9.6 Answers

9.6.1 Solution to 9.3 Example #1

$$\int \frac{1}{\sqrt{2x^2 + 12x}}\, dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2 + 6x}}\, dx$$

$$= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{(x+3)^2 - 3^2}}\, dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 - a^2}} \text{ integrates to } arcosh\left(\frac{x}{a}\right),\ x > a$$

$$= \frac{1}{\sqrt{2}}\, arcosh\left(\frac{x+3}{3}\right) + c$$

[4 marks]

9.6.2 Solution to 9.4 Example #2

$$\int_5^{13} \frac{100}{x^2 + 10x - 11}\, dx = 100 \int_5^{13} \frac{1}{(x+5)^2 - 6^2}\, dx$$

Standard Result

$$\frac{1}{x^2 - a^2} \text{ integrates to } \frac{1}{2a}\, ln\left|\frac{x-a}{x+a}\right|$$

$$= 100 \times \frac{1}{12}\left[ln\left|\frac{(x+5) - 6}{(x+5) + 6}\right| \right]_5^{13}$$

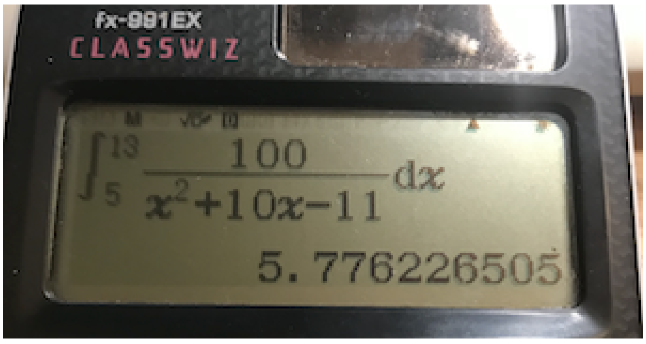
$$= \frac{25}{3}\left[ln\left|\frac{12}{24}\right| - ln\left|\frac{4}{16}\right| \right]$$

$$= \frac{25}{3}\left[ln\, 2^{-1} - ln\, 2^{-2} \right]$$

$$= \frac{25}{3}\left[-ln\, 2 + 2\, ln\, 2 \right]$$

$$= \frac{25\, ln\, 2}{3}$$

[6 marks]



9.6.3 Solutions to 9.5 Exercise

Answer 1

$$\int \frac{1}{\sqrt{x^2 - 2x + 10}}\, dx = \int \frac{1}{\sqrt{(x - 1)^2 + 3^2}} dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right), \quad ln\left\{ x + \sqrt{x^2 + a^2} \right\}$$

$$= arsinh\left(\frac{x - 1}{3} \right) + c$$

[2 marks]

Answer 2

$$\int_0^4 \frac{52}{\sqrt{x^2 + 10x + 169}}\, dx = 52 \int_0^4 \frac{1}{\sqrt{(x + 5)^2 - 25 + 169}}\, dx$$

$$= 52 \int_0^4 \frac{1}{\sqrt{(x + 5)^2 + 144}}\, dx$$

$$= 52 \int_0^4 \frac{1}{\sqrt{(x + 5)^2 + 12^2}}\, dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right), \quad ln\left\{ x + \sqrt{x^2 + a^2} \right\}$$

$$= 52 \left[ln\left\{ x + 5 + \sqrt{(x + 5)^2 + 12^2} \right\} \right]_0^4$$

$$= 52 \left[ln\left\{ 9 + \sqrt{9^2 + 12^2} \right\} - ln\left\{ 5 + \sqrt{5^2 + 12^2} \right\}\right]$$

$$= 52 \left[ln\{ 9 + 15\} - ln\{ 5 + 13 \} \right]$$

$$= 52\, ln\left(\frac{24}{18} \right)$$

$$= 52\, ln\left(\frac{4}{3} \right)$$

[7 marks]

Answer 3

$$\int_1^3 \frac{1}{\sqrt{x^2 - 2x + 2}}\,dx = \int_1^3 \frac{1}{\sqrt{(x-1)^2 + 1^2}}\,dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right),\ ln\left\{x + \sqrt{x^2 + a^2}\right\}$$

$$= \left[\, ln\left\{x - 1 + \sqrt{(x-1)^2 + 1^2}\right\}\right]_1^3$$

$$= ln\left\{2 + \sqrt{5}\right\} - ln\left\{0 + \sqrt{1}\right\}$$

$$= ln\left(2 + \sqrt{5}\right)$$

[5 marks]

Answer 4

(i) $4x^2 - 12x - 7 = 4\left(x^2 - 3x\right) - 7$

$$= 4\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}\right) - 7$$

$$= 2^2\left(x - \frac{3}{2}\right)^2 - 9 - 7$$

$$= (2x - 3)^2 - 4^2$$

So, $a = 2$, $b = 3$ and $c = 4$

[3 marks]

(ii) $\displaystyle\int \frac{1}{\sqrt{4x^2 - 12x - 7}}\,dx = \int \frac{1}{\sqrt{(2x-3)^2 - 4^2}}\,dx$

$$= \frac{1}{2}\int \frac{2}{(2x-3)^2 - 4^2}\,dx$$

Note the need to set up a "chain rule backwards" because of the $2x$

Standard Result

$$\frac{1}{\sqrt{x^2 - a^2}} \text{ integrates to } arcosh\left(\frac{x}{a}\right),\ x > a$$

$$= \frac{1}{2}\, arcosh\left(\frac{2x - 3}{4}\right),\ x > a$$

[4 marks]

