

2.4 Answers

Further A-Level Pure Mathematics, Core 2 Hyperbolic Functions

2.4.1 Solution to 2.2 Example (Teaching Video)

Prove that $\cosh^2 x - \sinh^2 x = 1$

$$\begin{aligned}\text{LHS} &= \cosh^2 x - \sinh^2 x \\&= \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2 \\&= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\&= \frac{4}{4} \\&= 1 \\&= \text{RHS} \quad \square\end{aligned}$$

[3 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

Prove that $\cosh^2 x + \sinh^2 x = \cosh 2x$

$$\begin{aligned}\text{LHS} &= \cosh^2 x + \sinh^2 x \\&= \left(\frac{e^x + e^{-x}}{2} \right)^2 + \left(\frac{e^x - e^{-x}}{2} \right)^2 \\&= \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} + \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} \\&= \frac{e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}}{4} \\&= \frac{2e^{2x} + 2e^{-2x}}{4} \\&= \frac{2(e^{2x} + e^{-2x})}{4} \\&= \frac{e^{2x} + e^{-2x}}{2} \\&= \cosh 2x \\&= \text{RHS} \quad \square\end{aligned}$$

[4 marks]

Answer 2

Prove that $2 \sinh x \cosh x = \sinh 2x$

$$\begin{aligned}
 \text{LHS} &= 2 \sinh x \cosh x \\
 &= 2 \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) \\
 &= \frac{1}{2} (e^x - e^{-x})(e^x + e^{-x}) \\
 &= \frac{1}{2} (e^{2x} - e^{-2x}) \quad \text{Difference of two squares} \\
 &= \frac{e^{2x} - e^{-2x}}{2} \\
 &= \sinh 2x \\
 &= \text{RHS} \quad \square
 \end{aligned}$$

[4 marks]

Answer 3

(i) $\cosh^2 x - \sinh^2 x = 1$

$$\cosh^2 x - \left(\frac{3}{4} \right)^2 = 1$$

$$\cosh^2 x = 1 + \frac{9}{16}$$

$$= \frac{25}{16}$$

$$\cosh x = \frac{5}{4} \quad \left(\cosh x \geq 1 \text{ so rejected } - \frac{5}{4} \right)$$

[2 mark]

(ii)

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$= \left(\frac{3}{4} \right) \div \left(\frac{5}{4} \right)$$

$$= \left(\frac{3}{4} \right) \times \left(\frac{4}{5} \right) \quad KOF$$

$$= \frac{3}{5}$$

[1 mark]

(iii)

$$\sinh 2x = 2 \sinh x \cosh x$$

$$= 2 \left(\frac{3}{4} \right) \left(\frac{5}{4} \right)$$

$$= \frac{15}{8}$$

[1 mark]

$$(iv) \quad \cosh^2 2x - \sinh^2 2x = 1$$

$$\cosh^2 2x - \left(\frac{15}{8}\right)^2 = 1$$

$$\cosh^2 2x = 1 + \frac{225}{64}$$

$$= \frac{289}{64}$$

$$\cosh 2x = \frac{17}{8} \quad \left(\cosh x \geq 1 \text{ so rejected } - \frac{17}{8} \right)$$

[2 marks]

$$\text{Alternatively: } \cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$= \frac{25}{16} + \frac{9}{16}$$

$$= \frac{17}{8}$$

[2 marks]

$$(v) \quad \sinh 4x = 2 \sinh 2x \cosh 2x$$

$$= 2 \left(\frac{15}{8}\right) \left(\frac{17}{8}\right)$$

$$= \frac{255}{32}$$

[1 mark]

Answer 4

$$(i) \quad \cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$$

$$\text{RHS} = \cosh A \cosh B + \sinh A \sinh B$$

$$= \left(\frac{e^A + e^{-A}}{2}\right) \left(\frac{e^B + e^{-B}}{2}\right) + \left(\frac{e^A - e^{-A}}{2}\right) \left(\frac{e^B - e^{-B}}{2}\right)$$

$$= \frac{e^{A+B} + e^{A-B} + e^{-A+B} + e^{-A-B} + e^{A+B} - e^{A-B} - e^{-A+B} + e^{-A-B}}{4}$$

$$= \frac{2e^{A+B} + 2e^{-A-B}}{4}$$

$$= \frac{2(e^{A+B} + e^{-(A+B)})}{4}$$

$$= \frac{e^{A+B} + e^{-(A+B)}}{2}$$

$$= \cosh(A+B)$$

$$= \text{LHS} \quad \square$$

[5 marks]

$$(ii) \quad \cosh 3A = 4 \cosh^3 A - 3 \cosh A$$

$$\begin{aligned} \text{RHS} &= 4 \cosh^3 A - 3 \cosh A \\ &= 4 \left(\frac{e^A + e^{-A}}{2} \right)^3 - 3 \left(\frac{e^A + e^{-A}}{2} \right) \\ &= \frac{1}{2} (e^A + e^{-A})^3 - \frac{3}{2} (e^A + e^{-A}) \\ &= \frac{1}{2} (e^{3A} + 3e^{2A}e^{-A} + 3e^Ae^{-2A} + e^{-3A}) - \frac{3}{2}e^A - \frac{3}{2}e^{-A} \\ &= \frac{1}{2}e^{3A} + \frac{3}{2}e^A + \frac{3}{2}e^{-A} + \frac{1}{2}e^{-3A} - \frac{3}{2}e^A - \frac{3}{2}e^{-A} \\ &= \frac{1}{2}e^{3A} + \frac{1}{2}e^{-3A} \\ &= \frac{e^{3A} + e^{-3A}}{2} \\ &= \cosh 3A \\ &= \text{LHS} \quad \square \end{aligned}$$

[5 marks]

$$(iii) \quad \sinh A - \sinh B = 2 \sinh \left(\frac{A-B}{2} \right) \cosh \left(\frac{A+B}{2} \right)$$

$$\begin{aligned} \text{RHS} &= 2 \sinh \left(\frac{A-B}{2} \right) \cosh \left(\frac{A+B}{2} \right) \\ &= 2 \times \frac{1}{2} \left(e^{\frac{A-B}{2}} - e^{-\frac{A-B}{2}} \right) \times \frac{1}{2} \left(e^{\frac{A+B}{2}} + e^{-\frac{A+B}{2}} \right) \\ &= \frac{1}{2} \left(e^{\frac{A-B}{2} + \frac{A+B}{2}} + e^{\frac{A-B}{2} - \frac{A+B}{2}} - e^{-\frac{A-B}{2} + \frac{A+B}{2}} - e^{-\frac{A-B}{2} - \frac{A+B}{2}} \right) \\ &= \frac{1}{2} (e^A + e^{-B} - e^B - e^{-A}) \\ &= \frac{e^A - e^{-A} - e^B + e^{-B}}{2} \\ &= \frac{e^A - e^{-A}}{2} - \frac{e^B - e^{-B}}{2} \\ &= \sinh A - \sinh B \\ &= \text{LHS} \quad \square \end{aligned}$$

[5 marks]

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