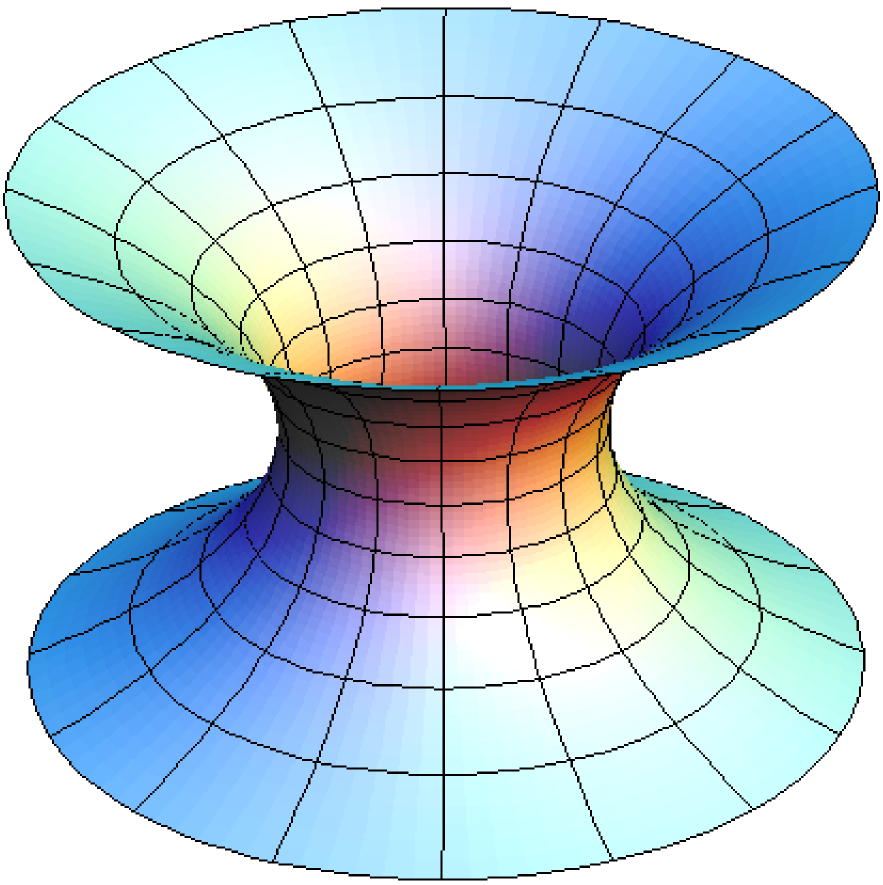
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

HYPERBOLIC FUNCTIONS



The Catenoid is an example of a minimal surface.
It occupies the least area between the two boundary circles,
one at the top and one at the bottom.
It is formed by rotation of the hyperbolic cosine curve about the y-axis.

HYPERBOLIC FUNCTIONS

Lesson 1

Further A-Level Pure Mathematics, Core 2
Hyperbolic Functions

1.1 Building With Exponentials

When a chain is hung between two fixed points the curve that naturally occurs due to gravity is called a catenary. Mathematicians call this curve "hyperbolic cosine" and as a function it has the abbreviation *cosh*

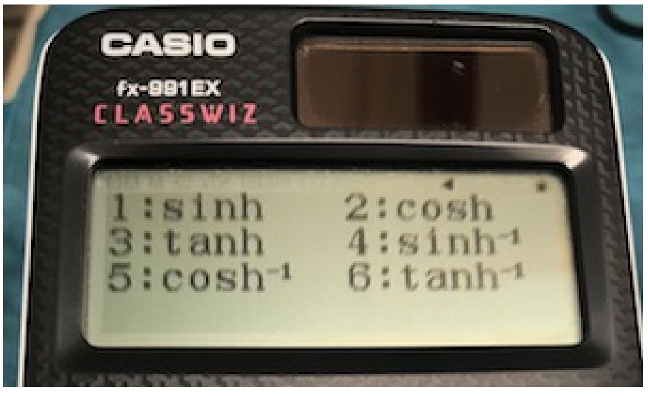
Photograph by Martin Hansen

The catenary is minimising the forces within and upon it's individual parts which is clearly going to give it some interesting features. Surprisingly, it turns out that the *cosh* function is built from a couple of exponential functions.

Definition of *cosh x*

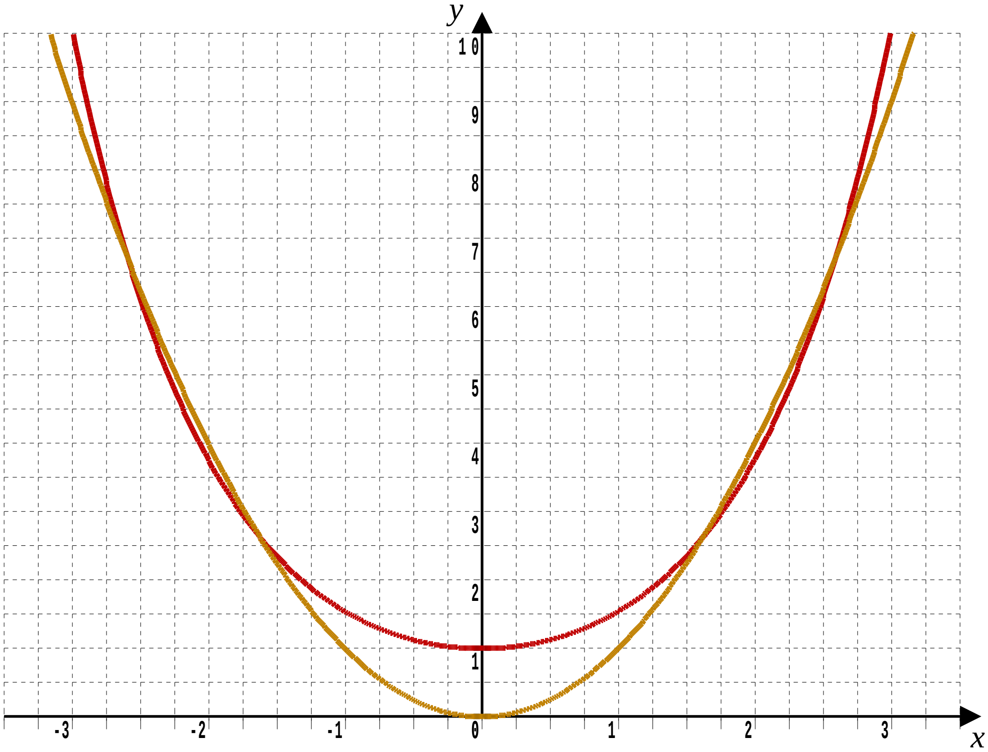
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Some calculators have buttons for the hyperbolic functions, others tuck them away in a menu. On the CASIO fx-991EX Classwiz, access them via the OPTN button.



Photograph by Martin Hansen

Although superficially similar to a parabola, the catenary is not the same curve.



In red, the catenary : $y = \cosh x$ In gold, the parabola : $y = x^2$

When inverted, a catenary arch is formed which has been used in the construction of bridges and buildings for hundreds of years. It's of interest to architects because of an ability to withstand the weight of the material from which it is constructed. Catenary arches feature in Barcelona's famous Masia Freixa, built in 1896.

Catenary arches and Parabolic arches feature in the Gaudi inspired Masia Freixa

An interior door of the Masia Freixa with a catenary arched door inside a parabolic arched frame

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Definition : The Six Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad x \in \mathbb{R}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad x \in \mathbb{R}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad x \in \mathbb{R}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad x \in \mathbb{R}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad x \in \mathbb{R},\ x \neq 0$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^{2x} + 1}{e^{2x} - 1} \qquad x \in \mathbb{R},\ x \neq 0$$

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Use your calculator to solve these equations, correct to 3 decimal places,

(i) $\cosh 2 = x$

[1 mark]

(ii) $\sinh (0.7) = 2x$

[1 mark]

(iii) $\tanh (-1) = x$

[1 mark]

(iv) $\cosh x = 4$

[1 mark]

(v) $\sinh x = 5$

[1 mark]

(vi) $\tanh 3x = 0.9$

[1 mark]

Question 2

Prove the following;

(i) That $\cosh x$ is an even function

[2 marks]

(ii) That $\sinh x$ is an odd function

[2 marks]

(iii) That $\tanh x$ is an odd function

[2 marks]

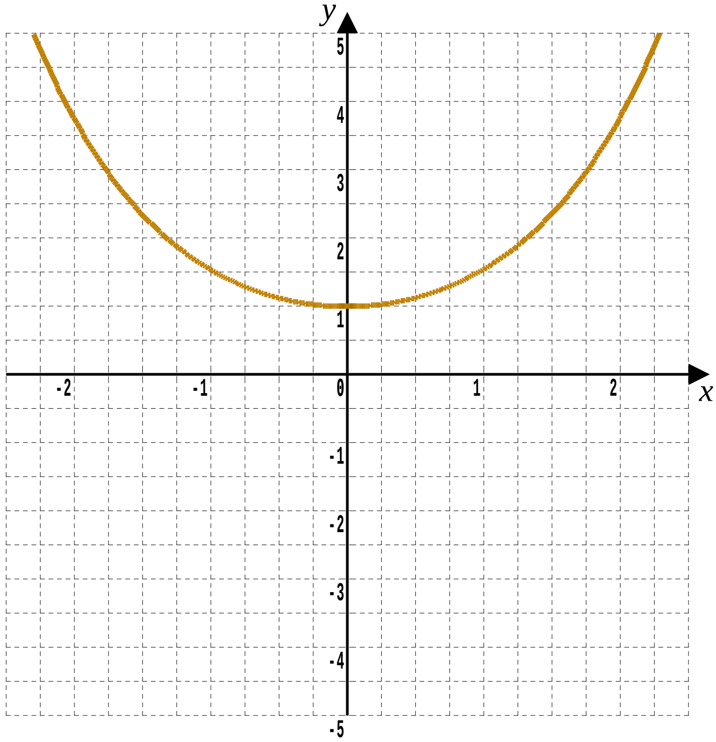
Question 3

(i) Complete the following table;

x	0	0.5	1	1.5	2	2.5
sinh x						

[3 marks]

(ii) Taking care over the different x-axis and y-axis scales, add a plot of $y = \sinh x$ to the graph below of $y = \cosh x$ using the results from the part (i) table and the fact that $y = \sinh x$ is an odd function



[3 marks]

Question 4

(i) Using the definitions of *cosh x* and *sinh x* explain why, for large positive values of x, the graph of $y = \sinh x$ is effectively the same as $y = \cosh x$

[2 marks]

(ii) Using the definitions of *cosh x* and *sinh x* explain why, for large negative values of x, the graph of $y = \sinh x$ is effectively a reflection in the x-axis of $y = \cosh x$

[3 marks]

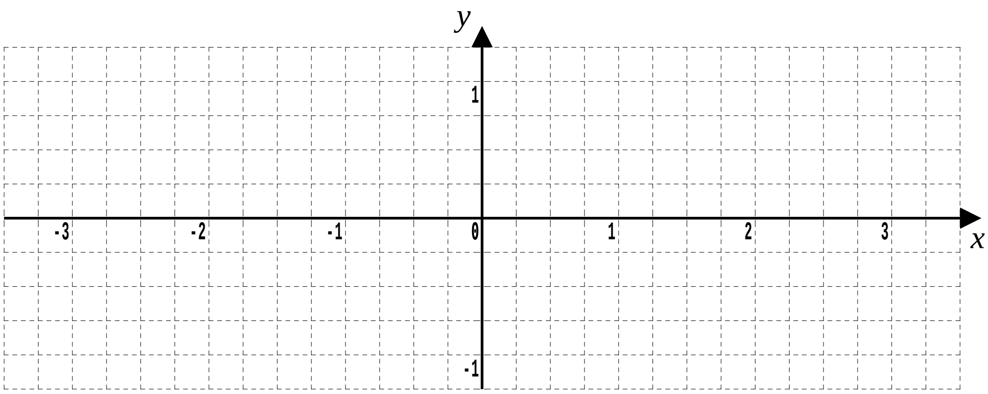
Question 5

(i) Working to two decimal places, complete the following table;

x	0	0.5	1	1.5	2	2.5	3	3.5
tanh x								

[3 marks]

(ii) Plot $y = tanh\ x$ on the graph below using the results from the part (i) table and the fact that $y = tanh\ x$ is an odd function



[2 marks]

Question 6

Using the definition of *tanh x* to explain why, for large positive values of x, the graph of $y = tanh\ x$ is effectively the same as $y = 1$

[2 marks]

1.3 Answers to 1.2 Exercise

Answers 1

(i) $cosh\ 2 = x \quad \Rightarrow \quad x = 3.762$

[1 mark]

(ii) $sinh\ (0.7) = 2x \quad \Rightarrow \quad x = 0.379$

[1 mark]

(iii) $tanh\ (-1) = x \quad \Rightarrow \quad x = -0.762$

[1 mark]

(iv) $cosh\ x = 4 \quad \Rightarrow \quad x = 2.063$

[1 mark]

(v) $sinh\ x = 5 \quad \Rightarrow \quad x = 2.312$

[1 mark]

(vi) $tanh\ 3x = 0.9 \quad \Rightarrow \quad x = 0.491$

[1 mark]

Answer 2

(i) Let $f(x) = cosh(x)$ in which case,

$$f(-x) = cosh(-x)$$
$$= \frac{e^{(-x)} + e^{-(-x)}}{2}$$
$$= \frac{e^{x} + e^{-x}}{2}$$
$$= cosh(x)$$
$$= f(x)$$ which is the definition of an even function □

[2 marks]

(ii) Let $f(x) = sinh(x)$ in which case,

$$f(-x) = sinh(-x)$$
$$= \frac{e^{(-x)} - e^{-(-x)}}{2}$$
$$= \frac{-(e^{x} - e^{-x})}{2}$$
$$= -sinh(x)$$
$$= -f(x)$$ which is the definition of an odd function □

[2 marks]

(iii) Let $f(x) = tanh(x)$ in which case,

$$f(-x) = tanh(-x)$$
$$= \frac{e^{2(-x)} - 1}{e^{2(-x)} + 1} \times \frac{e^{2x}}{e^{2x}}$$
$$= \frac{e^{0} - e^{2x}}{e^{0} + e^{2x}}$$
$$= \frac{1 - e^{2x}}{1 + e^{2x}}$$
$$= -\frac{e^{2x} - 1}{e^{2x} + 1}$$
$$= -tanh(x)$$
$= -f(x)$ which is the definition of an odd function □

[2 marks]

