12.2 Answers

Answer 1

$$\begin{aligned}
\int \cos^5 x \, dx &= \int \cos x \cos^4 x \, dx \\
&= \int \cos x \left(\cos^2 x \right)^2 \, dx \\
&= \int \cos x \left(1 - \sin^2 x \right)^2 \, dx \\
&= \int \cos x \left(1 - 2 \sin^2 x + \sin^4 x \right) dx \\
&= \int \cos x - 2 \cos x \sin^2 x + \cos x \sin^4 x \, dx \\
&= \sin x + \frac{2 \sin^3 x}{3} + \frac{\sin^5 x}{5} + c
\end{aligned}$$

[3 marks]

Answer 2

$$y = \operatorname{artanh}(\sin x)$$

Standard Result

$$\operatorname{artanh} x \text{ differentiates to } \frac{1}{1 - x^2}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1 - \sin^2 x} \times (\cos x) \\
&= \frac{1}{\cos^2 x} \times (\cos x) \\
&= \frac{1}{\cos x} \\
&= \sec x
\end{aligned}$$

[3 marks]

Answer 3

$$x = 2\cosh u \Rightarrow \frac{dx}{du} = 2\sinh u$$

$$\begin{aligned}\int \sqrt{x^2 - 4}\,dx &= \int \sqrt{4\cosh^2 u - 4} \times 2\sinh u\ du \\ &= \int 2\sqrt{\cosh^2 u - 1} \times 2\sinh u\,du \\ &= 4\int \sqrt{\sinh^2 u}\ \sinh u\,du \\ &= 4\int \sinh^2 u\,du\end{aligned}$$

$$\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$$

SUBTRACT $\cosh^2\theta - \sinh^2\theta = 1$

$$2\sinh^2\theta = \cosh 2\theta - 1$$

$$\begin{aligned}\int \sqrt{x^2 - 4}\,dx &= 2\int \cosh 2u - 1\,du \\ &= \int 2\cosh 2u - 2\,du \\ &= \sinh 2u - 2u + c \\ &= 2\sinh u\cosh u - 2u + c \\ &= 2\sqrt{\left(\frac{x}{2}\right)^2 - 1}\left(\frac{x}{2}\right) - 2\,\mathrm{arcosh}\left(\frac{x}{2}\right) + c \\ &= \frac{1}{2}x\sqrt{x^2 - 4} - 2\,\mathrm{arcosh}\left(\frac{x}{2}\right) + c\end{aligned}$$

[5 marks]

Answer 4

First to establish a useful identity,

$$\cosh^2\theta - \sinh^2\theta = 1$$

Divide through by $\cosh^2\theta$

$$\frac{\cosh^2\theta}{\cosh^2\theta} - \frac{\sinh^2\theta}{\cosh^2\theta} = \frac{1}{\cosh^2\theta}$$

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta$$

$$\tanh^2\theta = 1 - \operatorname{sech}^2\theta$$

$$2\tanh^2 x - \operatorname{sech} x = 1$$

$$2\left(1 - \operatorname{sech}^2 x\right) - \operatorname{sech} x = 1$$

$$2 - 2\operatorname{sech}^2 x - \operatorname{sech} x = 1$$

$$2\operatorname{sech}^2 x + \operatorname{sech} x - 1 = 0$$

$$(2\operatorname{sech} x - 1)(\operatorname{sech} x + 1) = 0$$

Either $\operatorname{sech} x = \dfrac{1}{2}$ or $\operatorname{sech} x = -1$

$$\frac{1}{\cosh x} = \frac{1}{2} \quad \text{or} \quad \frac{1}{\cosh x} = -1$$

$$\cosh x = 2 \quad \text{or} \quad \cosh x = -1$$

Thinking of the graph of $y = \cosh x$, it is always positive and so the equation $\cosh x = -1$ has no solutions whereas $\cosh x = 2$ will have two solutions given by $x = \pm \operatorname{arcosh} 2$

$$x = \pm \ln\left(2 + \sqrt{2^2 - 1}\right)$$

$$= \pm \ln\left(2 + \sqrt{3}\right)$$

That is, $a = 2$ and $b = 3$

[5 marks]

Answer 5

$$y = arsinh\, x$$

Standard Result

$$arsinh\, x \text{ differentiates to } \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{dy}{dx} = \left(x^2 + 1\right)^{-\frac{1}{2}}$$

$$= \frac{1}{\left(x^2 + 1\right)^{\frac{1}{2}}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\left(x^2 + 1\right)^{-\frac{3}{2}} 2x$$

$$= -x\left(x^2 + 1\right)^{-\frac{3}{2}}$$

$$= -\frac{x}{\left(x^2 + 1\right)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = (-x) \times \left(-\frac{3}{2}\right)\left(x^2 + 1\right)^{-\frac{5}{2}} 2x + (-1)\left(x^2 + 1\right)^{-\frac{3}{2}}$$

$$= \frac{3x^2}{\left(x^2 + 1\right)^{\frac{5}{2}}} - \frac{1}{\left(x^2 + 1\right)^{\frac{3}{2}}}$$

$$\text{LHS} = \left(x^2 + 1\right)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx}$$

$$= \left(\frac{3x^2\left(x^2 + 1\right)}{\left(x^2 + 1\right)^{\frac{5}{2}}} - \frac{1 \times \left(x^2 + 1\right)}{\left(x^2 + 1\right)^{\frac{3}{2}}}\right) + \left(-\frac{3x^2}{\left(x^2 + 1\right)^{\frac{3}{2}}}\right) + \frac{1}{\left(x^2 + 1\right)^{\frac{1}{2}}}$$

$$= \frac{3x^2}{\left(x^2 + 1\right)^{\frac{3}{2}}} - \frac{1}{\left(x^2 + 1\right)^{\frac{1}{2}}} - \frac{3x^2}{\left(x^2 + 1\right)^{\frac{3}{2}}} + \frac{1}{\left(x^2 + 1\right)^{\frac{1}{2}}}$$

$$= 0$$

$$= \text{RHS} \qquad \square$$

[7 marks]

