7.4 Answers to 7.3 Exercise

Answer 1

$$\cos^2 x + \sin^2 x = 1$$

$$\therefore 1 + \tan^2 x = \sec^2 x$$

$$\therefore 1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \text{(Osborn's Rule)}$$

$$15\operatorname{sech}^2 x + 7\tanh x = 13$$

$$15\left(1 - \tanh^2 x\right) + 7\tanh x = 13$$

$$15 - 15\tanh^2 x + 7\tanh x = 13$$

$$15\tanh^2 x - 7\tanh x - 2 = 0$$

$$\left(5\tanh x + 1\right)\left(3\tanh x - 2\right) = 0$$

Either $\tanh x = -\frac{1}{5}$ or $\tanh x = \frac{2}{3}$

$x = \operatorname{artanh}\left(-\frac{1}{5}\right)$ or $x = \operatorname{artanh}\left(\frac{2}{3}\right)$

$= \frac{1}{2}\ln\left(\frac{1 + \left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)}\right)$ $= \frac{1}{2}\ln\left(\frac{1 + \frac{2}{3}}{1 - \frac{2}{3}}\right)$

$= \frac{1}{2}\ln\left(\frac{2}{3}\right)$ $= \frac{1}{2}\ln 5$

[6 marks]

Answer 2

$$7\,sech\,x - tanh\,x = 5$$

$$\frac{7}{cosh\,x} - \frac{sinh\,x}{cosh\,x} = 5$$

$$7 - sinh\,x = 5\,cosh\,x$$

$$5\,cosh\,x + sinh\,x - 7 = 0$$

$$5 \times \frac{e^{x} + e^{-x}}{2} + \frac{e^{x} - e^{-x}}{2} - 7 = 0$$

multiply through by 2

$$5\,e^{x} + 5\,e^{-x} + e^{x} - e^{-x} - 14 = 0$$

$$6\,e^{x} + 4\,e^{-x} - 14 = 0$$

divide through by 2

$$3\,e^{x} + 2\,e^{-x} - 7 = 0$$

multiple through by e^{x}

$$3\left(e^{x}\right)^{2} - 7\,e^{x} + 2 = 0$$

$$\left(3\,e^{x} - 1\right)\left(e^{x} - 2\right) = 0$$

Either $e^{x} = \frac{1}{3}$ or $e^{x} = 2$

$$x = ln\left(\frac{1}{3}\right) \quad \text{or} \quad x = ln\,2$$

[5 marks]

Answer 3

$$y = \operatorname{artanh} x$$

$$\text{when } x = \frac{12}{13},\quad y = \frac{1}{2}\ln\left(\frac{1+\frac{12}{13}}{1-\frac{12}{13}} \times \frac{13}{13}\right)$$

$$= \frac{1}{2}\ln\left(\frac{13+12}{13-12}\right)$$

$$= \frac{1}{2}\ln(25)$$

$$= \ln 5$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

$$\text{when } x = \frac{12}{13},\quad \frac{dy}{dx} = \frac{1}{1-\left(\frac{12}{13}\right)^2} \times \frac{13^2}{13^2}$$

$$= \frac{13^2}{13^2 - 12^2}$$

$$= \frac{169}{25}$$

$$\text{Tangent :} \quad y = mx + c$$

$$\ln 5 = \frac{169}{25} \times \frac{12}{13} + c$$

$$\ln 5 = \frac{156}{25} + c$$

$$c = \ln 5 - \frac{156}{25}$$

$$\text{The tangent has equation } y = \frac{169}{25}x + \ln 5 - \frac{156}{25}$$

$$(y = 6.76x - 4.63)$$

$$(169x - 25y + 25\ln 5 - 156)$$

[3 marks]