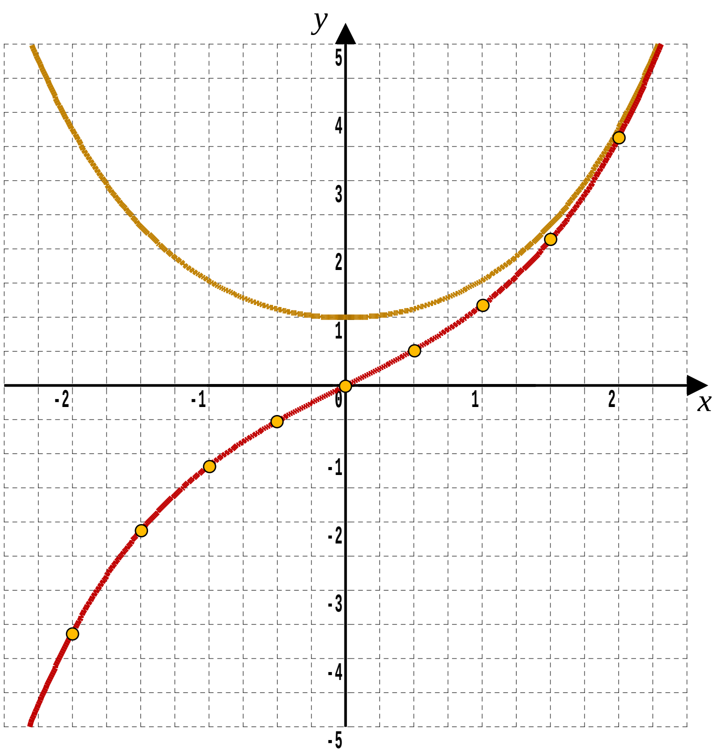
Answer 3

(i)

x	0	0.5	1	1.5	2	2.5
$sinh\ x$	0	0.52	1.18	2.13	3.63	6.05

[3 marks]

(ii)



[3 marks]

Answer 4

(i) Firstly, for *sinh x*

As x becomes large positive, e^{-x} becomes insignificant in comparison to e^{x}

Thus $\sinh x \; = \; \dfrac{e^{x} - e^{-x}}{2}$ becomes, in effect, $\dfrac{e^{x}}{2}$

Secondly, for *cosh x*

As x becomes large positive, e^{-x} becomes insignificant in comparison to e^{x}

Thus $\cosh x \; = \; \dfrac{e^{x} + e^{-x}}{2}$ becomes, in effect, $\dfrac{e^{x}}{2}$

$\therefore$ The two functions *cosh x* and *sinh x* are effectively the same for sufficiently large enough values of positive x

[2 marks]

(ii) Firstly, for *sinh x*

As x becomes large negative, e^{x} becomes insignificant in comparison to e^{-x}

Thus $\sinh x \; = \; \dfrac{e^{x} - e^{-x}}{2}$ becomes, in effect, $-\dfrac{e^{-x}}{2}$

Secondly, for *cosh x*

As x becomes large negative, e^{x} becomes insignificant in comparison to e^{-x}

Thus $\cosh x \; = \; \dfrac{e^{x} + e^{-x}}{2}$ becomes, in effect, $\dfrac{e^{-x}}{2}$

$\therefore$ Each of the functions *cosh x* and *sinh x* is effectively a reflection of the other in the x-axis for sufficiently large enough values of negative x

[3 marks]

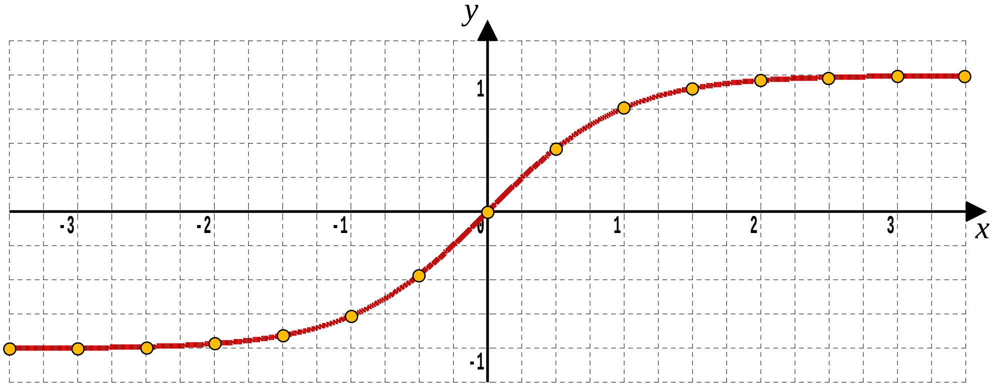
Answer 5

(i)

x	0	0.5	1	1.5	2	2.5	3	3.5
tanh x	0	0.46	0.76	0.91	0.96	0.99	1.00	1.00

[3 marks]

(ii)



[2 marks]

Answer 6

$$\lim_{x \to \infty} \tanh x = \lim_{x \to \infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \lim_{x \to \infty} \frac{e^{2x}}{e^{2x}} = \lim_{x \to \infty} 1 = 1$$

$\therefore$ For large positive values of x the graph of $y = \tanh x$ is effectively the same as the graph of $y = 1$ ☐

[2 marks]

2.1 Identities

The hyperbolic functions have identities that resonate with those already known for the trigonometric functions. For example there are hyperbolic addition formulae, hyperbolic double angle formulae and relationships between the squares of the hyperbolic functions that have echos with their trigonometric counterparts. Initially, hyperbolic identities are proven from the definitions in terms of exponentials but as a repertoire of relationships is built up many short cuts become available.

2.2 Example

Prove that $cosh^2 x - sinh^2 x = 1$

Teaching Video: http://www.NumberWonder.co.uk/v9102/2.mp4

[4 marks]

2.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Prove that $cosh^2 x + sinh^2 x = cosh\, 2x$

[4 marks]

Question 2

Prove that $2\,sinh\,x\ \ cosh\,x\ =\ sinh\,2x$

[4 marks]

Question 3

Given that $sinh\,x\ =\ \dfrac{3}{4}$ and by using the identities proven in the example and question 1 and question 2, find the exact value of,

(i) $cosh\,x$

[2 marks]

(ii) $tanh\,x$

[1 mark]

(iii) $sinh\,2x$

[1 mark]

(iv) $cosh\,2x$

[2 marks]

(v) $sinh\,4x$

[1 mark]

Question 4
Prove the following identities using the definitions of *cosh x* and *sinh x*
Top Tip : Start with the RHS and work towards the LHS

(i) $\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$

[5 marks]

(ii) $\cosh 3A = 4\cosh^3 A - 3\cosh A$

[5 marks]

(iii) $sinh\,A \,-\, sinh\,B \,=\, 2sinh\left(\frac{A\,-\,B}{2}\right)cosh\left(\frac{A\,+\,B}{2}\right)$

[5 marks]

2.4 Answers

2.4.1 Solution to 2.2 Example (Teaching Video)

Prove that $cosh^2 x - sinh^2 x = 1$

$$\begin{aligned}
\text{LHS} &= cosh^2 x - sinh^2 x \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2\,e^x\,e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2\,e^x\,e^{-x} + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} \\
&= \frac{4}{4} \\
&= 1 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

Prove that $cosh^2 x + sinh^2 x = cosh\,2x$

$$\begin{aligned}
\text{LHS} &= cosh^2 x + sinh^2 x \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2\,e^x\,e^{-x} + e^{-2x}}{4} + \frac{e^{2x} - 2\,e^x\,e^{-x} + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{2e^{2x} + 2e^{-2x}}{4} \\
&= \frac{2\left(e^{2x} + e^{-2x}\right)}{4} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= cosh\,2x \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

Prove that $2\,sinh\,x\;\;cosh\,x \;=\; sinh\,2x$

$$
\begin{aligned}
\text{LHS} &= 2\,sinh\,x\,cosh\,x \\
&= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{1}{2}\left(e^x - e^{-x}\right)\left(e^x + e^{-x}\right) \\
&= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \qquad \text{Difference of two squares} \\
&= \frac{e^{2x} - e^{-2x}}{2} \\
&= sinh\,2x \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[4 marks]

Answer 3

(i)

$$
\begin{aligned}
cosh^2 x - sinh^2 x &= 1 \\
cosh^2 x - \left(\frac{3}{4}\right)^2 &= 1 \\
cosh^2 x &= 1 + \frac{9}{16} \\
&= \frac{25}{16} \\
cosh\,x &= \frac{5}{4} \qquad \left(cosh\,x \geqslant 1 \;\; \text{so rejected} \; -\frac{5}{4}\right)
\end{aligned}
$$

[2 mark]

(ii)

$$
\begin{aligned}
tanh\,x &= \frac{sinh\,x}{cosh\,x} \\
&= \left(\frac{3}{4}\right) \div \left(\frac{5}{4}\right) \\
&= \left(\frac{3}{4}\right) \times \left(\frac{4}{5}\right) \qquad KOF \\
&= \frac{3}{5}
\end{aligned}
$$

[1 mark]

(iii)

$$
\begin{aligned}
sinh\,2x &= 2\,sinh\,x\,cosh\,x \\
&= 2\left(\frac{3}{4}\right)\left(\frac{5}{4}\right) \\
&= \frac{15}{8}
\end{aligned}
$$

[1 mark]

(iv) $$cosh^2 2x - sinh^2 2x = 1$$

$$cosh^2 2x - \left(\frac{15}{8}\right)^2 = 1$$

$$cosh^2 2x = 1 + \frac{225}{64}$$

$$= \frac{289}{64}$$

$$cosh\, 2x = \frac{17}{8} \qquad \left(cosh\, x \geqslant 1 \text{ so rejected } - \frac{17}{8} \right)$$

[2 marks]

Alternatively: $cosh\,(2x) = cosh^2 x + sinh^2 x$

$$= \frac{25}{16} + \frac{9}{16}$$

$$= \frac{17}{8}$$

[2 marks]

(v) $$sinh\, 4x = 2\, sinh\, 2x\, cosh\, 2x$$

$$= 2\left(\frac{15}{8}\right)\left(\frac{17}{8}\right)$$

$$= \frac{255}{32}$$

[1 mark]

Answer 4

(i) $cosh\,(A + B) = cosh\, A\, cosh\, B + sinh\, A\, sinh\, B$

RHS $= cosh\, A\, cosh\, B + sinh\, A\, sinh\, B$

$$= \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) + \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$$

$$= \frac{e^{A+B} + e^{A-B} + e^{-A+B} + e^{-A-B} + e^{A+B} - e^{A-B} - e^{-A+B} + e^{-A-B}}{4}$$

$$= \frac{2\, e^{A+B} + 2\, e^{-A-B}}{4}$$

$$= \frac{2\left(e^{A+B} + e^{-(A+B)}\right)}{4}$$

$$= \frac{e^{A+B} + e^{-(A+B)}}{2}$$

$$= cosh\,(A + B)$$

$= $ LHS $\square$

[5 marks]

(ii) $\cosh 3A = 4\cosh^3 A - 3\cosh A$

$$\begin{aligned}
\text{RHS} &= 4\cosh^3 A - 3\cosh A \\
&= 4\left(\frac{e^A + e^{-A}}{2}\right)^3 - 3\left(\frac{e^A + e^{-A}}{2}\right) \\
&= \frac{1}{2}\left(e^A + e^{-A}\right)^3 - \frac{3}{2}\left(e^A + e^{-A}\right) \\
&= \frac{1}{2}\left(e^{3A} + 3e^{2A}e^{-A} + 3e^{A}e^{-2A} + e^{-3A}\right) - \frac{3}{2}e^A - \frac{3}{2}e^{-A} \\
&= \frac{1}{2}e^{3A} + \frac{3}{2}e^{A} + \frac{3}{2}e^{-A} + \frac{1}{2}e^{-3A} - \frac{3}{2}e^{A} - \frac{3}{2}e^{-A} \\
&= \frac{1}{2}e^{3A} + \frac{1}{2}e^{-3A} \\
&= \frac{e^{3A} + e^{-3A}}{2} \\
&= \cosh 3A \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[5 marks]

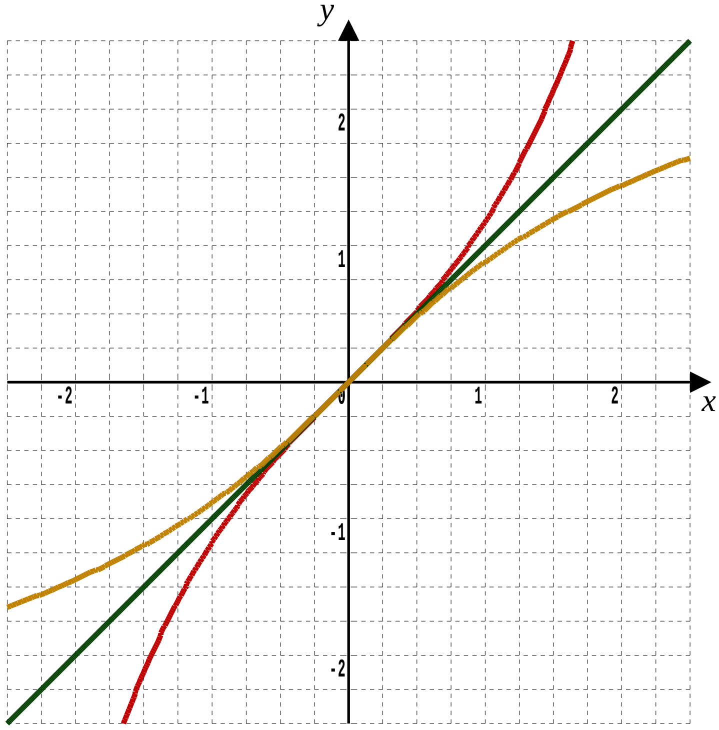
(iii) $\sinh A - \sinh B = 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$

$$\begin{aligned}
\text{RHS} &= 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right) \\
&= 2 \times \frac{1}{2}\left(e^{\frac{A-B}{2}} - e^{-\frac{A-B}{2}}\right) \times \frac{1}{2}\left(e^{\frac{A+B}{2}} + e^{-\frac{A+B}{2}}\right) \\
&= \frac{1}{2}\left(e^{\frac{A-B}{2}+\frac{A+B}{2}} + e^{\frac{A-B}{2}-\frac{A+B}{2}} - e^{-\frac{A-B}{2}+\frac{A+B}{2}} - e^{-\frac{A-B}{2}-\frac{A+B}{2}}\right) \\
&= \frac{1}{2}\left(e^A + e^{-B} - e^{B} - e^{-A}\right) \\
&= \frac{e^A - e^{-A} - e^B + e^{-B}}{2} \\
&= \frac{e^A - e^{-A}}{2} - \frac{e^B - e^{-B}}{2} \\
&= \sinh A - \sinh B \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[5 marks]

3.1 *arsinh x*

Any one-one function, such as *sinh x*, has an inverse that, graphically, is a reflection in the line $y = x$. The inverse of *sinh x* is called *arsinh x*.



Red : $y = \sinh x$ Green : $y = x$ Gold : $y = \operatorname{arsinh} x$

Given that *sinh x* is defined in terms of exponentials, the expectation would be that the inverse function, *arsinh x*, involves logarithms and this is the case.

The Inverse Of *sinh x* : *arsinh x*

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

A proof of this is written out on the next page which the following eight minute excellent video from *Exam Solutions* will talk through.