Answer 6

(i) $x = sinh^2 u \Rightarrow \dfrac{dx}{du} = 2\, sinh\, u\, cosh\, u$

$$\int \sqrt{\frac{x}{x+1}}\, dx = \int \sqrt{\frac{sinh^2 u}{sinh^2 u + 1}} \times 2\, sinh\, u\, cosh\, u\ \ du$$

$$= \int \sqrt{\frac{sinh^2 u}{cosh^2 u}} \times 2\, sinh\, u\, cosh\, u\ \ du$$

$$= \int \frac{sinh\, u}{cosh\, u} \times 2\, cosh\, u\, sinh\, u\ \ du$$

$$= \int 2\, sinh^2 u\, du$$

$$cosh^2 \theta + sinh^2 \theta = cosh\, 2\theta$$

SUBTRACT $cosh^2 \theta - sinh^2 \theta = 1$

$$2\, sinh^2 \theta = cosh\, 2\theta - 1$$

$$\int \sqrt{\frac{x}{x+1}}\, dx = \int cosh\, 2u - 1\, du$$

$$= \frac{1}{2} \int 2\, cosh\, 2u\, du - \int 1\, du$$

$$= \frac{1}{2}\, sinh\, 2u - u + c$$

$$= sinh\, u\, cosh\, u - u + c$$

$$= sinh\, u\, \sqrt{sinh^2 u + 1} - u + c$$

$$= \sqrt{x}\, \sqrt{x+1} - arsinh\left(\sqrt{x}\right) + c$$

[6 marks]

(ii) $$\int_0^1 \sqrt{\frac{x}{x+1}}\, dx = \left[\, \sqrt{x}\, \sqrt{x+1} - arsinh\left(\sqrt{x}\right) \right]_0^1$$

$$= \left[\, \sqrt{x}\, \sqrt{x+1}\, \sqrt{x} - ln\left\{ \sqrt{x} + \sqrt{x+1} \right\} \right]$$

$$= \sqrt{1}\sqrt{1+1} - ln\left\{ \sqrt{1} + \sqrt{1+1} \right\}$$

$$- \sqrt{0}\, \sqrt{0+1} + ln\left\{ \sqrt{0} + \sqrt{0+1} \right\}$$

$$= \sqrt{2} - ln\left\{ 1 + \sqrt{2} \right\} - 0 + ln\left\{ 1 \right\}$$

$$= \sqrt{2} - ln\left\{ 1 + \sqrt{2} \right\}$$

[4 marks]

Answer 7

(i) $$\int y\,dx = \int \frac{1}{\sqrt{x^2 + 2x - 3}}\,dx$$

$$= \int \frac{1}{\sqrt{(x^2 + 2x) - 3}}\,dx$$

$$= \int \frac{1}{\sqrt{((x + 1)^2 - 1) - 3}}\,dx$$

$$= \int \frac{1}{\sqrt{(x + 1)^2 - 2^2}}\,dx$$

Standard Result

$\frac{1}{\sqrt{x^2 - a^2}}$ integrates to $arcosh\left(\frac{x}{a}\right),\ x > a$

$$= arcosh\left(\frac{x + 1}{2}\right),\quad x > 2$$ $\Leftarrow$ Full marks from here on

$$= ln\left\{x + 1 + \sqrt{(x + 1)^2 - 2^2}\right\}$$

$$= ln\left\{x + 1 + \sqrt{x^2 + 2x - 3}\right\},\ x > 2$$

[3 marks]

(ii)

$$V = \pi \int_2^3 y^2 \, dx$$

$$= \pi \int_2^3 \frac{1}{x^2 + 2x - 3} \, dx$$

$$= \pi \int_2^3 \frac{1}{(x + 1)^2 - 2^2} \, dx$$

Standard Result

$$\frac{1}{x^2 - a^2} \text{ integrates to } \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right|$$

$$= \frac{\pi}{4} \ln \left| \frac{x + 1 - 2}{x + 1 + 2} \right|_2^3$$

$$= \frac{\pi}{4} \ln \left| \frac{x - 1}{x + 3} \right|_2^3$$

$$= \frac{\pi}{4} \left[\ln \left| \frac{2}{6} \right| - \ln \left| \frac{1}{5} \right| \right]$$

$$= \frac{\pi}{4} \ln \left| \frac{1 \times 5}{3 \times 1} \right|$$

$$= \frac{\pi}{4} \ln \left(\frac{5}{3} \right)$$

$$\therefore p = \frac{1}{4} \text{ and } q = \frac{5}{3}$$

[4 marks]