8.4 Answers

8.4.1 Solution to 8.2 Example

$$\int \frac{2 + 5x}{\sqrt{x^2 + 1}}\, dx = \int \frac{2}{\sqrt{x^2 + 1}}\, dx + \int \frac{5x}{\sqrt{x^2 + 1}}\, dx$$

$$= 2 \int \frac{1}{\sqrt{x^2 + 1}}\, dx + 5 \int x \left(x^2 + 1 \right)^{-\frac{1}{2}}\, dx$$

$$= 2 \int \frac{1}{\sqrt{x^2 + 1}}\, dx + 5 \times \frac{1}{2} \int 2\, x \left(x^2 + 1 \right)^{-\frac{1}{2}}\, dx$$

$$= 2\, arsinh\, x + \frac{5}{2} \times \frac{2 \left(x^2 + 1 \right)^{\frac{1}{2}}}{1} + c$$

$$= 2\, arsinh\, x + 5 \sqrt{x^2 + 1} + c$$

[6 marks]

8.4.2 Solution to 8.3 Exercise

Answer 1

(i) $$\int sinh\,(5x)\, dx = \frac{1}{5} \int 5\, sinh\,(5x)\, dx = \frac{1}{5} cosh\,(5x) + c$$

[2 marks]

(ii) $$\int \frac{1}{cosh^2\,(4x)}\, dx = \frac{1}{4} \int 4\, sech^2\,(4x)\, dx = \frac{1}{4} tanh\,(4x) + c$$

[3 marks]

Answer 2

$$\int x^2\, cosh \left(x^3 + 4 \right) dx = \frac{1}{3} \int 3\, x^2\, cosh \left(x^3 + 4 \right) dx = \frac{1}{3} sinh \left(x^3 + 4 \right) + c$$

[2 marks]

Answer 3

(i) $$\int sinh\,(5x)\; cosh^6\,(5x)\, dx = \frac{1}{5} \int 5\, sinh\,(5x)\; cosh^6\,(5x)\, dx$$

$$= \frac{1}{35} cosh^7\,(5x) + c$$

[2 marks]

(ii) $$\int \frac{sinh\,(5x)}{cosh^6\,(5x)}\, dx = \frac{1}{5} \int 5\, sinh\,(5x)\; \left(cosh\,(5x) \right)^{-6}\, dx$$

$$= -\frac{1}{25} \left(cosh\,(5x) \right)^{-5} + c$$

$$= -\frac{1}{25\, cosh^5\,(5x)} + c$$

[2 marks]

Answer 4

(i)

$$\int tanh\, x\, dx \;=\; \int \frac{sinh\, x}{cosh\, x}\, dx$$

$$= \int sinh\, x\, (\, cosh\, x\,)^{-1}\; dx$$

$$= \; ln\left|\, cosh\, x\, \right| \,+\, c$$

$$= \; ln\, (\, cosh\, x\,) \,+\, c$$

[2 marks]

(ii) The modulus signs were removed from the end result because the *cosh x* function is always positive, easily seen from its graph or from its definition in terms of exponentials.

[1 mark]

Answer 5

(i)

$$\int coth\, x\, dx \;=\; \int \frac{cosh\, x}{sinh\, x}\, dx$$

$$= \int cosh\, x\;\, (\, sinh\, x\,)^{-1}\; dx$$

$$= \; ln\left|\, sinh\, x\, \right| \,+\, c$$

[2 marks]

(ii) This time the modulus signs are needed because the *sinh x* function is negative for $x < 0$

[1 mark]

Answer 6

$$\int_0^{\sqrt{2}} \frac{6}{\sqrt{1 \,+\, 4x^2}}\, dx \;=\; 3\int_0^{\sqrt{2}} \frac{2}{\sqrt{1 \,+\, (2x)^2}}\, dx$$

$$= \; 3\; \left[\; arsinh\, (2x)\; \right]_0^{\sqrt{2}}$$

$$= \; 3\left[\; ln\left(\, 2x \,+\, \sqrt{4x^2 \,+\, 1}\;\right)\; \right]_0^{\sqrt{2}}$$

$$= \; 3\left[\; ln\left(\, 2\sqrt{2} \,+\, 3\,\right) \,-\, ln\left(\, 0 \,+\, \sqrt{1}\;\right)\; \right]$$

$$= \; 3\, ln\left(\, 2\sqrt{2} \,+\, 3\,\right)$$

[5 marks]