Teaching Video: http://www.NumberWonder.co.uk/v9102/3.mp4

3.2 The Proof

$$
\begin{aligned}
y &= arsinh\,x \\
\therefore\ x &= sinh\,y \\
&= \frac{e^{y} - e^{-y}}{2} && \text{From the definition of } sinh \\
2x &= e^{y} - e^{-y} \\
2x\,e^{y} &= (e^{y})^{2} - 1 && \text{From multiplying through by } e^{y} \\
(e^{y})^{2} - 2x(e^{y}) - 1 &= 0 && \text{Which is a “quadratic in disguise”} \\
e^{y} &= \frac{2x \pm \sqrt{(-2x)^{2} - 4(1)(-1)}}{2(1)} \\
&= \frac{2x \pm \sqrt{4x^{2} + 4}}{2} \\
&= \frac{2x \pm \sqrt{4}\sqrt{x^{2} + 1}}{2} \\
&= x \pm \sqrt{x^{2} + 1} \\
\text{Now, } e^{y} &> 0 \text{ since } \sqrt{x^{2} + 1} > x \\
\therefore\ e^{y} &= x + \sqrt{x^{2} + 1} \\
y &= ln\left(x + \sqrt{x^{2} + 1}\right) \\
\text{That is, } arsinh\,x &= ln\left(x + \sqrt{x^{2} + 1}\right) && \square
\end{aligned}
$$

3.3 Example

Determine the exact value of

(i) $arsinh(2)$ **(ii)** $arsinh(-3)$

[2 marks]

Solution :

(i) $arsinh(2) = ln\left(2 + \sqrt{5}\right)$ About 1.44

(ii) $arsinh(-3) = ln\left(-3 + \sqrt{10}\right)$ About -1.81

This example is also covered in the teaching video.

3.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Express each of the following as a natural logarithm,

(i) $arsinh(1)$

[1 mark]

(ii) $arsinh(\sqrt{3})$

[1 mark]

(iii) $arsinh(2\sqrt{2})$

[2 marks]

Question 2

Find the exact value of $arsinh\left(\frac{3}{4}\right)$ in as simple a form as possible

[2 marks]

Question 3

Find the exact value of $arsinh\left(\frac{1}{\sqrt{3}}\right)$ in as simple a form as possible

[2 marks]

Question 4

With the aid of the identity $cosh^2 x - sinh^2 x = 1$ solve the following equation, giving exact answers as natural logarithms.

$$2\,cosh^2 x - 5\,sinh\,x = 5$$

[6 marks]

Question 5

Further A-Level Examination Question from June 2015, FP3, Q1 (Edexcel)

Solve the equation

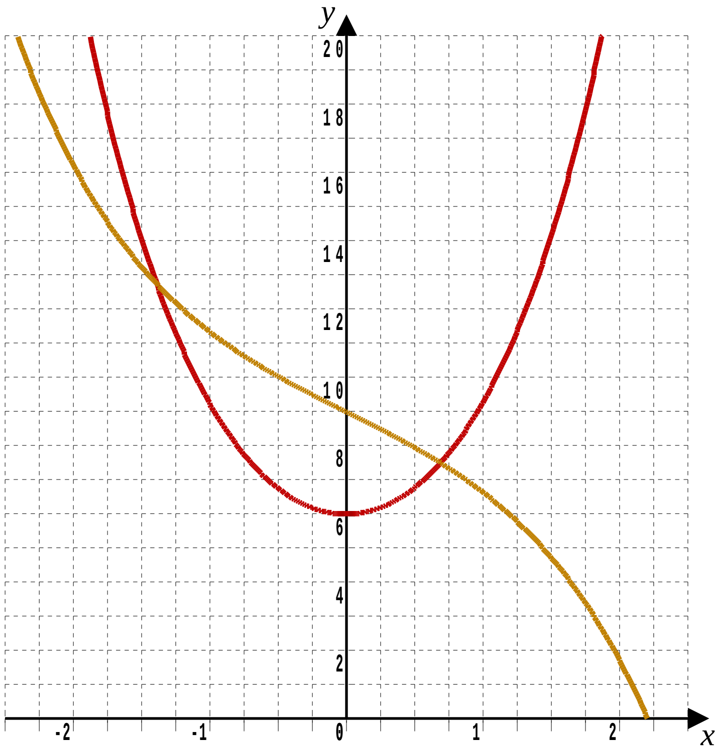
$$2\,cosh^2 x - 3\,sinh\,x = 1$$

giving your answers in terms of natural logarithms

[6 marks]

Question 6

Further A-Level Examination Question from June 2013, FP3(R), Q7(a) (Edexcel)



The curves have equations, $y = 6\cosh x$ and $y = 9 - 2\sinh x$

Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect.

[6 marks]

Question 7

Further A-Level Examination Question from June 2002, P6, Q1 (Edexcel)

Prove that $sinh\left(i\pi\ -\ \theta\right)\ =\ sinh\,\theta$

[4 marks]

3.5 Answers to 3.4 Exercise

Answer 1

(i) $arsinh(1) = ln(1 + \sqrt{2})$

[1 mark]

(ii) $arsinh(\sqrt{3}) = ln(\sqrt{3} + 2)$

[1 mark]

(iii) $arsinh(2\sqrt{2}) = ln(2\sqrt{2} + 3)$

[2 marks]

Answer 2

$$\begin{aligned} arsinh\left(\frac{3}{4}\right) &= ln\left(\frac{3}{4} + \sqrt{\left(\frac{3}{4}\right)^2 + 1}\right) \\ &= ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + \frac{16}{16}}\right) \\ &= ln\left(\frac{3}{4} + \sqrt{\frac{25}{16}}\right) \\ &= ln\left(\frac{3}{4} + \frac{5}{4}\right) \\ &= ln\,2 \end{aligned}$$

[2 marks]

Answer 3

$$\begin{aligned} arsinh\left(\frac{1}{\sqrt{3}}\right) &= ln\left(\frac{1}{\sqrt{3}} + \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1}\right) \\ &= ln\left(\frac{1}{\sqrt{3}} + \sqrt{\frac{1}{3} + \frac{3}{3}}\right) \\ &= ln\left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right) \\ &= ln\left(\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right) \\ &= ln\sqrt{3} \quad \text{or} \quad \frac{1}{2}\,ln\,3 \end{aligned}$$

[2 marks]

Answer 4

$$2\cosh^2 x - 5\sinh x = 5$$

$$2\left(1 + \sinh^2 x\right) - 5\sinh x = 5$$

$$2\sinh^2 x - 5\sinh x - 3 = 0$$

$$\left(2\sinh x + 1\right)\left(\sinh x - 3\right) = 0$$

Either $\sinh x = -\dfrac{1}{2}$ or $\sinh x = 3$

From which is deduced that

$$x = arsinh\left(-\frac{1}{2}\right) \qquad \text{or} \quad x = arsinh(3)$$

$$x = ln\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4}{4}}\right) \qquad x = ln\left(3 + \sqrt{10}\right)$$

$$= ln\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$$

[6 marks]

Answer 5

$$2\cosh^2 x - 3\sinh x = 1$$

$$2\left(1 + \sinh^2 x\right) - 3\sinh x = 1$$

$$2\sinh^2 x - 3\sinh x + 1 = 0$$

$$\left(2\sinh x - 1\right)\left(\sinh x - 1\right) = 0$$

Either $\sinh x = \dfrac{1}{2}$ or $\sinh x = 1$

From which is deduced that

$$x = arsinh\left(\frac{1}{2}\right) \qquad \text{or} \quad x = arsinh(1)$$

$$x = ln\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4}{4}}\right) \qquad x = ln\left(1 + \sqrt{2}\right)$$

$$= ln\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$$

[6 marks]

Answer 6

The graphs will intersect when,

$$6\cosh x = 9 - 2\sinh x$$

$$6\left(\frac{e^x + e^{-x}}{2}\right) = 9 - 2\left(\frac{e^x - e^{-x}}{2}\right)$$

$$3e^x + 3e^{-x} = 9 - e^x + e^{-x}$$

$$4e^x - 9 + 2e^{-x} = 0$$

Multiply through by e^x

$$4(e^x)^2 - 9e^x + 2 = 0$$

$$(4e^x - 1)(e^x - 2) = 0$$

Either $\quad e^x = \frac{1}{4} \quad$ or $\quad e^x = 2$

$$x = ln\left(\frac{1}{4}\right) \qquad x = ln(2)$$

$$= ln\, 2^{-2}$$

$$= -2\, ln\, 2$$

[6 marks]

Answer 7

Prove that $sinh(i\pi - \theta) = sinh\,\theta$

$$\text{LHS} = sinh(i\pi - \theta)$$

$$= \frac{1}{2}\left(e^{i\pi - \theta} - e^{-i\pi + \theta}\right)$$

$$= \frac{1}{2}\left(e^{i\pi} e^{-\theta} - e^{-i\pi} e^{\theta}\right)$$

$$= \frac{1}{2}\left((-1)\, e^{-\theta} - (-1)\, e^{\theta}\right)$$

$$= \frac{1}{2}\left(e^{\theta} - e^{-\theta}\right)$$

$$= sinh\,\theta$$

$$= \text{RHS} \qquad \square$$

[4 marks]

4.1 Osborn's Rule

There are obvious similarities between identities involving the circular trigonometric functions (*cos, sin, tan, sec, csc* and *cot*) and the hyperbolic functions (*cosh, sinh, tanh, sech, csch, coth*).

Previously† the following where proven,

Hyperbolic	Circular
$\cosh^2 x - \sinh^2 x = 1$	$\cos^2 x + \sin^2 x = 1$
$\cosh^2 x + \sinh^2 x = \cosh 2x$	$\cos^2 x - \sin^2 x = \cos 2x$
$2 \sinh x \cosh x = \sinh 2x$	$2 \sin x \cos x = \sin 2x$
$\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$	$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\cosh 3A = 4\cosh^3 A - 3\cosh A$	$\cos 3A = 4\cos^3 A - 3\cos A$
$\sinh A - \sinh B = 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$	$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

Given that the circular trigonometric identities are already known, it would be ideal if the hyperbolic identities could be obtained from them. In some of the comparisons in the above table the hyperbolic identity is identical when *cos* is replaced with *cosh* and *sin* replaced with *sinh*. However, in other comparisons when the same swap is made, the odd sign needs changing. Exactly when a sign needs changing is given by Osborn's Rule. It allows any circular trigonometric identity to be taken and the corresponding hyperbolic identity easily written down.

Osborn's Rule

- Replace *cos* by *cosh* : $\cos A \to \cosh A$
- Replace *sin* by *sinh* : $\sin A \to \sinh A$

  However ...
- Replace any product (or implied product) of two *sin* terms by minus the product of two *sinh* terms

  e.g. $\sin A \sin B \to -\sinh A \sinh B$

  $$\tan^2 A \to -\tanh^2 A \qquad \left(\text{an implied product as } \tan^2 A = \frac{\sin^2 A}{\cos^2 A}\right)$$

---

† In Lesson 2, Example 2.2 and Exercise 2.3

4.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

(i) Use Osborn's Rule to write down the hyperbolic identity corresponding to,

$$1 + tan^2 x = sec^2 x$$

[2 marks]

(ii) Prove that your part (i) answer is correct using the facts that,

$$tanh\, x = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \text{and} \quad sech\, x = \frac{2}{e^x + e^{-x}}$$

[6 marks]

Question 2

Use Osborn's Rule to write down the hyperbolic identity corresponding to,

$$cos\, 2x = 1 - 2\, sin^2 x$$

[2 marks]

Question 3

Further A-Level Examination Question from June 2010, FP3, Q3 (Edexcel)

(a) Starting from the definitions of *sinh x* and *cosh x* in terms of exponentials, prove that,

$$\cosh 2x = 1 + 2\sinh^2 x$$

[3 marks]

(b) Solve the equation

$$\cosh 2x - 3\sinh x = 15$$

giving your answer as exact logarithms

[5 marks]

Question 4

Use Osborn's Rule to write down the hyperbolic identity corresponding to,

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

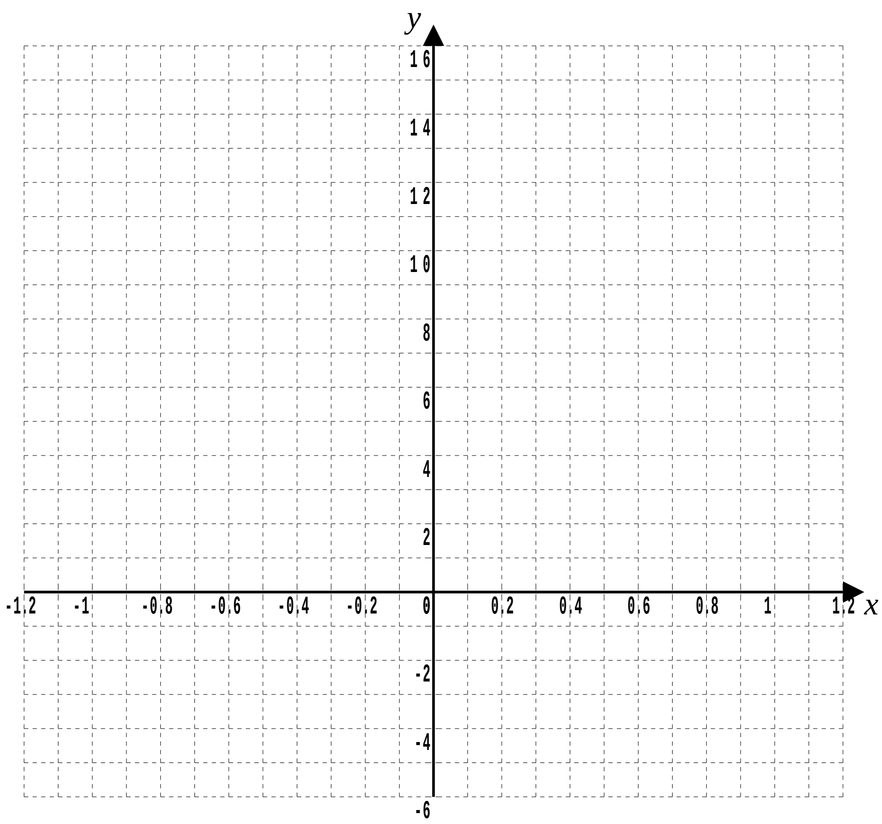
[3 marks]

Question 5

Further A-Level Examination Question from June 2011, FP3, Q5 (Edexcel)

Curve C_1 has equation $y = 3\sinh 2x$, and curve C_2 has equation $y = 13 - 3e^{2x}$

(a) Sketch the graphs of the curves C_1 and C_2 on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes



[4 marks]

(b) Solve the equation $3\,sinh\,2x = 13 - 3\,e^{2x}$ giving your answer in the form $\frac{1}{2}\,ln\,k$, where k is an integer

[5 marks]

4.3 Answers to 4.2 Exercise

Answer 1

(i) $1 - tanh^2 x = sech^2 x$

[2 marks]

