4.3 Answers to 4.2 Exercise

Further A-Level Pure Mathematics, Core 2
Hyperbolic Functions

Answer 1

(i) $1 - \tanh^2 x = \operatorname{sech}^2 x$

[2 marks]

(ii)

$$\begin{aligned}
\text{RHS} &= \operatorname{sech}^2 x \\
&= \left(\frac{2}{e^x + e^{-x}} \right)^2 \\
&= \left(\frac{2}{e^x + e^{-x}} \times \frac{e^x}{e^x} \right)^2 \\
&= \left(\frac{2\,e^x}{e^{2x} + 1} \right)^2 \\
&= \frac{4\,e^{2x}}{\left(e^{2x} + 1 \right)^2}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= 1 - \tanh^2 x \\
&= 1 - \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right)^2 \\
&= \left(\frac{e^{2x} + 1}{e^{2x} + 1} \right)^2 - \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right)^2 \\
&= \frac{\left(e^{2x} + 1 \right)^2 - \left(e^{2x} - 1 \right)^2}{\left(e^{2x} + 1 \right)^2} \qquad \text{Difference of two squares} \\
&= \frac{\left(e^{2x} + 1 + e^{2x} - 1 \right)\left(e^{2x} + 1 - e^{2x} + 1 \right)}{\left(e^{2x} + 1 \right)^2} \\
&= \frac{4\,e^{2x}}{\left(e^{2x} + 1 \right)^2} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[6 marks]

Answer 2

$$\cosh 2x = 1 + 2\sinh^2 x$$

[2 marks]

Answer 3

(a)

$$\begin{aligned} \text{RHS} &= 1 + 2\sinh^2 x \\ &= 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{2}{2} + \frac{e^{2x} - 2 + e^{-2x}}{2} \\ &= \frac{e^{2x} + e^{-2x}}{2} \\ &= \cosh 2x \\ &= \text{LHS} \qquad \square \end{aligned}$$

[3 marks]

(b)

$$\begin{aligned} \cosh 2x - 3\sinh x &= 15 \\ 1 + 2\sinh^2 x - 3\sinh x &= 15 \\ 2\sinh^2 x - 3\sinh x - 14 &= 0 \\ (2\sinh x - 7)(\sinh x + 2) &= 0 \end{aligned}$$

Either $\sinh x = \frac{7}{2}$ or $\sinh x = -2$

$$x = \ln\left(\frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 + 1}\right) \qquad \text{or} \qquad x = \ln\left(-2 + \sqrt{5}\right)$$

$$= \ln\left(\frac{7}{2} + \frac{\sqrt{53}}{2}\right)$$

[5 marks]

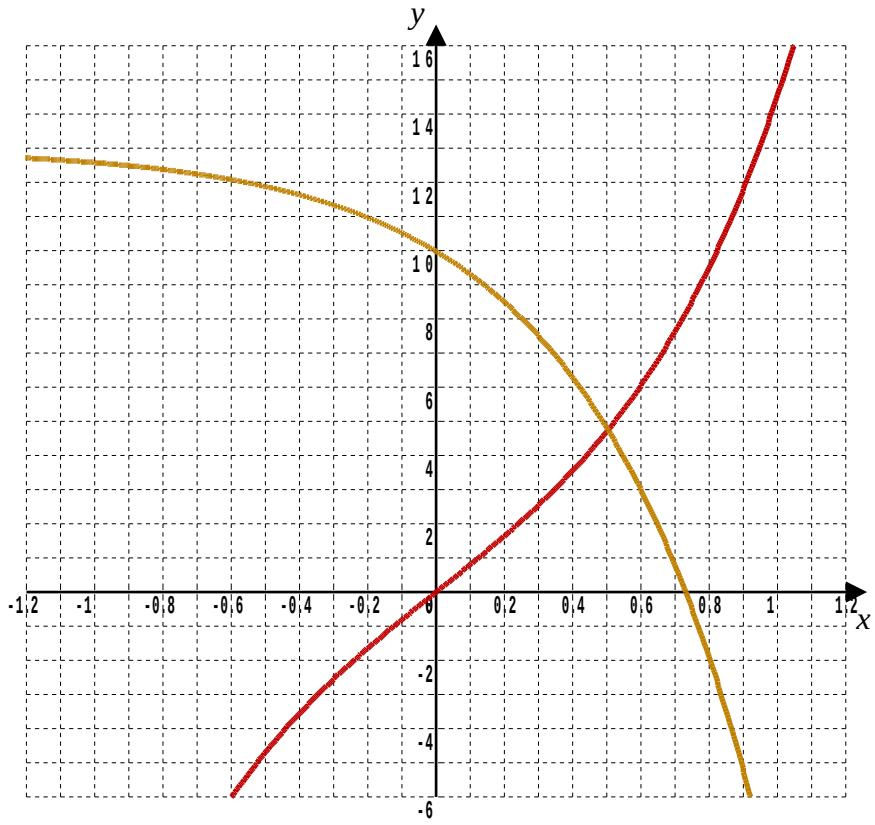
Answer 4

$$\tanh(A - B) = \frac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$$

[2 marks]

Answer 5

(a)



C_1 : Red curve $y = 3\sinh 2x$ passes through $(0,\ 0)$

C_2 : Gold curve $y = 13 - 3e^{2x}$: Asymptote $y = 13$

crosses y-axis $(0,\ 10)$, and crosses x-axis $\left(\ln\left(\sqrt{\frac{13}{3}}\right),\ 0\right)$

[4 marks]

(b)

$$3\sinh 2x = 13 - 3e^{2x}$$

$$\frac{3}{2}\left(e^{2x} - e^{-2x}\right) = 13 - 3e^{2x}$$

$$3e^{2x} - 3e^{-2x} = 26 - 6e^{2x}$$

$$9e^{2x} - 26 - 3e^{-2x} = 0$$

Multiply through by e^{2x}

$$9\left(e^{2x}\right)^2 - 26e^{2x} - 3 = 0$$

$$\left(9e^{2x} + 1\right)\left(e^{2x} - 3\right) = 0$$

Either $e^{2x} = -\frac{1}{9}$ or $e^{2x} = 3$

No solutions

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

That is, $k = 3$

[5 marks]

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