Answer 7

$$\int_{-1}^{\sqrt{3}} \frac{1 + x}{\sqrt{x^2 + 1}}\, dx = \int_{-1}^{\sqrt{3}} \frac{1}{\sqrt{x^2 + 1}}\, dx + \int_{-1}^{\sqrt{3}} \frac{x}{\sqrt{x^2 + 1}}\, dx$$

$$= \int_{-1}^{\sqrt{3}} \frac{1}{\sqrt{x^2 + 1}}\, dx + \frac{1}{2} \int_{-1}^{\sqrt{3}} 2x\left(x^2 + 1 \right)^{-\frac{1}{2}}\, dx$$

$$= \left[arsinh\, x + \sqrt{x^2 + 1} \right]_{-1}^{\sqrt{3}}$$

If not convinced by the preceding line, differentiate it

to check it goes back from whence it came !

$$= \left[ln\left(x + \sqrt{x^2 + 1} \right) + \sqrt{x^2 + 1} \right]_{-1}^{\sqrt{3}}$$

$$= ln\left(\sqrt{3} + 2 \right) + 2 - ln\left(-1 + \sqrt{2} \right) - \sqrt{2}$$

$$= ln\left(\frac{\sqrt{3} + 2}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} \right) + 2 - \sqrt{2}$$

$$= ln\left(\left(\sqrt{3} + 2 \right)\left(\sqrt{2} + 1 \right) \right) + 2 - \sqrt{2}$$

(It's about 2.784...)

[8 marks]

Answer 8

$$\int_0^{ln\, 3} e^x \cosh(2x)\;\; dx = \int_0^{ln\, 3} e^x \left(\frac{e^{2x} + e^{-2x}}{2} \right) dx$$

$$= \frac{1}{2} \int_0^{ln\, 3} e^{3x}\, dx + \frac{1}{2} \int_0^{ln\, 3} e^{-x}\, dx$$

$$= \frac{1}{6} \int_0^{ln\, 3} 3\, e^{3x}\, dx - \frac{1}{2} \int_0^{ln\, 3} (-1)\; e^{-x}\, dx$$

$$= \left[\frac{1}{6} e^{3x} - \frac{1}{2} e^{-x} \right]_0^{ln\, 3}$$

$$= \frac{e^{3\, ln\, 3}}{6} - \frac{e^{-ln\, 3}}{2} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{e^{ln\, 27}}{6} - \frac{e^{ln\, 3^{-1}}}{2} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{27}{6} - \frac{1}{6} - \frac{1}{6} + \frac{3}{6}$$

$$= \frac{14}{3}$$

[4 marks]

Answer 9

$$\begin{aligned}
\int_0^1 e^{2x} \sinh x \; dx &= \int_0^1 e^{2x}\left(\frac{e^{x} - e^{-x}}{2} \right) dx \\
&= \frac{1}{2} \int_0^1 e^{3x}\, dx - \frac{1}{2} \int_0^1 e^{x}\, dx \\
&= \frac{1}{6} \int_0^1 3\, e^{3x}\, dx - \frac{1}{2} \int_0^1 e^{x}\, dx \\
&= \left[\frac{1}{6} e^{3x} - \frac{1}{2} e^{x} \right]_0^1 \\
&= \frac{e^3}{6} - \frac{e}{2} - \frac{1}{6} + \frac{1}{2} \\
&= \frac{e^3}{6} - \frac{e}{2} + \frac{1}{3}
\end{aligned}$$

[4 marks]

Answer 10

(i)

$$6\cosh x = 9 - 2\sinh x$$

$$6\left(\frac{e^x + e^{-x}}{2}\right) = 9 - 2\left(\frac{e^x - e^{-x}}{2}\right)$$

$$3e^x + 3e^{-x} = 9 - e^x + e^{-x}$$

$$4e^x + 2e^{-x} - 9 = 0$$

Multiply through by e^x

$$4\left(e^x\right)^2 - 9e^x + 2 = 0$$

$$\left(4e^x - 1\right)\left(e^x - 2\right) = 0$$

Either $e^x = \frac{1}{4}$ in which case $x = ln\left(\frac{1}{4}\right)$

Or $e^x = 2$ in which case $x = ln\,2$

[6 marks]

(ii)

$$\int_{ln\,0.25}^{ln\,2} 9 - 2\sinh x - 6\cosh x \;\; dx$$

$$= \left[\, 9x - 2\cosh x - 6\sinh x \,\right]_{ln\,0.25}^{ln\,2}$$

$$= 9\,ln\,2 - e^{ln\,2} - e^{-ln\,2} - 3e^{ln\,2} + 3e^{-ln\,2}$$

$$- 9\,ln\,0.25 + e^{ln\,0.25} + e^{-ln\,0.25} + 3e^{ln\,0.25} - 3e^{-ln\,0.25}$$

$$= 9\,ln\,2 - 2 - \frac{1}{2} - 6 + \frac{3}{2} + 18\,ln\,2 + \frac{1}{4} + 4 + \frac{3}{4} - 12$$

$$= 27\,ln\,2 - 14$$

[6 marks]