(ii)

$$\begin{aligned} \text{RHS} &= sech^2 x \\ &= \left(\frac{2}{e^x + e^{-x}} \right)^2 \\ &= \left(\frac{2}{e^x + e^{-x}} \times \frac{e^x}{e^x} \right)^2 \\ &= \left(\frac{2\, e^x}{e^{2x} + 1} \right)^2 \\ &= \frac{4\, e^{2x}}{(e^{2x} + 1)^2} \end{aligned}$$

$$\begin{aligned} \text{LHS} &= 1 - tanh^2 x \\ &= 1 - \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right)^2 \\ &= \left(\frac{e^{2x} + 1}{e^{2x} + 1} \right)^2 - \left(\frac{e^{2x} - 1}{e^{2x} + 1} \right)^2 \\ &= \frac{(e^{2x} + 1)^2 - (e^{2x} - 1)^2}{(e^{2x} + 1)^2} \qquad \text{Difference of two squares} \\ &= \frac{(e^{2x} + 1 + e^{2x} - 1)(e^{2x} + 1 - e^{2x} + 1)}{(e^{2x} + 1)^2} \\ &= \frac{4\, e^{2x}}{(e^{2x} + 1)^2} \\ &= \text{RHS} \qquad \square \end{aligned}$$

[6 marks]

Answer 2

$$cosh\, 2x = 1 + 2\, sinh^2 x$$

[2 marks]

Answer 3

(a)

$$\begin{aligned} \text{RHS} &= 1 + 2\,sinh^2 x \\ &= 1 + 2\left(\frac{e^x - e^{-x}}{2} \right)^2 \\ &= \frac{2}{2} + \frac{e^{2x} - 2 + e^{-2x}}{2} \\ &= \frac{e^{2x} + e^{-2x}}{2} \\ &= cosh\,2x \\ &= \text{LHS} \qquad \square \end{aligned}$$

[3 marks]

(b)

$$cosh\,2x - 3\,sinh\,x = 15$$

$$1 + 2\,sinh^2 x - 3\,sinh\,x = 15$$

$$2\,sinh^2 x - 3\,sinh\,x - 14 = 0$$

$$(2\,sinh\,x - 7)(sinh\,x + 2) = 0$$

$$\text{Either} \quad sinh\,x = \frac{7}{2} \quad \text{or} \quad sinh\,x = -2$$

$$x = ln\left(\frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 + 1} \right) \qquad \text{or} \qquad x = ln\left(-2 + \sqrt{5}\right)$$

$$= ln\left(\frac{7}{2} + \frac{\sqrt{53}}{2} \right)$$

[5 marks]

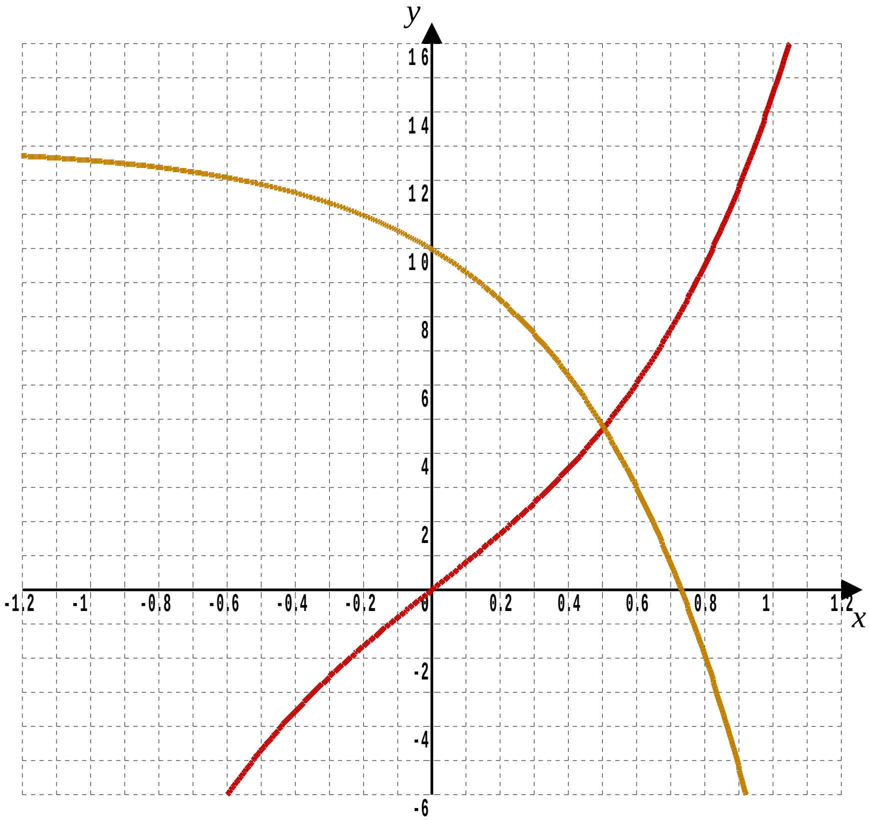
Answer 4

$$tanh\,(A - B) = \frac{tanh\,A - tanh\,B}{1 - tanh\,A\,tanh\,B}$$

[2 marks]

Answer 5

(a)



C_1 : Red curve $y = 3\,sinh\,2x$ passes through $(0,\ 0)$

C_2 : Gold curve $y = 13 - 3\,e^{2x}$: Asymptote $y = 13$

crosses y-axis $(0,\ 10)$, and crosses x-axis $\left(ln\left(\sqrt{\frac{13}{3}}\right),\ 0\right)$

[4 marks]

(b)

$$3\,sinh\,2x = 13 - 3\,e^{2x}$$

$$\frac{3}{2}\left(e^{2x} - e^{-2x} \right) = 13 - 3\,e^{2x}$$

$$3\,e^{2x} - 3\,e^{-2x} = 26 - 6\,e^{2x}$$

$$9\,e^{2x} - 26 - 3\,e^{-2x} = 0$$

Multiply through by e^{2x}

$$9\left(e^{2x} \right)^2 - 26\,e^{2x} - 3 = 0$$

$$\left(9\,e^{2x} + 1 \right)\left(e^{2x} - 3 \right) = 0$$

Either $e^{2x} = -\frac{1}{9}$ or $e^{2x} = 3$

No solutions $2x = ln\,3$

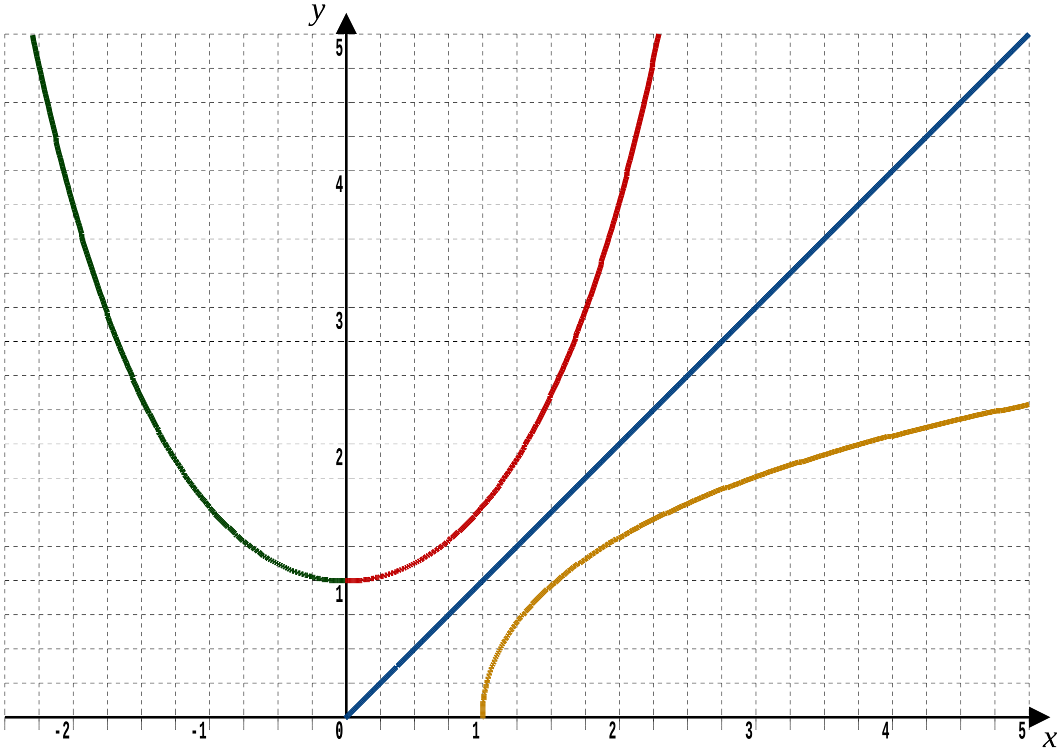
$$x = \frac{1}{2}\,ln\,3$$

That is, $k = 3$

[5 marks]

5.1 *arcosh x*

Before the inverse of the *cosh* function can be found, the domain must be restricted to obtain a one-to-one function. On the graph below the curve $y = cosh\ x$ is coloured green and red, the green part being excluded by the restriction on the domain $x \geqslant 0$. The red part is then reflected in the line $y = x$ to obtain $y = arcosh\ x$, coloured gold.



The Inverse Of *cosh x* : *arcosh x*

$$arcosh\, x = ln\left(x + \sqrt{x^2 - 1} \right) \qquad x \in \mathbb{R},\ x \geqslant 1$$

A proof is on the next page and talked through in the teaching video from *Exam Solutions*. It's similar to the *arsinh* proof but with a few extra technicalities.

Teaching Video: http://www.NumberWonder.co.uk/v9102/5.mp4

5.2 The Proof

$$y = arcosh\, x$$

$$\therefore x = cosh\, y$$

$$= \frac{e^y + e^{-y}}{2} \qquad \text{From the definition of } cosh$$

$$2x = e^y + e^{-y}$$

$$2x\, e^y = \left(e^y \right)^2 + 1 \qquad \text{From multiplying through by } e^y$$

$$\left(e^y \right)^2 - 2x \left(e^y \right) + 1 = 0 \qquad \text{Which is a “quadratic in disguise”}$$

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$= \frac{2x \pm \sqrt{4}\sqrt{x^2 - 1}}{2}$$

$$= x \pm \sqrt{x^2 - 1}$$

$$y = ln\left(x \pm \sqrt{x^2 - 1} \right)$$

Now, as the *arcosh* function is one-to-one only one of these two possibilities can apply. The one to select, corresponding to the gold curve that in turn came from the red half of the *cosh* curve is the following,

$$arcosh\, x = ln\left(x + \sqrt{x^2 - 1} \right) \qquad x \in \mathbb{R},\ x \geqslant 1 \qquad \square$$

(The other solution would correspond to the inverse of the green half of the *cosh* curve, and would be a reflection of the gold curve in the x-axis)

5.3 Example

(i) Determine the exact value of $arcosh(4)$

(ii) Solve, to three decimal places, $cosh\, w = 2$

[3 marks]

Solution :

(i) $arcosh(4) = ln\left(4 + \sqrt{15} \right)$ About 2.06

(ii) $w = arcosh(2)$ i.e. $x = 2$

$$= ln\left(x \pm \sqrt{x^2 - 1} \right)$$

$$= ln\left(2 \pm \sqrt{3} \right)$$

$$= \pm 1.316$$

This example is also covered in the *Exam Solutions* teaching video.

5.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Express each of the following as a natural logarithm,

(i) $arcosh\left(3\right)$

[1 mark]

(ii) $arcosh\left(\sqrt{3}\right)$

[1 mark]

(iii) $arcosh\left(3\sqrt{2}\right)$

[2 marks]

Question 2

Find the exact value of $arcosh\left(\frac{13}{5}\right)$ in as simple a form as possible

[3 marks]

Question 3

Solve the equation $cosh\, w = 12$ giving both solutions to 3 decimal places

[3 marks]

Question 4

Further A-Level Examination Question from June 2017, FP3, Q3 (Edexcel)

(a) Using the definitions for *cosh x* in terms of exponentials, show that

$$\cosh 2x = 2\cosh^2 x - 1$$

[3 marks]

(b) Find the exact values of x for which

$$29\cosh x - 3\cosh 2x = 38$$

giving your answers in terms of natural logarithms

[6 marks]

Question 5

Further A-Level Examination Question from June 2008, FP2, Q2 (Edexcel)

Find the values of x for which

$$8\, cosh\, x \;-\; 4\, sinh\, x \;=\; 13$$

giving your answers as natural logarithms.

[6 marks]

Question 6

Further A-Level Examination Question from June 2004, P5. Q1(b) (Edexcel)

Solve,

$$csch\, x \;-\; 2\, coth\, x \;=\; 2$$

giving your answer in the form $k \ln a$ where k and a are integers

[5 marks]

5.5 Answers to 5.4 Exercise

Answer 1

(i) $arcosh(3) = ln(3 + 2\sqrt{2})$

[1 mark]

(ii) $arcosh(\sqrt{3}) = ln(\sqrt{3} + \sqrt{2})$

[1 mark]

(iii) $arcosh(3\sqrt{2}) = ln(3\sqrt{2} + \sqrt{17})$

[2 marks]

Answer 2

$$\begin{aligned}
arcosh\left(\frac{13}{5}\right) &= ln\left(\frac{13}{5} + \sqrt{\left(\frac{13}{5}\right)^2 - 1}\right) \\
&= ln\left(\frac{13}{5} + \sqrt{\frac{169}{25} - \frac{25}{25}}\right) \\
&= ln\left(\frac{13}{5} + \sqrt{\frac{144}{25}}\right) \\
&= ln\left(\frac{13}{5} + \frac{12}{5}\right) \\
&= ln\left(\frac{25}{5}\right) \\
&= ln\,5
\end{aligned}$$

[3 marks]

Answer 3

$$\begin{aligned}
cosh\,w &= 12 \\
w &= arcosh(12) \qquad \text{i.e. } x = 12 \\
&= ln\left(x \pm \sqrt{x^2 - 1}\right) \\
&= ln\left(12 \pm \sqrt{143}\right) \\
&= \pm 3.176
\end{aligned}$$

[3 marks]

Answer 4

(a) Prove $cosh\, 2x = 2\, cosh^2 x - 1$

$$\begin{aligned}
\text{RHS} &= 2\, cosh^2 x - 1 \\
&= 2\left(\frac{e^x + e^{-x}}{2} \right)^2 - 1 \\
&= \frac{e^{2x} + 2\, e^x e^{-x} + e^{-2x}}{2} - \frac{2}{2} \\
&= \frac{e^{2x} + 2 + e^{-2x} - 2}{2} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= cosh\, 2x \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[3 marks]

(b)

$$\begin{aligned}
29\, cosh\, x - 3\, cosh\, 2x &= 38 \\
29\, cosh\, x - 3\left(2\, cosh^2 x - 1 \right) &= 38 \\
29\, cosh\, x - 6\, cosh^2 x + 3 &= 38 \\
6\, cosh^2 x - 29\, cosh\, x + 35 &= 0 \\
\left(3\, cosh\, x - 7\right)\left(2\, cosh\, x - 5 \right) &= 0
\end{aligned}$$

Either $cosh\, x = \dfrac{7}{3}$ or $cosh\, x = \dfrac{5}{2}$

$$x = ln\left(\frac{7}{3} \pm \sqrt{\frac{49}{9} - \frac{9}{9}} \right) \qquad x = ln\left(\frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{4}{4}} \right)$$

$$= ln\left(\frac{7}{3} \pm \frac{2\sqrt{10}}{3} \right) \qquad = ln\left(\frac{5}{2} \pm \frac{\sqrt{21}}{2} \right)$$

[6 marks]

Answer 5

$$8\,cosh\,x - 4\,sinh\,x = 13$$

$$8\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^x - e^{-x}}{2}\right) = 13$$

$$4\left(e^x + e^{-x}\right) - 2\left(e^x - e^{-x}\right) = 13$$

$$4\,e^x + 4\,e^{-x} - 2\,e^x + 2\,e^{-x} = 13$$

$$2\,e^x - 13 + 6\,e^{-x} = 0$$

Multiply through by e^x

$$2\,\left(e^x\right)^2 - 13\left(e^x\right) + 6 = 0$$

$$\left(2\,e^x - 1\right)\left(e^x - 6\right) = 0$$

Either $\quad e^x = \frac{1}{2} \quad$ or $\quad e^x = 6$

$$x = ln\left(\frac{1}{2}\right) \quad \text{or} \quad x = ln\,(6)$$

[6 marks]

Answer 6

$$csch\,x - 2\,coth\,x = 2$$

$$\frac{1}{sinh\,x} - \frac{2\,cosh\,x}{sinh\,x} = 2$$

Multiply through by $sinh\,x$

$$1 - 2\,cosh\,x = 2\,sinh\,x$$

$$2\,sinh\,x + 2\,cosh\,x - 1 = 0$$

$$2\left(\frac{e^x - e^{-x}}{2}\right) + 2\left(\frac{e^x + e^{-x}}{2}\right) - 1 = 0$$

$$2\,e^x - 1 = 0$$

$$e^x = \frac{1}{2}$$

$$x = ln\left(\frac{1}{2}\right)$$

[5 marks]

6.1 Differentiation

The hyperbolic functions and their inverse functions are differentiable. As no new ideas are involved this provides a welcome opportunity to refresh the various differentiation techniques of A-Level and Further A-Level mathematics.

