2.4 Answers

2.4.1 Solution to 2.2 Example (Teaching Video)

Prove that $\cosh^2 x - \sinh^2 x = 1$

$$\begin{aligned}
\text{LHS} &= \cosh^2 x - \sinh^2 x \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2\,e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2\,e^x e^{-x} + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\
&= \frac{4}{4} \\
&= 1 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

Prove that $\cosh^2 x + \sinh^2 x = \cosh 2x$

$$\begin{aligned}
\text{LHS} &= \cosh^2 x + \sinh^2 x \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2\,e^x e^{-x} + e^{-2x}}{4} + \frac{e^{2x} - 2\,e^x e^{-x} + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{2e^{2x} + 2e^{-2x}}{4} \\
&= \frac{2\left(e^{2x} + e^{-2x}\right)}{4} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= \cosh 2x \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

Prove that $2\sinh x \cosh x = \sinh 2x$

$$\begin{aligned}
\text{LHS} &= 2\sinh x \cosh x \\
&= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{1}{2}\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right) \\
&= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \qquad \text{Difference of two squares} \\
&= \frac{e^{2x} - e^{-2x}}{2} \\
&= \sinh 2x \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 3

(i) $\cosh^2 x - \sinh^2 x = 1$

$$\begin{aligned}
\cosh^2 x - \left(\frac{3}{4}\right)^2 &= 1 \\
\cosh^2 x &= 1 + \frac{9}{16} \\
&= \frac{25}{16} \\
\cosh x &= \frac{5}{4} \qquad \left(\cosh x \geqslant 1 \text{ so rejected } -\frac{5}{4}\right)
\end{aligned}$$

[2 mark]

(ii)
$$\begin{aligned}
\tanh x &= \frac{\sinh x}{\cosh x} \\
&= \left(\frac{3}{4}\right) \div \left(\frac{5}{4}\right) \\
&= \left(\frac{3}{4}\right) \times \left(\frac{4}{5}\right) \qquad KOF \\
&= \frac{3}{5}
\end{aligned}$$

[1 mark]

(iii)
$$\begin{aligned}
\sinh 2x &= 2\sinh x \cosh x \\
&= 2\left(\frac{3}{4}\right)\left(\frac{5}{4}\right) \\
&= \frac{15}{8}
\end{aligned}$$

[1 mark]

(iv) $\cosh^2 2x - \sinh^2 2x = 1$

$$\cosh^2 2x - \left(\frac{15}{8}\right)^2 = 1$$

$$\cosh^2 2x = 1 + \frac{225}{64}$$

$$= \frac{289}{64}$$

$$\cosh 2x = \frac{17}{8} \qquad \left(\cosh x \geqslant 1 \text{ so rejected } -\frac{17}{8}\right)$$

[2 marks]

Alternatively: $\cosh(2x) = \cosh^2 x + \sinh^2 x$

$$= \frac{25}{16} + \frac{9}{16}$$

$$= \frac{17}{8}$$

[2 marks]

(v) $\sinh 4x = 2 \sinh 2x \cosh 2x$

$$= 2\left(\frac{15}{8}\right)\left(\frac{17}{8}\right)$$

$$= \frac{255}{32}$$

[1 mark]

Answer 4

(i) $\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$

RHS $= \cosh A \cosh B + \sinh A \sinh B$

$$= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) + \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$$

$$= \frac{e^{A+B} + e^{A-B} + e^{-A+B} + e^{-A-B} + e^{A+B} - e^{A-B} - e^{-A+B} + e^{-A-B}}{4}$$

$$= \frac{2\,e^{A+B} + 2\,e^{-A-B}}{4}$$

$$= \frac{2\left(e^{A+B} + e^{-(A+B)}\right)}{4}$$

$$= \frac{e^{A+B} + e^{-(A+B)}}{2}$$

$$= \cosh(A + B)$$

$$= \text{LHS} \qquad \square$$

[5 marks]

(ii) $\cosh 3A = 4\cosh^3 A - 3\cosh A$

$$\begin{aligned}
\text{RHS} &= 4\cosh^3 A - 3\cosh A \\
&= 4\left(\frac{e^A + e^{-A}}{2}\right)^3 - 3\left(\frac{e^A + e^{-A}}{2}\right) \\
&= \frac{1}{2}\left(e^A + e^{-A}\right)^3 - \frac{3}{2}\left(e^A + e^{-A}\right) \\
&= \frac{1}{2}\left(e^{3A} + 3e^{2A}e^{-A} + 3e^A e^{-2A} + e^{-3A}\right) - \frac{3}{2}e^A - \frac{3}{2}e^{-A} \\
&= \frac{1}{2}e^{3A} + \frac{3}{2}e^A + \frac{3}{2}e^{-A} + \frac{1}{2}e^{-3A} - \frac{3}{2}e^A - \frac{3}{2}e^{-A} \\
&= \frac{1}{2}e^{3A} + \frac{1}{2}e^{-3A} \\
&= \frac{e^{3A} + e^{-3A}}{2} \\
&= \cosh 3A \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[5 marks]

(iii) $\sinh A - \sinh B = 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$

$$\begin{aligned}
\text{RHS} &= 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right) \\
&= 2 \times \frac{1}{2}\left(e^{\frac{A-B}{2}} - e^{-\frac{A-B}{2}}\right) \times \frac{1}{2}\left(e^{\frac{A+B}{2}} + e^{-\frac{A+B}{2}}\right) \\
&= \frac{1}{2}\left(e^{\frac{A-B}{2}+\frac{A+B}{2}} + e^{\frac{A-B}{2}-\frac{A+B}{2}} - e^{-\frac{A-B}{2}+\frac{A+B}{2}} - e^{-\frac{A-B}{2}-\frac{A+B}{2}}\right) \\
&= \frac{1}{2}\left(e^A + e^{-B} - e^B - e^{-A}\right) \\
&= \frac{e^A - e^{-A} - e^B + e^{-B}}{2} \\
&= \frac{e^A - e^{-A}}{2} - \frac{e^B - e^{-B}}{2} \\
&= \sinh A - \sinh B \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[5 marks]