dx$$

$$= \int sinh\, x \left(cosh^2 x \;-\; 1 \right) x\, dx$$

$$= \int sinh\, x\, cosh^2 x \;-\; sinh\, x \;\; dx$$

$$= \frac{1}{3}\, cosh^3 x \;-\; cosh\, x \;+\; c$$

[3 marks]

10.6.3 Solution to 10.4 Example #3 (Substitution)

Setting up the substitution...

Differentiating:

$$x = 3\,sinh\,u \;\Rightarrow\; \frac{dx}{du} = 3\,cosh\,u$$

Limits:

$$u = arsinh\left(\frac{x}{3}\right)$$

When $x = 0$ then $u = 0$,

When $x = 6$ then $u = arsinh\,2$

Making the substitution...

$$\int_{x=0}^{x=6} \frac{x^3}{\sqrt{x^2 + 9}}\,dx = \int_{u=0}^{u=arsinh\,2} \frac{27\,sinh^3 u}{\sqrt{9\,sinh^2 u + 9}}\,3\,cosh\,u\,du$$

$$= \int_0^{arsinh\,2} \frac{27\,sinh^3 u\,cosh\,u}{\sqrt{sinh^2 u + 1}}\,du$$

$$= \int_0^{arsinh\,2} \frac{27\,sinh^3 u\,cosh\,u}{\sqrt{cosh^2 u}}\,du$$

$$= 27\int_0^{arsinh\,2} sinh^3 u\,du$$

Using the result from Example #2...

$$= 27\left[\frac{1}{3}\,cosh^3 u - cosh\,u\right]_0^{u\,=\,arsinh\,2}$$

Some clever manipulation of the limits...

From $u = arsinh\,2$ we can extract that $sinh\,u = 2$

Identity: $cosh^2 u - sinh^2 u = 1$

$$cosh^2 u = 1 + sinh^2 u$$

$\therefore$ when $sinh\,u = 2$, $cosh\,u = \sqrt{5}$

Picking up the main thread...

$$\int_{x=0}^{x=6} \frac{x^3}{\sqrt{x^2 + 9}}\,dx = 27\left[\frac{5\sqrt{5}}{3} - \sqrt{5} - \frac{1}{3} + 1\right]$$

$$= 27 \times \frac{1}{3}\left[5\sqrt{5} - 3\sqrt{5} - 1 + 3\right]$$

$$= 9\left[2\sqrt{5} + 2\right]$$

$$= 18\left(\sqrt{5} + 1\right)$$

[11 marks]

10.6.4 Solutions to 10.5 Exercise

Answer 1

The integral for the shaded area will be,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} cos^3 x\, dx &= \int_0^{\frac{\pi}{2}} cos\, x\, cos^2\, x\, dx \\
&= \int_0^{\frac{\pi}{2}} cos\, x \left(1 - sin^2\, x \right) dx \\
&= \int_0^{\frac{\pi}{2}} cos\, x\, dx - \int_0^{\frac{\pi}{2}} cos\, x\, sin^2\, x\, dx \\
&= \left[sin\, x - \frac{1}{3} sin^3\, x \right]_0^{\frac{\pi}{2}} \\
&= sin\left(\frac{\pi}{2}\right) - \frac{1}{3} sin^3 \left(\frac{\pi}{2}\right) - sin\, 0 + \frac{1}{3} sin^3\, 0 \\
&= \frac{2}{3}
\end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned}
\int 25 x^2 cosh(5x)\; dx &= 5 \int x^2 \; (5\, cosh(5x) \;)\; dx \\
&= 5\left[x^2 sinh(5x) - \int 2x\, sinh(5x)\; dx \right] \\
&= 5x^2 sinh(5x) - 2 \int x \; (5\, sinh(5x) \;)\; dx \\
&= 5x^2 sinh(5x) - 2\left[x\, cosh(5x) - \int 1\, cosh(5x)\; dx \right] \\
&= 5x^2 sinh(5x) - 2x\, cosh(5x) + \frac{2}{5} \int 5\, cosh(5x)\; dx \\
&= 5x^2 sinh(5x) - 2x\, cosh(5x) + \frac{2}{5} sinh(5x) + c
\end{aligned}$$

[5 marks]

Answer 3

$$\begin{aligned}
\int cosh^3 x\, dx &= \int cosh\, x\, cosh^2\, x\, dx \\
&= \int cosh\, x \left(1 + sinh^2\, x \right) dx \\
&= \int cosh\, x\, dx + \int cosh\, x\, sinh^2\, x\, dx \\
&= sinh\, x + \frac{1}{3} sinh^3\, x + c
\end{aligned}$$

[3 marks]

Answer 4

Setting up the substitution...