Answer 5

(i) $$16x^2 - 40x + 9 = 16\left(x^2 - \frac{5}{2}x \right) + 9$$

$$= 16\left(\left(x - \frac{5}{4} \right)^2 - \frac{25}{16} \right) + 9$$

$$= 4^2\left(x - \frac{5}{4} \right)^2 - 25 + 9$$

$$= (4x - 5)^2 - 4^2$$

So, $a = 4$, $b = 5$ and $c = 4$

[3 marks]

(ii) $$\int_{-0.25}^{0} \frac{1}{16x^2 - 40x + 9}\, dx = \int_{-0.25}^{0} \frac{1}{\sqrt{(4x - 5)^2 - 4^2}}\, dx$$

$$= \frac{1}{4}\int_{-0.25}^{0} \frac{4}{(4x - 5)^2 - 4^2}\, dx$$

Note the need to set up a "chain rule backwards" because of the $4x$

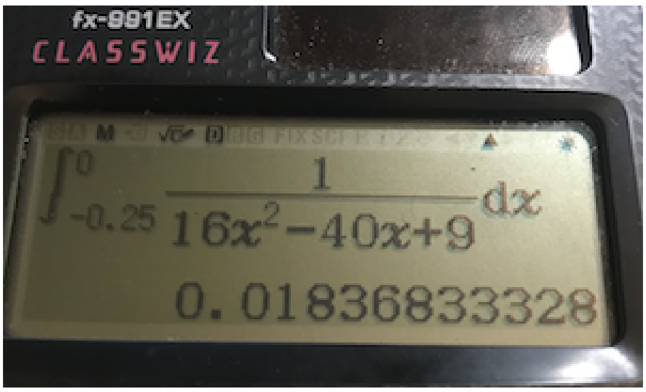
Standard Result

$$\frac{1}{x^2 - a^2} \text{ integrates to } \frac{1}{2a}\, ln\left| \frac{x - a}{x + a} \right|$$

$$= \frac{1}{4}\left[\frac{1}{8}\, ln\left| \frac{(4x - 5) - 4}{(4x - 5) + 4} \right| \right]_{-0.25}^{0}$$

$$= \frac{1}{32}\, ln\left| \frac{4x - 9}{4x - 1} \right|_{-0.25}^{0}$$

$$= \frac{1}{32}\left[ln\, 9 - ln\, 5 \right]$$



[5 marks]

Answer 6

(a) $$\int_5^8 \frac{1}{x^2 - 10x + 34}\,dx = \int_5^8 \frac{1}{(x-5)^2 - 25 + 34}\,dx$$

$$= \int_5^8 \frac{1}{(x-5)^2 + 3^2}\,dx$$

Standard Result

$$\frac{1}{a^2 + x^2} \text{ integrates to } \frac{1}{a}\,arctan\left(\frac{x}{a}\right)$$

$$= \left[\frac{1}{3}\,arctan\left(\frac{x-5}{3}\right)\right]_5^8$$

$$= \frac{1}{3}\left[\,arctan(1) - arctan(0)\,\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{12} \qquad \therefore k = \frac{1}{12}$$

[**5 marks]**

(b) $$\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,dx = \int_5^8 \frac{1}{\sqrt{(x-5)^2 + 3^2}}\,dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right),\ ln\left\{x + \sqrt{x^2 + a^2}\right\}$$

$$= \left[\,ln\left\{(x-5) + \sqrt{(x-5)^2 + 3^2}\right\}\right]_5^8$$

$$= ln\left\{3 + \sqrt{3^2 + 3^2}\right\} - ln\left\{0 + \sqrt{0^2 + 3^2}\right\}$$

$$= ln\left\{3 + 3\sqrt{2}\right\} - ln\,3$$

$$= ln\left(\frac{3 + 3\sqrt{2}}{3}\right)$$

$$= ln\left(1 + \sqrt{2}\right) \qquad \therefore A = 1,\ n = 2$$

[4 marks]

Answer 7

(i) $$\int_3^5 \frac{1}{\sqrt{15 + 2x - x^2}}\,dx = \int_3^5 \frac{1}{\sqrt{15 - (x^2 - 2x)}}\,dx$$

$$= \int_3^5 \frac{1}{\sqrt{15 - (\,(x-1)^2 - 1\,)}}\,dx$$

$$= \int_3^5 \frac{1}{\sqrt{4^2 - (x-1)^2}}\,dx$$

Standard Result

$$\frac{1}{\sqrt{a^2 - x^2}} \text{ integrates to } arcsin\left(\frac{x}{a}\right), \quad |x| < a$$

$$= \left[\, arcsin\left(\frac{x-1}{4}\right)\right]_3^5$$

$$= arcsin(1) - arcsin\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

[5 marks]

(ii) **(a)** $$\text{LHS} = 5\,cosh\,x - 4\,sinh\,x$$

$$= 5\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{e^x + 9\,e^{-x}}{2}\right) \times \frac{e^x}{e^x}$$

$$= \frac{e^{2x} + 9}{2\,e^x}$$

$$= \text{RHS} \qquad \square$$

[3 marks]

(b) $$\int \frac{1}{5\cosh x - 4\sinh x}\,dx = \int \frac{2e^x}{e^{2x} + 9}\,dx$$

$$= 2\int \frac{e^x}{(e^x)^2 + 3^2}\,dx$$

Notice that this is already set up as a “chain rule backwards” because the derivative of e^x is e^x but by all means use the suggested substitution if you find this hard to see.

Standard Result

$$\frac{1}{a^2 + x^2} \text{ integrates to } \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$= \frac{2}{3}\arctan\left(\frac{e^x}{3}\right) + c$$

[4 marks]