6.2 Table of standard derivatives (Hyperbolic Functions)

$f(x)$	$f'(x)$	In Formula Book ?
$\sinh x$	$\cosh x$	Yes
$\cosh x$	$\sinh x$	Yes
$\tanh x$	$\operatorname{sech}^2 x$	Yes
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$	Yes
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}} \quad x > 1$	Yes
$\operatorname{artanh} x$	$\frac{1}{1-x^2} \quad \lvert x \rvert < 1$	Yes

6.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Differentiate with respect to x,

(i) $y = \sinh(5x)$

[1 mark]

(ii) $y = \tanh\left(\frac{x}{3}\right)$

[1 mark]

(iii) $y = \operatorname{arcosh}(2x)$

[1 mark]

(iv) $y = \operatorname{arsinh}\left(\frac{x}{3}\right)$

[2 mark]

Question 2

If $y = a\cosh(nx) + b\sinh(nx)$ where *a* and *b* are constants, prove that,

$$\frac{d^2y}{dx^2} = n^2 y$$

[4 marks]

Question 3

Given that $y = (\operatorname{arcosh} x)^2$ prove that

$$\left(x^2 - 1\right)\left(\frac{dy}{dx}\right)^2 = 4y$$

[5 marks]

Question 4
Differentiate with respect to *x*,

(i) $y = \sinh(2x)\cosh(3x)$

[2 marks]

(ii) $y = \dfrac{\cosh x}{4x}$

[2 marks]

(iii) $y = x^2 \operatorname{arcosh} x$

[2 marks]

Question 5
Further A-Level Examination Question from June 2012, FP3, Q5(a) (Edexcel)
Differentiate $y = x \operatorname{arsinh}(2x)$ with respect to x

[3 marks]

Question 6

Differentiate with respect to x,

$$y = sech(2x)$$

[3 marks]

Question 7

Further A-Level Examination Question from June 2014, FP3, Q5 (Edexcel)

Given that $y = artanh\left(\frac{x}{\sqrt{1 + x^2}}\right)$ show that $\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$

[4 marks]

6.4 Answers to 6.3 Exercise

Answer 1

(i) $y = \sinh(5x) \Rightarrow \dfrac{dy}{dx} = 5\cosh(5x)$

[1 mark]

(ii) $y = \tanh\left(\dfrac{x}{3}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3}\,\operatorname{sech}^2\left(\dfrac{x}{3}\right)$

[1 mark]

(iii) $y = \operatorname{arcosh}(2x) \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{\sqrt{4x^2 - 1}}$

[1 mark]

(iv) $y = \operatorname{arsinh}\left(\dfrac{x}{3}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3}\,\dfrac{1}{\sqrt{\left(\frac{x}{3}\right)^2 + 1}}$

$$= \frac{1}{3} \times \frac{3}{\sqrt{x^2 + 9}}$$

$$= \frac{1}{\sqrt{x^2 + 9}}$$

[2 mark]

Answer 2

$$\begin{aligned}
y &= a\cosh(nx) + b\sinh(nx) \\
\frac{dy}{dx} &= an\sinh(nx) + bn\cosh(nx) \\
\frac{d^2y}{dx^2} &= an^2\cosh(nx) + bn^2\sinh(nx) \\
&= n^2(\,a\cosh(nx) + b\sinh(nx)) \\
&= n^2 y \qquad \square
\end{aligned}$$

[4 marks]

Answer 3

$$y = (arcosh\,x)^2$$

$$\frac{dy}{dx} = 2\,(arcosh\,x)^1 \times \frac{1}{\sqrt{x^2 - 1}}$$

multiply both sides by $\sqrt{x^2 - 1}$

$$\sqrt{x^2 - 1}\,\frac{dy}{dx} = 2\,arcosh\,x$$

square both sides

$$\left(x^2 - 1\right)\left(\frac{dy}{dx}\right)^2 = 4\,(arcosh\,x)^2$$

$$\left(x^2 - 1\right)\left(\frac{dy}{dx}\right)^2 = 4\,y \qquad \square$$

[5 marks]

Answer 4

(i) $y = sinh\left(2x\right)\,cosh\left(3x\right)$

$$\frac{dy}{dx} = 3\,sinh\left(2x\right)sinh\left(3x\right) + 2\,cosh\left(2x\right)cosh\left(3x\right)$$

[2 marks]

(ii) $y = \dfrac{cosh\,x}{4x}$

$$\frac{dy}{dx} = \frac{4x\,sinh\,x - 4\,cosh\,x}{16\,x^2}$$

$$= \frac{x\,sinh\,x - cosh\,x}{4\,x^2}$$

[2 marks]

(iii) $y = x^2\,arcosh\,x$

$$\frac{dy}{dx} = \frac{x^2}{\sqrt{x^2 - 1}} + 2x\,arcosh\,x$$

[2 marks]

Answer 5

$$y = x\,arsinh\left(2x\right)$$

$$\frac{dy}{dx} = \frac{2x}{\sqrt{4\,x^2 + 1}} + arsinh\left(2x\right)$$

[3 marks]

Answer 6

$$\begin{aligned} y &= sech\left(2x\right) \\ &= (\ cosh\,(2x))^{-1} \\ \frac{dy}{dx} &= \left(-1\right)(\ cosh\,(2x))^{-2}\left(\ sinh\left(2x\right)\right)\ 2 \\ &= -\frac{2\ sinh\,(2x)}{cosh^2\left(2x\right)} \\ &= -2\ sech\left(2x\right)tanh\left(2x\right) \end{aligned}$$

[3 marks]

Answer 7

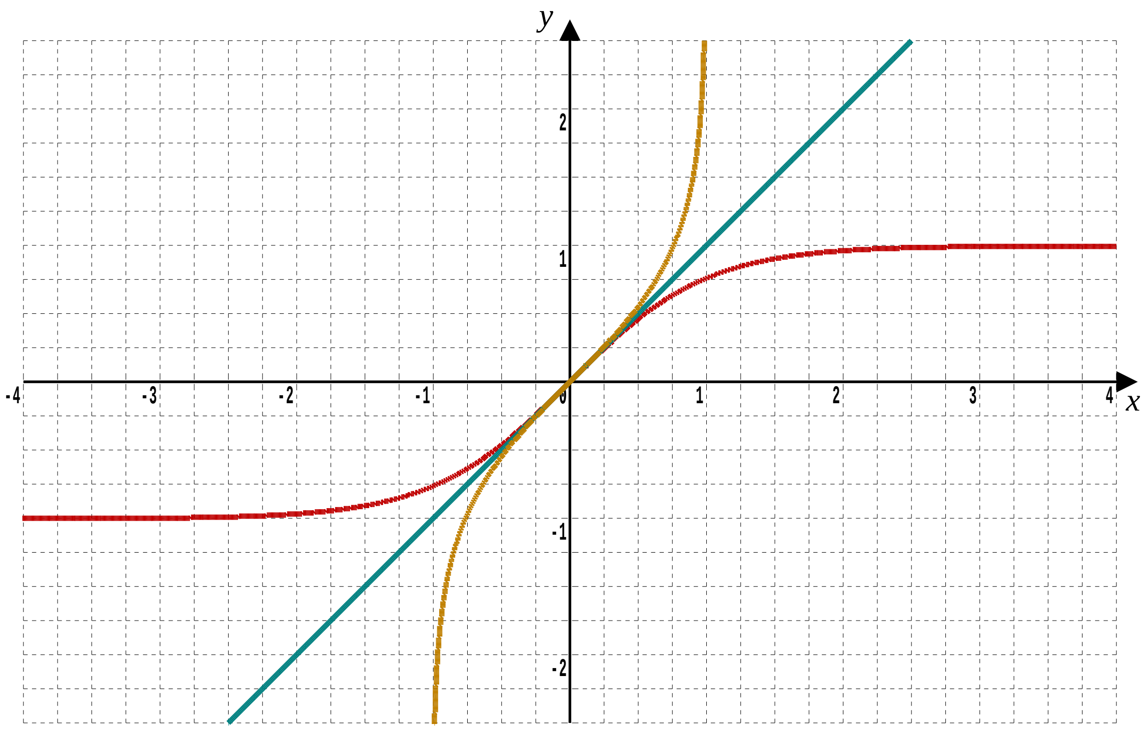
$$y = artanh\left(\frac{x}{\sqrt{1 + x^2}}\right)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{1 - \left(\frac{x}{\sqrt{1+x^2}}\right)^2} \times \frac{\sqrt{1 + x^2} \times 1 - \frac{1}{2}\left(1 + x^2\right)^{-\frac{1}{2}} \times 2x \times x}{1 + x^2} \\ &= \frac{1}{1 - \frac{x^2}{1+x^2}} \times \frac{1 + x^2 - x^2}{\left(1 + x^2\right)\sqrt{1 + x^2}} \\ &= \frac{1 + x^2}{1 + x^2 - x^2} \times \frac{1}{\left(1 + x^2\right)\sqrt{1 + x^2}} \\ &= \frac{1 + x^2}{1} \times \frac{1}{\left(1 + x^2\right)\sqrt{1 + x^2}} \\ &= \frac{1}{\sqrt{1 + x^2}} \qquad \square \end{aligned}$$

[4 marks]

7.1 *artanh x*

Like *sinh x*, the function *tanh x* is one-to-one on an unrestricted domain although, unlike *sinh x* it has horizontal asymptotes at $y = \pm 1$. Like any one-to-one function, it has an inverse that, graphically, is a reflection in $y = x$.
The inverse of *tanh x* is called *artanh x*.



Red : $y = \tanh x$ Blue : $y = x$ Gold : $y = \text{artanh}\, x$

Like *sinh x* and *cosh x*, the *tanh x* function is defined in terms of exponentials, and its inverse, *artanh x*, involves logarithms.

The Inverse Of *tanh x* : *artanh x*

$$\text{artanh}\, x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \qquad x \in \mathbb{R},\ |x| < 1$$

A proof of this is written out on the next page.
An excellent video from *Exam Solutions* talks through the proof

Teaching Video: http://www.NumberWonder.co.uk/v9102/7.mp4

7.2 The Proof

$$
\begin{aligned}
y &= artanh\, x \\
\therefore x &= tanh\, y \\
&= \frac{sinh\, y}{cosh\, y} \\
&= \frac{e^y - e^{-y}}{e^y + e^{-y}} \times \frac{e^y}{e^y} \\
&= \frac{(e^y)^2 - 1}{(e^y)^2 + 1} \\
x(e^y)^2 + x &= (e^y)^2 - 1 \\
1 + x &= (e^y)^2 - x(e^y)^2 \\
1 + x &= (e^y)^2 (1 - x) \\
e^y &= \pm\sqrt{\frac{1 + x}{1 - x}} \\
\text{Now, since } e^y > 0, \quad e^y &= \sqrt{\frac{1 + x}{1 - x}} \\
&= \left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}} \\
y &= ln\left(\frac{1 + x}{1 - x}\right)^{\frac{1}{2}} \\
\text{That is, } artanh\, x &= \frac{1}{2}\, ln\left(\frac{1 + x}{1 - x}\right) \qquad \square
\end{aligned}
$$

The domain of the inverse function is the range of the original function

That is $x \in \mathbb{R}, \ |x| < 1$

7.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 35

Question 1

Further A-Level Examination Question from June 2018, FP3, Q1 (Edexcel)

Solve the equation

$$15\,sech^2 x + 7\,tanh\,x = 13$$

Give your answers in terms of simplified natural logarithms

[6 marks]

Question 2

Further A-Level Examination Question from June 2009, FP3, Q1 (Edexcel)

Solve the equation

$$7\,sech\,x - tanh\,x = 5$$

Give your answers in the form *ln a* where *a* is a rational number

[5 marks]

Question 3

Find the equation of the tangent at the point where $x = \frac{12}{13}$ on the curve with equation $y = artanh\,x$

[3 marks]

Question 4

Further A-Level Examination Question from June 2006, FP2, Q5 (Edexcel)

The curve with equation

$$y = -x + tanh(4x) \qquad x \geqslant 0$$

has a maximum turning point A

(a) Find, in exact logarithmic form, the x-coordinate of A

[4 marks]

(b) Show that the y-coordinate of A is $\frac{1}{4}\left\{ 2\sqrt{3} - ln\left(2 + \sqrt{3} \right) \right\}$

[3 marks]

Question 5

Given that,

$$artanh\, x + artanh\, y = ln\sqrt{3}$$

prove that

$$y = \frac{2x - 1}{x - 2}$$

[6 marks]

Question 6

Further A-Level Examination Question from June 2008, FP2, Q1 (Edexcel)

Show that $\dfrac{d}{dx}\left[\, ln\left(tanh\, x \right) \right] = 2\, csch\left(2x\right) \qquad x > 0$

[4 marks]

Question 7

Further A-Level Examination Question from June 2004, P5, Q1(b) (Edexcel)

Solve $csch\, x - 2\, coth\, x = 2$ giving your answer in the form $k\, ln\, a$ where k and a are integers