Differentiating:

$$x = 6\,cosh\,u \;\Rightarrow\; \frac{dx}{du} = 6\,sinh\,u$$

Limits:

$$u = arcosh\left(\frac{x}{6}\right)$$

When $x = 12$ then $u = arcosh\,2$

When $x = 18$ then $u = arcosh\,3$

Making the substitution...

$$\int_{x=12}^{x=18} \frac{x^3}{\sqrt{x^2 - 36}}\,dx = \int_{u=arcosh\,2}^{u=arcosh\,3} \frac{216\,cosh^3 u}{\sqrt{36\,cosh^2 u - 36}}\,6\,sinh\,u\,du$$

$$= \int_{arcosh\,2}^{arcosh\,3} \frac{216\,cosh^3 u\,sinh\,u}{\sqrt{cosh^2 u - 1}}\,du$$

$$= \int_{arcosh\,2}^{arcosh\,3} \frac{216\,cosh^3 u\,sinh\,u}{\sqrt{sinh^2 u}}\,du$$

$$= 216 \int_{arcosh\,2}^{arcosh\,3} cosh^3 u\,du$$

Using the result from Question 2...

$$= 216\left[sinh\,u + \frac{1}{3}\,sinh^3 u\right]_{arcosh\,2}^{arcosh\,3}$$

Some clever manipulation of the limits...

From $u = arcosh\,3$ we can extract that $cosh\,u = 3$

and from $u = arcosh\,2$ we can extract that $cosh\,u = 2$

Identity: $cosh^2 u - sinh^2 u = 1$

$$sinh^2 u = 1 + sinh^2 u - 1$$

$\therefore$ when $cosh\,u = 3$, $sinh\,u = 2\sqrt{2}$ and when $cosh\,u = 2$, $sinh\,u = \sqrt{3}$

Picking up the main thread...

$$\int_{x=12}^{x=18} \frac{x^3}{\sqrt{x^2 - 36}}\,dx = 216\left[2\sqrt{2} + \frac{16\sqrt{2}}{3} - \sqrt{3} - \frac{3\sqrt{3}}{3}\right]$$

$$= 72\left[6\sqrt{2} + 16\sqrt{2} - 3\sqrt{3} - 3\sqrt{3}\right]$$

$$= 72\left[22\sqrt{2} - 6\sqrt{3}\right]$$

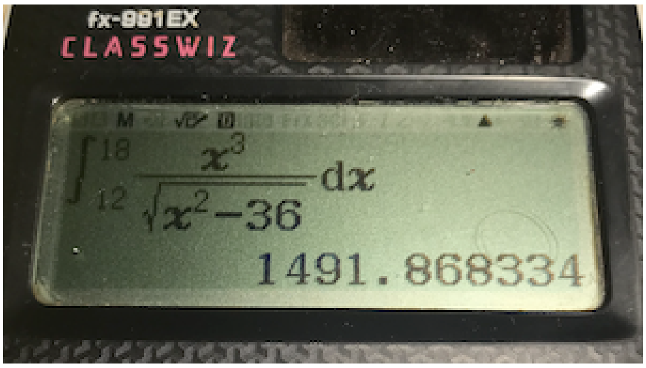
$$= 144\left(11\sqrt{2} - 3\sqrt{3}\right)$$

[11 marks]

Calculator check...

As a decimal $144\left(11\sqrt{2} - 3\sqrt{3}\right) = 1491.868334...$

Getting the calculator to do the integral directly...



Answer 5

(i) $x = \sec\theta \Rightarrow \dfrac{dx}{d\theta} = \sec\theta\tan\theta$

$$\int \frac{1}{x\sqrt{x^2 - 1}}dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta - 1}}\,d\theta$$

$$= \int \frac{\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta$$

$$= \int 1\,d\theta$$

$$= \theta + c$$

$= \operatorname{arcsec} x + c$, where c is a constant

[2 marks]

(ii) $x = \sec\theta \Rightarrow \dfrac{dx}{d\theta} = \sec\theta\tan\theta$

$$\int \frac{\sqrt{x^2 - 1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \times \sec\theta\tan\theta\,d\theta$$

$$= \int \sqrt{\tan^2\theta}\,\tan\theta\,d\theta$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int \sec^2\theta - 1\,d\theta$$

$$= \tan\theta - \theta + c$$

$= \sqrt{x^2 - 1} - \operatorname{arcsec} x + c$, where c is a constant

[4 marks]