11.2 Answers

Answer 1

$$\begin{aligned}
\int \sin^5 x \, dx &= \int \sin x \sin^4 x \, dx \\
&= \int \sin x \left(\sin^2 x \right)^2 \, dx \\
&= \int \sin x \left(1 - \cos^2 x \right)^2 \, dx \\
&= \int \sin x \left(1 - 2\cos^2 x + \cos^4 x \right) dx \\
&= \int \sin x + 2(-\sin x)\cos^2 x - (-\sin x)\cos^4 x \, dx \\
&= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + c
\end{aligned}$$

[3 marks]

Answer 2

$$y = \operatorname{artanh}(\cos x)$$

Standard Result

$$\operatorname{artanh} x \text{ differentiates to } \frac{1}{1 - x^2}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1 - \cos^2 x} \times (-\sin x) \\
&= \frac{1}{\sin^2 x} \times (-\sin x) \\
&= -\frac{1}{\sin x} \\
&= -\csc x
\end{aligned}$$

[3 marks]

Answer 3

$$\text{Preparation: } x = 5\,sinh\,u \iff \frac{x}{5} = sinh\,u$$

Use the identity

$$cosh^2 u - sinh^2 u = 1$$
$$cosh^2 u = 1 - sinh^2 u$$
$$cosh\,u = \sqrt{1 + \left(\frac{x}{5}\right)^2} \quad \text{Positive as } cosh\,u \text{ is never negative}$$

$$x = 5\,sinh\,u \Rightarrow \frac{dx}{du} = 5\,cosh\,u$$

$$\int \sqrt{x^2 + 25}\,dx = \int \sqrt{25\,sinh^2 u + 25} \times 5\,cosh\,u\;du$$
$$= \int 5\sqrt{sinh^2 u + 1} \times 5\,cosh\,u\,du$$
$$= 25 \int \sqrt{cosh^2 u}\;cosh\,u\,du$$
$$= 25 \int cosh^2 u\,du$$

$$cosh^2 \theta + sinh^2 \theta = cosh\,2\theta$$
$$\text{ADD} \quad cosh^2 \theta - sinh^2 \theta = 1$$

$$2\,cosh^2 \theta = cosh\,2\theta + 1$$

$$\int \sqrt{x^2 + 25}\,dx = \frac{25}{2} \int cosh\,2u + 1\,du$$
$$= \frac{25}{4} \int 2\,cosh\,2u + 2\,du$$
$$= \frac{25}{4}\left(sinh\,2u + 2u\right) + c$$
$$= \frac{25}{4}\left(2\,sinh\,u\,cosh\,u + 2u\right) + c$$
$$= \frac{25}{4}\left(2\left(\frac{x}{5}\right)\sqrt{\left(\frac{x}{5}\right)^2 + 1} + 2\,arsinh\left(\frac{x}{5}\right)\right) + c$$
$$= \frac{25}{4}\left(\frac{2x}{5}\sqrt{\frac{x^2 + 25}{25}} + 2\,arsinh\left(\frac{x}{5}\right)\right) + c$$
$$= \frac{1}{2}x\sqrt{x^2 + 25} + \frac{25}{2}\,arsinh\left(\frac{x}{5}\right) + c$$

[5 marks]

Answer 4

First to establish a useful identity,

$$\cosh^2\theta - \sinh^2\theta = 1$$

Divide through by $\cosh^2\theta$

$$\frac{\cosh^2\theta}{\cosh^2\theta} - \frac{\sinh^2\theta}{\cosh^2\theta} = \frac{1}{\cosh^2\theta}$$

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta$$

$$\tanh^2\theta = 1 - \operatorname{sech}^2\theta$$

$$3\tanh^2 x - 2\operatorname{sech} x = 2$$

$$3\left(1 - \operatorname{sech}^2 x\right) - 2\operatorname{sech} x = 2$$

$$3 - 3\operatorname{sech}^2 x - 2\operatorname{sech} x = 2$$

$$3\operatorname{sech}^2 x + 2\operatorname{sech} x - 1 = 0$$

$$(3\operatorname{sech} x - 1)(\operatorname{sech} x + 1) = 0$$

Either $\operatorname{sech} x = \dfrac{1}{3}$ or $\operatorname{sech} x = -1$

$$\frac{1}{\cosh x} = \frac{1}{3} \quad \text{or} \quad \frac{1}{\cosh x} = -1$$

$$\cosh x = 3 \quad \text{or} \quad \cosh x = -1$$

Thinking of the graph of $y = \cosh x$ it is always positive and so the equation $\cosh x = -1$ has no solutions whereas $\cosh x = 3$ will have two solutions given by $x = \pm \operatorname{arcosh} 3$

$$x = \pm \ln\left(3 + \sqrt{3^2 - 1}\right)$$

$$= \pm \ln\left(3 + \sqrt{8}\right)$$

That is, $a = 3$ and $b = 8$

[5 marks]