[4 marks]

7.4 Answers to 7.3 Exercise

Further A-Level Pure Mathematics, Core 2
Hyperbolic Functions

Answer 1

$$\cos^2 x + \sin^2 x = 1$$
$$\therefore 1 + \tan^2 x = \sec^2 x$$
$$\therefore 1 - \tanh^2 x = \text{sech}^2 x \qquad \text{(Osborn's Rule)}$$

$$15\,\text{sech}^2 x + 7\tanh x = 13$$
$$15\left(1 - \tanh^2 x\right) + 7\tanh x = 13$$
$$15 - 15\tanh^2 x + 7\tanh x = 13$$
$$15\tanh^2 x - 7\tanh x - 2 = 0$$
$$\left(5\tanh x + 1\right)\left(3\tanh x - 2\right) = 0$$

$$\text{Either } \tanh x = -\frac{1}{5} \qquad \text{or} \qquad \tanh x = \frac{2}{3}$$

$$x = \text{artanh}\left(-\frac{1}{5}\right) \qquad \text{or} \qquad x = \text{artanh}\left(\frac{2}{3}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1 + \left(-\frac{1}{5}\right)}{1 - \left(-\frac{1}{5}\right)}\right) \qquad = \frac{1}{2}\ln\left(\frac{1 + \frac{2}{3}}{1 - \frac{2}{3}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{2}{3}\right) \qquad = \frac{1}{2}\ln 5$$

[6 marks]

Answer 2

$$7\,sech\,x - tanh\,x = 5$$

$$\frac{7}{cosh\,x} - \frac{sinh\,x}{cosh\,x} = 5$$

$$7 - sinh\,x = 5\,cosh\,x$$

$$5\,cosh\,x + sinh\,x - 7 = 0$$

$$5 \times \frac{e^{x} + e^{-x}}{2} + \frac{e^{x} - e^{-x}}{2} - 7 = 0$$

multiply through by 2

$$5\,e^{x} + 5\,e^{-x} + e^{x} - e^{-x} - 14 = 0$$

$$6\,e^{x} + 4\,e^{-x} - 14 = 0$$

divide through by 2

$$3\,e^{x} + 2\,e^{-x} - 7 = 0$$

multiple through by e^{x}

$$3\left(e^{x}\right)^{2} - 7\,e^{x} + 2 = 0$$

$$\left(3\,e^{x} - 1\right)\left(e^{x} - 2\right) = 0$$

Either $e^{x} = \frac{1}{3}$ or $e^{x} = 2$

$x = ln\left(\frac{1}{3}\right)$ or $x = ln\,2$

[5 marks]

Answer 3

$$y = artanh\, x$$

$$\text{when } x = \frac{12}{13},\quad y = \frac{1}{2}\, ln\left(\frac{1 + \frac{12}{13}}{1 - \frac{12}{13}} \times \frac{13}{13} \right)$$

$$= \frac{1}{2}\, ln\left(\frac{13 + 12}{13 - 12} \right)$$

$$= \frac{1}{2}\, ln\left(25 \right)$$

$$= ln\, 5$$

$$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

$$\text{when } x = \frac{12}{13},\quad \frac{dy}{dx} = \frac{1}{1 - \left(\frac{12}{13}\right)^2} \times \frac{13^2}{13^2}$$

$$= \frac{13^2}{13^2 - 12^2}$$

$$= \frac{169}{25}$$

$$\text{Tangent :} \qquad y = mx + c$$

$$ln\, 5 = \frac{169}{25} \times \frac{12}{13} + c$$

$$ln\, 5 = \frac{156}{25} + c$$

$$c = ln\, 5 - \frac{156}{25}$$

$$\text{The tangent has equation } y = \frac{169}{25}\, x + ln 5 - \frac{156}{25}$$

$$\left(y = 6.76x - 4.63 \right)$$

$$\left(169x - 25y + 25\, ln\, 5 - 156 \right)$$

[3 marks]

Answer 4

(a)

$$y = -x + tanh(4x)$$

$$\frac{dy}{dx} = -1 + 4\,sech^2(4x)$$

For a turning point $\frac{dy}{dx} = 0$

$$0 = -1 + \frac{4}{cosh^2(4x)}$$

multiply through by $cosh^2(4x)$

$$0 = -cosh^2(4x) + 4$$

$$cosh^2(4x) = 4$$

$$cosh(4x) = 2 \qquad \text{because } cosh\,x \text{ is always positive}$$

$$4x = arcosh(2)$$

$$= ln(2 + \sqrt{3})$$

$$x = \frac{1}{4}\,ln(2 + \sqrt{3})$$

[4 marks]

(b)

$$cosh^2(4x) - sinh^2(4x) = 1$$

$$sinh^2(4x) = cosh^2(4x) - 1$$

$$= 4 - 1$$

$$= 3$$

$$sinh\,x = \sqrt{3}$$

$\left(\text{As } x = \frac{1}{4}\,ln(2 + \sqrt{3}) \text{ the other possibility of } sinh\,x = -\sqrt{3} \text{ is rejected}\right)$

$$y = -x + tanh(4x)$$

$$= -x + \frac{sinh(4x)}{cosh(4x)}$$

$$= -\frac{1}{4}\,ln(2 + \sqrt{3}) + \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}\left\{2\sqrt{3} - ln(2 + \sqrt{3})\right\}$$

[3 marks]

Answer 5

$$artanh\,x + artanh\,y = ln\sqrt{3}$$

$$\Rightarrow \frac{1}{2}ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}ln\left(\frac{1+y}{1-y}\right) = \frac{1}{2}\,ln\,3$$

multiply through by 2

$$ln\left(\frac{1+x}{1-x}\right) + ln\left(\frac{1+y}{1-y}\right) = ln\,3$$

$$ln\left(\frac{(1+x)(1+y)}{(1-x)(1-y)}\right) = ln\,3$$

$$\frac{(1+x)(1+y)}{(1-x)(1-y)} = 3$$

$$(1+x)(1+y) = 3(1-x)(1-y)$$

$$1 + x + y + xy = 3 - 3x - 3y + 3xy$$

$$1 + x - 3 + 3x = -3y + 3xy - y - xy$$

$$-2 + 4x = -4y + 2xy$$

divide through by 2

$$-1 + 2x = -2y + xy$$

$$2x - 1 = y(x-2)$$

$$y = \frac{2x-1}{x-2}$$

[6 marks]

Answer 6

$$\text{LHS} = \frac{d}{dx}\left[\,ln\left(tanh\,x\right)\,\right]$$

$$= \frac{1}{tanh\,x} \times sech^2\,x$$

$$= \frac{cosh\,x}{sinh\,x} \times \frac{1}{cosh^2\,x}$$

$$= \frac{1}{sinh\,x\;cosh\,x}$$

$$= \frac{2}{2\,sinh\,x\;cosh\,x}$$

$$= \frac{2}{sinh\left(2x\right)}$$

$$= 2\,csch\left(2x\right)$$

$$= \text{RHS} \qquad \square$$

[4 marks]

Answer 7

$$csch\, x - 2\, coth\, x = 2$$

$$\frac{1}{sinh\, x} - \frac{2\, cosh\, x}{sinh\, x} = 2$$

multiply through by $sinh\, x$

$$1 - 2\, cosh\, x = 2\, sinh\, x$$

$$1 - \left(e^{x} + e^{-x} \right) = \left(e^{x} - e^{-x} \right)$$

$$1 - e^{x} - e^{-x} = e^{x} - e^{-x}$$

$$1 - e^{x} = e^{x}$$

$$2\, e^{x} = 1$$

$$e^{x} = \frac{1}{2}$$

$$x = ln\left(\frac{1}{2}\right)$$

$$= ln\, 2^{-1}$$

$$= (-1)\; ln\, 2$$

That is, $k = -1$ and $a = 2$

[4 marks]

8.1 Integration (Chain Rule Backwards)

In principle, given we've already[†] looked at differentiation questions, tackling an exercise on integration is just a case of using the following, previously supplied, table backwards, and remembering that if it's an indefinite integral (one without limits) to put in the constant of integration, *c*.

$f(x)$	$f'(x)$	In Formula Book ?
$\sinh x$	$\cosh x$	Yes
$\cosh x$	$\sinh x$	Yes
$\tanh x$	$\operatorname{sech}^2 x$	Yes
$\operatorname{arsinh} x$	$\dfrac{1}{\sqrt{x^2+1}}$	Yes
$\operatorname{arcosh} x$	$\dfrac{1}{\sqrt{x^2-1}} \quad x > 1$	Yes
$\operatorname{artanh} x$	$\dfrac{1}{1-x^2} \quad \lvert x \rvert < 1$	Yes

As with any non-trivial integration, always scan straight away for a possible "Chain Rule Backwards" before reaching for the much bigger weapons in the armoury; Integration by Substitution and Integration by Parts.

8.2 Example

Find $\displaystyle\int \frac{2 + 5x}{\sqrt{x^2 + 1}}\, dx$

Teaching Video : http://www.NumberWonder.co.uk/v9102/8.mp4

[6 marks]

† Hyperbolic Functions, Lesson 6

8.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Integrate the following with respect to x,

(i) $\sinh(5x)$ **(ii)** $\dfrac{1}{\cosh^2(4x)}$

[2, 3 marks]

Question 2

Marvin is thinking of using integration by parts twice to find the following integral,

$$\int x^2 \cosh\left(x^3 + 4\right)\, dx$$

but Giles points out that it can far more easily be done as a “chain rule backwards”. Following Giles' advice, carry out the integration.

[2 marks]

Question 3

By setting up a “chain rule backwards” find,

(i) $\displaystyle\int \sinh(5x)\, \cosh^6(5x)\, dx$

(ii) $\displaystyle\int \frac{\sinh(5x)}{\cosh^6(5x)}\, dx$

[4 marks]

Question 4

Frederic wishes to integrate the *tanh x* function and has begun by setting up a "chain rule backwards".

(i) Complete Fredric's solution,

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$

$$= \int \sinh x \, (\cosh x)^{-1} \, dx$$

[2 marks]

(ii) Justify why no modulus signs are needed in the final result.

[1 mark]

Question 5

(i) Use the approach outlined in question 4 to find $\int \coth x \, dx$

[2 marks]

(ii) Are modulus signs are needed in the final result ?

[1 mark]

Question 6

Find the exact value of $\displaystyle \int_0^{\sqrt{2}} \frac{6}{\sqrt{1 + 4x^2}} \, dx$

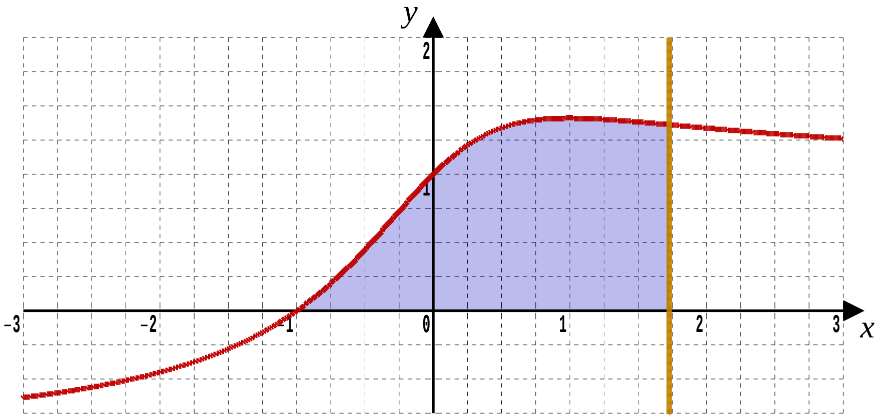
[5 marks]

Question 7

The graph is of the function $f(x) = \dfrac{1 + x}{\sqrt{x^2 + 1}}$ (in red) along with the vertical line (in gold) with equation $x = \sqrt{3}$. Show that the exact value of the area bounded by this vertical line, $f(x)$, and the x-axis is,

$$ln\left(\left(\sqrt{3} + 2\right)\left(\sqrt{2} + 1\right) \right) + 2 - \sqrt{2}$$

The techniques of example 8.2 will be useful !



[8 marks]

Question 8

By replacing the $\cosh x$ function with exponentials, find $\int_0^{\ln 3} e^x \cosh(2x) \; dx$

[4 marks]

Question 9

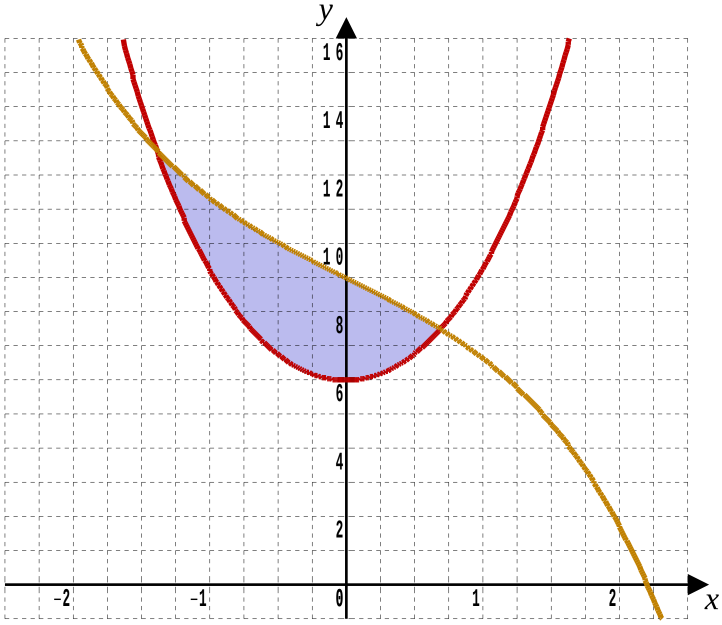
FM A-Level Examination Question from June 2014, Paper FP3, Q3(b) (Edexcel)

Using calculus, find the exact value of $\int_0^{1} e^{2x} \sinh x \; dx$

[4 marks]

Question 10

FM A-Level Examination Question from June 2013, Paper FP3, Q7 (Edexcel)



The curves shown are $y = 6\cosh x$ (red) and $y = 9 - 2\sinh x$ (gold).

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinate of the two points where the curves intersect.

[6 marks]

The finite region between the two curves is shown shaded.

(b) Using calculus, find the area of the shaded region, giving your answer in the form $a \ln b + c$, where a, b and c are integers.