Answer 4

(a)

$$y = -x + \tanh(4x)$$

$$\frac{dy}{dx} = -1 + 4\,\mathrm{sech}^2(4x)$$

For a turning point $\frac{dy}{dx} = 0$

$$0 = -1 + \frac{4}{\cosh^2(4x)}$$

multiply through by $\cosh^2(4x)$

$$0 = -\cosh^2(4x) + 4$$

$$\cosh^2(4x) = 4$$

$$\cosh(4x) = 2 \qquad \text{because } \cosh x \text{ is always positive}$$

$$4x = \operatorname{arcosh}(2)$$

$$= \ln\left(2 + \sqrt{3}\right)$$

$$x = \frac{1}{4}\ln\left(2 + \sqrt{3}\right)$$

[4 marks]

(b)

$$\cosh^2(4x) - \sinh^2(4x) = 1$$

$$\sinh^2(4x) = \cosh^2(4x) - 1$$

$$= 4 - 1$$

$$= 3$$

$$\sinh x = \sqrt{3}$$

$\left(\text{As } x = \frac{1}{4}\ln\left(2 + \sqrt{3}\right) \text{ the other possibility of } \sinh x = -\sqrt{3} \text{ is rejected}\right)$

$$y = -x + \tanh(4x)$$

$$= -x + \frac{\sinh(4x)}{\cosh(4x)}$$

$$= -\frac{1}{4}\ln\left(2 + \sqrt{3}\right) + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}\left\{2\sqrt{3} - \ln\left(2 + \sqrt{3}\right)\right\}$$

[3 marks]

Answer 5

$$artanh\, x + artanh\, y = ln\sqrt{3}$$

$$\Rightarrow \frac{1}{2}ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}ln\left(\frac{1+y}{1-y}\right) = \frac{1}{2}ln\, 3$$

multiply through by 2

$$ln\left(\frac{1+x}{1-x}\right) + ln\left(\frac{1+y}{1-y}\right) = ln\, 3$$

$$ln\left(\frac{(1+x)(1+y)}{(1-x)(1-y)}\right) = ln\, 3$$

$$\frac{(1+x)(1+y)}{(1-x)(1-y)} = 3$$

$$(1+x)(1+y) = 3(1-x)(1-y)$$

$$1 + x + y + xy = 3 - 3x - 3y + 3xy$$

$$1 + x - 3 + 3x = -3y + 3xy - y - xy$$

$$-2 + 4x = -4y + 2xy$$

divide through by 2

$$-1 + 2x = -2y + xy$$

$$2x - 1 = y(x - 2)$$

$$y = \frac{2x-1}{x-2}$$

[6 marks]

Answer 6

$$\text{LHS} = \frac{d}{dx}\left[\, ln\left(\, tanh\, x\,\right)\,\right]$$

$$= \frac{1}{tanh\, x} \times sech^2\, x$$

$$= \frac{cosh\, x}{sinh\, x} \times \frac{1}{cosh^2\, x}$$

$$= \frac{1}{sinh\, x\ cosh\, x}$$

$$= \frac{2}{2\, sinh\, x\ cosh\, x}$$

$$= \frac{2}{sinh\,(2x)}$$

$$= 2\, csch\,(2x)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 7

$$csch\,x - 2\,coth\,x = 2$$

$$\frac{1}{sinh\,x} - \frac{2\,cosh\,x}{sinh\,x} = 2$$

multiply through by $sinh\,x$

$$1 - 2\,cosh\,x = 2\,sinh\,x$$

$$1 - \left(e^{x} + e^{-x}\right) = \left(e^{x} - e^{-x}\right)$$

$$1 - e^{x} - e^{-x} = e^{x} - e^{-x}$$

$$1 - e^{x} = e^{x}$$

$$2\,e^{x} = 1$$

$$e^{x} = \frac{1}{2}$$

$$x = ln\left(\frac{1}{2}\right)$$

$$= ln\,2^{-1}$$

$$= (-1)\,ln\,2$$

That is, $k = -1$ and $a = 2$

[4 marks]