Answer 5

$$y = arcosh\, x$$

Standard Result

$$arcosh\, x \text{ differentiates to } \frac{1}{\sqrt{x^2 - 1}} \quad x > 1$$

$$\frac{dy}{dx} = \left(x^2 - 1\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\left(x^2 - 1\right)^{\frac{1}{2}}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\left(x^2 - 1\right)^{-\frac{3}{2}} 2x$$

$$= -x\left(x^2 - 1\right)^{-\frac{3}{2}}$$

$$= -\frac{x}{\left(x^2 - 1\right)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = (-x) \times \left(-\frac{3}{2}\right)\left(x^2 - 1\right)^{-\frac{5}{2}} 2x + (-1)\left(x^2 - 1\right)^{-\frac{3}{2}}$$

$$= \frac{3x^2}{\left(x^2 - 1\right)^{\frac{5}{2}}} - \frac{1}{\left(x^2 - 1\right)^{\frac{3}{2}}}$$

$$\text{LHS} = \left(x^2 - 1\right)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx}$$

$$= \left(\frac{3x^2\left(x^2 - 1\right)}{\left(x^2 - 1\right)^{\frac{5}{2}}} - \frac{1 \times \left(x^2 - 1\right)}{\left(x^2 - 1\right)^{\frac{3}{2}}}\right) + \left(-\frac{3x^2}{\left(x^2 - 1\right)^{\frac{3}{2}}}\right) + \frac{1}{\left(x^2 - 1\right)^{\frac{1}{2}}}$$

$$= \frac{3x^2}{\left(x^2 - 1\right)^{\frac{3}{2}}} - \frac{1}{\left(x^2 - 1\right)^{\frac{1}{2}}} - \frac{3x^2}{\left(x^2 - 1\right)^{\frac{3}{2}}} + \frac{1}{\left(x^2 - 1\right)^{\frac{1}{2}}}$$

$$= 0$$

$$= \text{RHS} \qquad \square$$

[7 marks]

Answer 6

(i) $x = cosh^2 u \Rightarrow \dfrac{dx}{du} = 2\, cosh\, u\, sinh\, u$

$$\int \sqrt{\frac{x}{x-1}}\, dx = \int \sqrt{\frac{cosh^2 u}{cosh^2 u - 1}} \times 2\, cosh\, u\, sinh\, u\ \ du$$

$$= \int \sqrt{\frac{cosh^2 u}{sinh^2 u}} \times 2\, cosh\, u\, sinh\, u\ \ du$$

$$= \int \frac{cosh\, u}{sinh\, u} \times 2\, cosh\, u\, sinh\, u\ \ du$$

$$= \int 2\, cosh^2 u\, du$$

$$cosh^2 \theta + sinh^2 \theta = cosh\, 2\theta$$

$$\text{ADD}\quad cosh^2 \theta - sinh^2 \theta = 1$$

$$2\, cosh^2 \theta = cosh\, 2\theta + 1$$

$$\int \sqrt{\frac{x}{x-1}}\, dx = \int cosh\, 2u + 1\, du$$

$$= \frac{1}{2} \int 2\, cosh\, 2u\, du + \int 1\, du$$

$$= \frac{1}{2}\, sinh\, 2u + u + c$$

$$= sinh\, u\, cosh\, u + u + c$$

$$= \sqrt{cosh^2 u - 1}\ cosh\, u + u + c$$

$$= \sqrt{x-1}\,\sqrt{x} + arcosh\left(\sqrt{x}\right) + c$$

[6 marks]

(ii) $\displaystyle\int_2^4 \sqrt{\frac{x}{x-1}}\, dx = \left[\sqrt{x-1}\,\sqrt{x} + arcosh\left(\sqrt{x}\right) \right]_2^4$

$$= \left[\sqrt{x-1}\,\sqrt{x} + ln\left\{ \sqrt{x} + \sqrt{x-1} \right\} \right]$$

$$= \sqrt{4-1}\,\sqrt{4} + ln\left\{ 2 + \sqrt{4-1} \right\}$$

$$- \sqrt{2-1}\,\sqrt{2} - ln\left\{ \sqrt{2} + \sqrt{2-1} \right\}$$

$$= 2\sqrt{3} + ln\left\{ 2 + \sqrt{3} \right\} - \sqrt{2} - ln\left\{ \sqrt{2} + 1 \right\}$$

$$= 2\sqrt{3} - \sqrt{2} + ln\left\{ \frac{2+\sqrt{3}}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} \right\}$$

$$= 2\sqrt{3} - \sqrt{2} + ln\left\{ \left(2+\sqrt{3}\right)\left(\sqrt{2}-1\right) \right\}$$

[4 marks]

Answer 7

$$\int_1^3 \frac{1}{\sqrt{3x^2 - 6x + 7}}\,dx = \int_1^3 \frac{1}{\sqrt{3\,(x^2 - 2x) + 7}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{3\,(\,(x - 1)^2 - 1\,) + 7}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{3(x - 1)^2 - 3 + 7}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{(\sqrt{3})^2 (x - 1)^2 + 4}}\,dx$$

$$= \frac{1}{\sqrt{3}} \int_1^3 \frac{\sqrt{3}}{\sqrt{(\sqrt{3}\,x - \sqrt{3})^2 + 2^2}}\,dx$$

Standard Result

$\dfrac{1}{\sqrt{x^2 + a^2}}$ integrates to $arsinh\left(\dfrac{x}{a}\right)$, $ln\left\{ x + \sqrt{x^2 + a^2} \right\}$

$$= \frac{1}{\sqrt{3}} \left[ln\left\{ (\sqrt{3}\,x - \sqrt{3}) + \sqrt{(\sqrt{3}\,x - \sqrt{3})^2 + 2^2} \right\} \right]_1^3$$

$$= \frac{1}{\sqrt{3}} \left[ln\left\{ 2\sqrt{3} + \sqrt{12 + 4} \right\} - ln\left\{ 0 + \sqrt{0^2 + 4} \right\} \right]$$

$$= \frac{1}{\sqrt{3}} \left[ln\left\{ 2\sqrt{3} + 4 \right\} - ln\left\{ 2 \right\} \right]$$

$$= \frac{1}{\sqrt{3}} \left[ln\left\{ 2 + \sqrt{3} \right\} \right]$$

[7 marks]