[6 marks]

8.4 Answers

8.4.1 Solution to 8.2 Example

$$\int \frac{2 + 5x}{\sqrt{x^2 + 1}}\, dx = \int \frac{2}{\sqrt{x^2 + 1}}\, dx + \int \frac{5x}{\sqrt{x^2 + 1}}\, dx$$

$$= 2\int \frac{1}{\sqrt{x^2 + 1}}\, dx + 5\int x\left(x^2 + 1\right)^{-\frac{1}{2}}\, dx$$

$$= 2\int \frac{1}{\sqrt{x^2 + 1}}\, dx + 5 \times \frac{1}{2}\int 2\,x\left(x^2 + 1\right)^{-\frac{1}{2}}\, dx$$

$$= 2\, arsinh\, x + \frac{5}{2} \times \frac{2\left(x^2 + 1\right)^{\frac{1}{2}}}{1} + c$$

$$= 2\, arsinh\, x + 5\sqrt{x^2 + 1} + c$$

[6 marks]

8.4.2 Solution to 8.3 Exercise

Answer 1

(i) $\int sinh\,(5x)\; dx = \frac{1}{5}\int 5\, sinh\,(5x)\; dx = \frac{1}{5}\, cosh\,(5x) + c$

[2 marks]

(ii) $\int \frac{1}{cosh^2\,(4x)}\, dx = \frac{1}{4}\int 4\, sech^2\,(4x)\; dx = \frac{1}{4}\, tanh\,(4x) + c$

[3 marks]

Answer 2

$$\int x^2\, cosh\left(x^3 + 4\right)\, dx = \frac{1}{3}\int 3\,x^2\, cosh\left(x^3 + 4\right)\, dx = \frac{1}{3}\, sinh\left(x^3 + 4\right) + c$$

[2 marks]

Answer 3

(i) $\int sinh\,(5x)\; cosh^6\,(5x)\; dx = \frac{1}{5}\int 5\, sinh\,(5x)\; cosh^6\,(5x)\; dx$

$$= \frac{1}{35}\, cosh^7\,(5x) + c$$

[2 marks]

(ii) $\int \frac{sinh\,(5x)}{cosh^6\,(5x)}\, dx = \frac{1}{5}\int 5\, sinh\,(5x)\;\left(\, cosh\,(5x)\,\right)^{-6}\, dx$

$$= -\frac{1}{25}\left(\, cosh\,(5x)\right)^{-5} + c$$

$$= -\frac{1}{25\, cosh^5\,(5x)} + c$$

[2 marks]

Answer 4

(i)

$$\int tanh\, x\, dx = \int \frac{sinh\, x}{cosh\, x}\, dx$$

$$= \int sinh\, x\, (\, cosh\, x\,)^{-1}\, dx$$

$$= ln\, |\, cosh\, x\, | + c$$

$$= ln\, (\, cosh\, x\,) + c$$

[2 marks]

(ii) The modulus signs were removed from the end result because the *cosh x* function is always positive, easily seen from its graph or from its definition in terms of exponentials.

[1 mark]

Answer 5

(i)

$$\int coth\, x\, dx = \int \frac{cosh\, x}{sinh\, x}\, dx$$

$$= \int cosh\, x\, (\, sinh\, x\,)^{-1}\, dx$$

$$= ln\, |\, sinh\, x\, | + c$$

[2 marks]

(ii) This time the modulus signs are needed because the *sinh x* function is negative for $x < 0$

[1 mark]

Answer 6

$$\int_0^{\sqrt{2}} \frac{6}{\sqrt{1 + 4x^2}}\, dx = 3 \int_0^{\sqrt{2}} \frac{2}{\sqrt{1 + (2x)^2}}\, dx$$

$$= 3\, \left[\, arsinh\, (2x)\, \right]_0^{\sqrt{2}}$$

$$= 3 \left[\, ln \left(2x + \sqrt{4x^2 + 1}\, \right) \right]_0^{\sqrt{2}}$$

$$= 3 \left[\, ln \left(2\sqrt{2} + 3 \right) - ln \left(0 + \sqrt{1}\, \right) \right]$$

$$= 3\, ln \left(2\sqrt{2} + 3 \right)$$

[5 marks]

Answer 7

$$\int_{-1}^{\sqrt{3}} \frac{1 + x}{\sqrt{x^2 + 1}}\, dx = \int_{-1}^{\sqrt{3}} \frac{1}{\sqrt{x^2 + 1}}\, dx + \int_{-1}^{\sqrt{3}} \frac{x}{\sqrt{x^2 + 1}}\, dx$$

$$= \int_{-1}^{\sqrt{3}} \frac{1}{\sqrt{x^2 + 1}}\, dx + \frac{1}{2}\int_{-1}^{\sqrt{3}} 2x\left(x^2 + 1\right)^{-\frac{1}{2}}\, dx$$

$$= \left[\, arsinh\, x + \sqrt{x^2 + 1}\,\right]_{-1}^{\sqrt{3}}$$

If not convinced by the preceding line, differentiate it

to check it goes back from whence it came !

$$= \left[\, ln\left(x + \sqrt{x^2 + 1}\right) + \sqrt{x^2 + 1}\,\right]_{-1}^{\sqrt{3}}$$

$$= ln\left(\sqrt{3} + 2\right) + 2 - ln\left(-1 + \sqrt{2}\right) - \sqrt{2}$$

$$= ln\left(\frac{\sqrt{3} + 2}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}\right) + 2 - \sqrt{2}$$

$$= ln\left(\left(\sqrt{3} + 2\right)\left(\sqrt{2} + 1\right)\right) + 2 - \sqrt{2}$$

(It's about 2.784...)

[8 marks]

Answer 8

$$\int_0^{ln\,3} e^x\, cosh\,(2x)\ \ dx = \int_0^{ln\,3} e^x\left(\frac{e^{2x} + e^{-2x}}{2}\right)dx$$

$$= \frac{1}{2}\int_0^{ln\,3} e^{3x}\, dx + \frac{1}{2}\int_0^{ln\,3} e^{-x}\, dx$$

$$= \frac{1}{6}\int_0^{ln\,3} 3\, e^{3x}\, dx - \frac{1}{2}\int_0^{ln\,3} (-1)\ e^{-x}\, dx$$

$$= \left[\, \frac{1}{6}\, e^{3x} - \frac{1}{2}\, e^{-x}\,\right]_0^{ln\,3}$$

$$= \frac{e^{3\,ln\,3}}{6} - \frac{e^{-\,ln\,3}}{2} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{e^{ln\,27}}{6} - \frac{e^{ln\,3^{-1}}}{2} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{27}{6} - \frac{1}{6} - \frac{1}{6} + \frac{3}{6}$$

$$= \frac{14}{3}$$

[4 marks]

Answer 9

$$\int_0^1 e^{2x} \sinh x \; dx = \int_0^1 e^{2x}\left(\frac{e^x - e^{-x}}{2} \right) dx$$

$$= \frac{1}{2} \int_0^1 e^{3x}\, dx - \frac{1}{2} \int_0^1 e^x\, dx$$

$$= \frac{1}{6} \int_0^1 3\, e^{3x}\, dx - \frac{1}{2} \int_0^1 e^x\, dx$$

$$= \left[\frac{1}{6} e^{3x} - \frac{1}{2} e^x \right]_0^1$$

$$= \frac{e^3}{6} - \frac{e}{2} - \frac{1}{6} + \frac{1}{2}$$

$$= \frac{e^3}{6} - \frac{e}{2} + \frac{1}{3}$$

[4 marks]

Answer 10

(i)

$$6 \cosh x = 9 - 2 \sinh x$$

$$6\left(\frac{e^{x} + e^{-x}}{2} \right) = 9 - 2\left(\frac{e^{x} - e^{-x}}{2} \right)$$

$$3 e^{x} + 3 e^{-x} = 9 - e^{x} + e^{-x}$$

$$4 e^{x} + 2 e^{-x} - 9 = 0$$

Multiply through by e^{x}

$$4 \left(e^{x} \right)^{2} - 9 e^{x} + 2 = 0$$

$$\left(4 e^{x} - 1 \right)\left(e^{x} - 2 \right) = 0$$

Either $e^{x} = \frac{1}{4}$ in which case $x = ln\left(\frac{1}{4}\right)$

Or $e^{x} = 2$ in which case $x = ln\, 2$

[6 marks]

(ii)

$$\int_{ln\, 0.25}^{ln\, 2} 9 - 2 \sinh x - 6 \cosh x \quad dx$$

$$= \left[9x - 2 \cosh x - 6 \sinh x \right]_{ln\, 0.25}^{ln\, 2}$$

$$= 9\, ln\, 2 - e^{ln\, 2} - e^{-ln\, 2} - 3 e^{ln\, 2} + 3 e^{-ln\, 2}$$

$$- 9\, ln\, 0.25 + e^{ln\, 0.25} + e^{-ln\, 0.25} + 3 e^{ln\, 0.25} - 3 e^{-ln\, 0.25}$$

$$= 9\, ln\, 2 - 2 - \frac{1}{2} - 6 + \frac{3}{2} + 18\, ln\, 2 + \frac{1}{4} + 4 + \frac{3}{4} - 12$$

$$= 27\, ln\, 2 - 14$$

[6 marks]

9.1 Integration of Quadratic Denominators

The adsorption of the hyperbolic functions into the body of knowledge with which we are familiar has put us in a position to be able to integrate any function that is the reciprocal of a quadratic, even when that quadratic is square rooted, That is, functions of the forms,

$$f(x) = \frac{1}{a x^2 + bx + c}, \qquad g(x) = \frac{1}{\sqrt{ax^2 + bx + c}}$$

This has become about because the hyperbolics have allowed us to have the following "complete list" of standard integrations.

$f(x)$	$f'(x)$	In Formula Book ?
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}} \quad \lvert x \rvert < 1$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}} \quad \lvert x \rvert < 1$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes
$\text{arsinh}\, x$	$\frac{1}{\sqrt{x^2+1}}$	Yes
$\text{arcosh}\, x$	$\frac{1}{\sqrt{x^2-1}} \quad x > 1$	Yes
$\text{artanh}\, x$	$\frac{1}{1-x^2} \quad \lvert x \rvert < 1$	Yes

9.2 The Better Version "Completed List"

In fact, the examination formulae book provides an even better list of standard integrations that may be quoted without proof. Proving these results is reasonably straight forward, should you be asked to do so, using "integration by substitution" where the substitution involves a trigonometric function.

$f(x)$	$\int f(x)\, dx$ (Constant of integration omitted)	In Formula Book ?
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad \lvert x \rvert < a$	Yes
$\frac{1}{a^2+x^2}$	$\frac{1}{a} \arctan\left(\frac{x}{a}\right)$	Yes
$\frac{1}{\sqrt{x^2-a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right),\ \ln\left\{ x + \sqrt{x^2 - a^2} \right\} \quad x > a$	Yes
$\frac{1}{\sqrt{a^2+x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right),\ \ln\left\{ x + \sqrt{x^2 + a^2} \right\}$	Yes
$\frac{1}{a^2-x^2}$	$\frac{1}{2a} \ln\left\lvert \frac{a+x}{a-x} \right\rvert = \frac{1}{a} \text{artanh}\left(\frac{x}{a}\right) \quad \lvert x \rvert < a$	Yes
$\frac{1}{x^2-a^2}$	$\frac{1}{2a} \ln\left\lvert \frac{x-a}{x+a} \right\rvert$	Yes

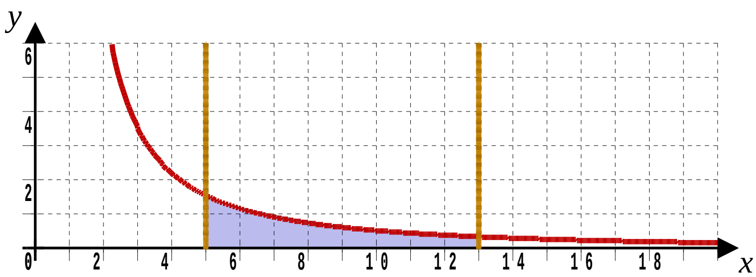
9.3 Example #1

Find $\displaystyle\int \frac{1}{\sqrt{2x^2 + 12x}}\,dx$

The strategy is to, in the denominator, pull out a 2, then complete the square, ahead of using one of the standard results from the previous table.

[4 marks]

9.4 Example #2



Find the area indicated by evaluating $\displaystyle\int_5^{13} \frac{100}{x^2 + 10x - 11}\,dx$

[6 marks]

9.5 Exercise

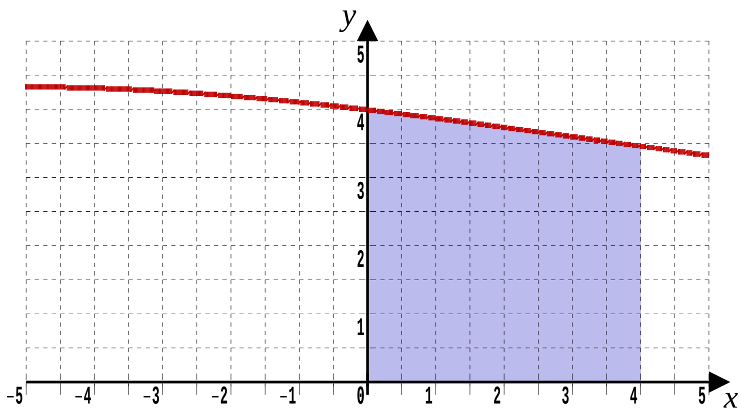
Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Find $\int \frac{1}{\sqrt{x^2 - 2x + 10}}\, dx$ giving an answer in terms of the *arsinh* function.

[2 marks]

Question 2



Find the area indicated by evaluating $\int_0^4 \frac{52}{\sqrt{x^2 + 10x + 169}}\, dx$

Give your answer in term of a natural logarithm.

[7 marks]

Question 3

Evaluate $\int_1^3 \frac{1}{\sqrt{x^2 - 2x + 2}}\, dx$, giving your answer as a single natural logarithm.

[5 marks]

Question 4

(i) Write $4x^2 - 12x - 7$ in the form $(ax - b)^2 - c^2$, $a, b, c \in \mathbb{Z}^+$

[3 marks]

(ii) Hence, or otherwise, find $\int \frac{1}{\sqrt{4x^2 - 12x - 7}}\, dx$

[4 marks]

Question 5

(i) Write $16x^2 - 40x + 9$ in the form $(ax - b)^2 - c^2, \quad a, b, c \in \mathbb{Z}^+$

[3 marks]

(ii) Hence, or otherwise, find the exact value of $\int_{-0.25}^{0} \frac{1}{16x^2 - 40x + 9}\, dx$

[5 marks]

Question 6

FM A-Level Examination Question from June 2011, Paper FP3, Q3 (Edexcel)

Show that,

(a) $\displaystyle\int_5^8 \frac{1}{x^2 - 10x + 34}\,dx = k\pi$, giving the value of the fraction k

[5 marks]

(b) $\displaystyle\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,dx = ln\left(A + \sqrt{n}\right)$ giving the values of the integers A and n

[4 marks]

Question 7

FM A-Level Examination Question from June 2016, Paper FP3, Q4 (Edexcel)

(i) Find, without using a calculator, $\int_3^5 \frac{1}{\sqrt{15 + 2x - x^2}}\, dx$

giving your answer as a multiple of π

[5 marks]

(ii) **(a)** Show that $5\cosh x - 4\sinh x = \dfrac{e^{2x} + 9}{2\, e^x}$

[3 marks]

(b) Hence, using the substitution $u = e^x$ or otherwise, find

$$\int \frac{1}{5\cosh x - 4\sinh x}\, dx$$

[4 marks]

9.6 Answers

9.6.1 Solution to 9.3 Example #1

$$\int \frac{1}{\sqrt{2x^2 + 12x}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2 + 6x}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(x+3)^2 - 3^2}}\,dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 - a^2}} \text{ integrates to } arcosh\left(\frac{x}{a}\right),\ x > a$$

$$= \frac{1}{\sqrt{2}}\,arcosh\left(\frac{x+3}{3}\right) + c$$

[4 marks]

9.6.2 Solution to 9.4 Example #2

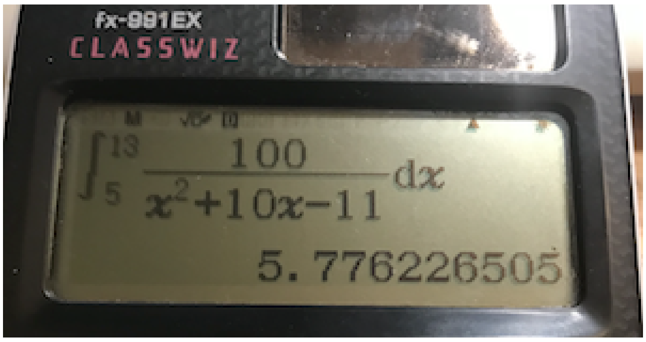
$$\int_5^{13} \frac{100}{x^2 + 10x - 11}\,dx = 100\int_5^{13} \frac{1}{(x+5)^2 - 6^2}\,dx$$

Standard Result

$$\frac{1}{x^2 - a^2} \text{ integrates to } \frac{1}{2a}\,ln\left|\frac{x-a}{x+a}\right|$$

$$= 100 \times \frac{1}{12}\left[ln\left|\frac{(x+5)-6}{(x+5)+6}\right|\right]_5^{13}$$
$$= \frac{25}{3}\left[ln\left|\frac{12}{24}\right| - ln\left|\frac{4}{16}\right|\right]$$
$$= \frac{25}{3}\left[ln\,2^{-1} - ln\,2^{-2}\right]$$
$$= \frac{25}{3}\left[-ln\,2 + 2\,ln\,2\right]$$
$$= \frac{25\,ln\,2}{3}$$

[6 marks]



9.6.3 Solutions to 9.5 Exercise

Answer 1

$$\int \frac{1}{\sqrt{x^2 - 2x + 10}}\, dx = \int \frac{1}{\sqrt{(x-1)^2 + 3^2}}\, dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right), \quad ln\left\{ x + \sqrt{x^2 + a^2} \right\}$$

$$= arsinh\left(\frac{x - 1}{3} \right) + c$$

[2 marks]

Answer 2

$$\int_0^4 \frac{52}{\sqrt{x^2 + 10x + 169}}\, dx = 52 \int_0^4 \frac{1}{\sqrt{(x+5)^2 - 25 + 169}}\, dx$$

$$= 52 \int_0^4 \frac{1}{\sqrt{(x+5)^2 + 144}}\, dx$$

$$= 52 \int_0^4 \frac{1}{\sqrt{(x+5)^2 + 12^2}}\, dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right), \quad ln\left\{ x + \sqrt{x^2 + a^2} \right\}$$

$$= 52 \left[ln\left\{ x + 5 + \sqrt{(x+5)^2 + 12^2} \right\} \right]_0^4$$

$$= 52 \left[ln\left\{ 9 + \sqrt{9^2 + 12^2} \right\} - ln\left\{ 5 + \sqrt{5^2 + 12^2} \right\} \right]$$

$$= 52 \left[ln\{ 9 + 15 \} - ln\{ 5 + 13 \} \right]$$

$$= 52\, ln\left(\frac{24}{18} \right)$$

$$= 52\, ln\left(\frac{4}{3} \right)$$

[7 marks]

Answer 3

$$\int_1^3 \frac{1}{\sqrt{x^2 - 2x + 2}}\, dx = \int_1^3 \frac{1}{\sqrt{(x-1)^2 + 1^2}}\, dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right),\ ln\left\{ x + \sqrt{x^2 + a^2} \right\}$$

$$= \left[ln\left\{ x - 1 + \sqrt{(x-1)^2 + 1^2} \right\} \right]_1^3$$

$$= ln\left\{ 2 + \sqrt{5} \right\} - ln\left\{ 0 + \sqrt{1} \right\}$$

$$= ln\left(2 + \sqrt{5} \right)$$

[5 marks]

Answer 4

(i) $4x^2 - 12x - 7 = 4\left(x^2 - 3x\right) - 7$

$$= 4\left(\left(x - \frac{3}{2} \right)^2 - \frac{9}{4} \right) - 7$$

$$= 2^2\left(x - \frac{3}{2} \right)^2 - 9 - 7$$

$$= (2x - 3)^2 - 4^2$$

So, $a = 2$, $b = 3$ and $c = 4$

[3 marks]

(ii) $\displaystyle\int \frac{1}{\sqrt{4x^2 - 12x - 7}}\, dx = \int \frac{1}{\sqrt{(2x-3)^2 - 4^2}}\, dx$

$$= \frac{1}{2} \int \frac{2}{(2x-3)^2 - 4^2}\, dx$$

Note the need to set up a "chain rule backwards" because of the 2*x*

Standard Result

$$\frac{1}{\sqrt{x^2 - a^2}} \text{ integrates to } arcosh\left(\frac{x}{a}\right),\ x > a$$

$$= \frac{1}{2}\, arcosh\left(\frac{2x - 3}{4}\right),\ x > a$$

[4 marks]

Answer 5

(i) $16x^2 - 40x + 9 = 16\left(x^2 - \frac{5}{2}x \right) + 9$

$$= 16\left(\left(x - \frac{5}{4} \right)^2 - \frac{25}{16} \right) + 9$$

$$= 4^2 \left(x - \frac{5}{4} \right)^2 - 25 + 9$$

$$= (4x - 5)^2 - 4^2$$

So, $a = 4$, $b = 5$ and $c = 4$

[3 marks]

(ii) $\int_{-0.25}^{0} \frac{1}{16x^2 - 40x + 9}\, dx = \int_{-0.25}^{0} \frac{1}{\sqrt{(4x-5)^2 - 4^2}}\, dx$

$$= \frac{1}{4} \int_{-0.25}^{0} \frac{4}{(4x-5)^2 - 4^2}\, dx$$

Note the need to set up a “chain rule backwards” because of the $4x$

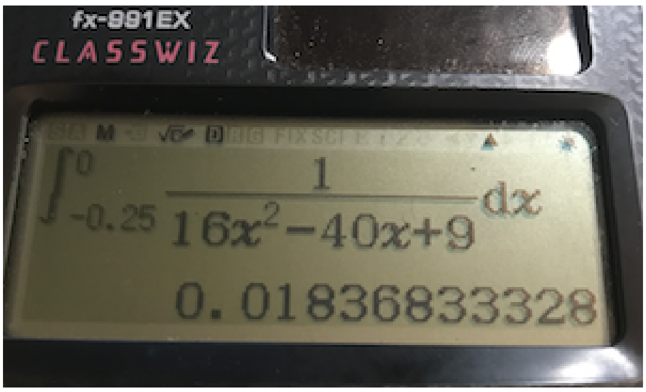
Standard Result

$$\frac{1}{x^2 - a^2} \text{ integrates to } \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right|$$

$$= \frac{1}{4} \left[\frac{1}{8} \ln \left| \frac{(4x - 5) - 4}{(4x - 5) + 4} \right| \right]_{-0.25}^{0}$$

$$= \frac{1}{32} \ln \left| \frac{4x - 9}{4x - 1} \right|_{-0.25}^{0}$$

$$= \frac{1}{32} [\ln 9 - \ln 5]$$



[5 marks]

Answer 6

(a) $\displaystyle\int_5^8 \frac{1}{x^2 - 10x + 34}\,dx = \int_5^8 \frac{1}{(x-5)^2 - 25 + 34}\,dx$

$\displaystyle = \int_5^8 \frac{1}{(x-5)^2 + 3^2}\,dx$

Standard Result

$$\frac{1}{a^2 + x^2} \text{ integrates to } \frac{1}{a}\,arctan\left(\frac{x}{a}\right)$$

$$= \left[\frac{1}{3}\,arctan\left(\frac{x-5}{3}\right)\right]_5^8$$

$$= \frac{1}{3}\left[\,arctan(1) - arctan(0)\,\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{12} \qquad \therefore k = \frac{1}{12}$$

[**5 marks]**

(b) $\displaystyle\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,dx = \int_5^8 \frac{1}{\sqrt{(x-5)^2 + 3^2}}\,dx$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right),\ ln\left\{x + \sqrt{x^2 + a^2}\right\}$$

$$= \left[\,ln\left\{(x-5) + \sqrt{(x-5)^2 + 3^2}\right\}\right]_5^8$$

$$= ln\left\{3 + \sqrt{3^2 + 3^2}\right\} - ln\left\{0 + \sqrt{0^2 + 3^2}\right\}$$

$$= ln\left\{3 + 3\sqrt{2}\right\} - ln\,3$$

$$= ln\left(\frac{3 + 3\sqrt{2}}{3}\right)$$

$$= ln\left(1 + \sqrt{2}\right) \qquad \therefore A = 1,\ n = 2$$

[4 marks]

Answer 7

(i)

$$\int_3^5 \frac{1}{\sqrt{15 + 2x - x^2}}\,dx = \int_3^5 \frac{1}{\sqrt{15 - (x^2 - 2x)}}\,dx$$

$$= \int_3^5 \frac{1}{\sqrt{15 - (\,(x-1)^2 - 1\,)}}\,dx$$

$$= \int_3^5 \frac{1}{\sqrt{4^2 - (x-1)^2}}\,dx$$

Standard Result

$$\frac{1}{\sqrt{a^2 - x^2}} \text{ integrates to } \arcsin\left(\frac{x}{a}\right), \quad |x| < a$$

$$= \left[\arcsin\left(\frac{x-1}{4}\right)\right]_3^5$$

$$= \arcsin(1) - \arcsin\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

[5 marks]

(ii) **(a)**

$$\text{LHS} = 5\cosh x - 4\sinh x$$

$$= 5\left(\frac{e^x + e^{-x}}{2}\right) - 4\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{e^x + 9e^{-x}}{2}\right) \times \frac{e^x}{e^x}$$

$$= \frac{e^{2x} + 9}{2e^x}$$

$$= \text{RHS} \qquad \square$$

[3 marks]

(b) $$\int \frac{1}{5\cosh x - 4\sinh x}\,dx = \int \frac{2e^x}{e^{2x} + 9}\,dx$$

$$= 2\int \frac{e^x}{(e^x)^2 + 3^2}\,dx$$

Notice that this is already set up as a "chain rule backwards" because the derivative of e^x is e^x but by all means use the suggested substitution if you find this hard to see.

Standard Result

$$\frac{1}{a^2 + x^2} \text{ integrates to } \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$= \frac{2}{3}\arctan\left(\frac{e^x}{3}\right) + c$$

[4 marks]

10.1 By Parts and Substitution

This lesson is a revision of the two topics Integration by Parts and Integration by Substitution but with an emphasis on situations that involve the hyperbolic functions and their associated trigonometric identities.

Keep in mind that carrying out an integration using parts (LIDI) or a substitution can be a lot of work and there is a definite time saving in opting for a "chain rule backwards" if a question is amenable to that approach. However, an examination question may prescribe that the solution method has to be "by parts" or "by substitution", or the parts or substitution methods may genuinely be the easier method of doing the question.

10.2 Example #1 (By Parts)

Use integration by parts to find $\int x \sinh 3x \, dx$

[3 marks]

10.3 Example #2 (Trigonometric Odd Powers)

In general to integrate a trigonometric function raised to an odd power the following simple manoeuvre can be very effective.

Trigonometric Odd Powers Integration

For small odd values of n, use the cunning moves,

$$\int \sinh^n x \, dx = \int \sinh x \sinh^{n-1} x \, dx$$

$$\int \cosh^n x \, dx = \int \cosh x \cosh^{n-1} x \, dx$$

Find $\int \sinh^3 x \, dx$

[3 marks]

10.4 Example #3 (Substitution)

When carrying out an integration involving a substitution $x = f(u)$ remember that three items need attending to:

- Making the substitution into the function being integrated
- Replacing the dx with du having first worked out $\frac{dx}{du}$
- Replacing the "$x =$" limits with "$u =$" limits

Use the substitution $x = 3\,sinh\,u$ to find the exact value of $\int_0^6 \frac{x^3}{\sqrt{x^2 + 9}}\,dx$

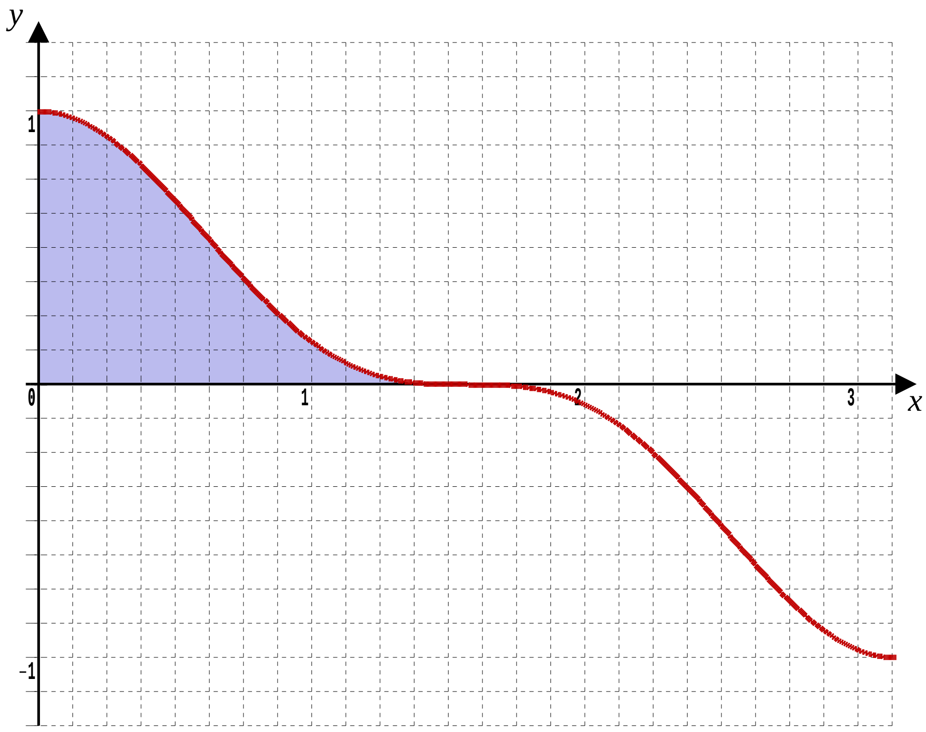
[11 marks]

10.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

The graph is of the function $f(x) = cos^3 x, \quad 0 \leqslant x \leqslant \pi$



Find the area shaded which is bounded by $f(x)$ and the positive coordinate axes.

[5 marks]

Question 2

Use integration by parts to find $\int 25x^2 \, cosh\,(5x)\; dx$

[5 marks]

Question 3

Find $\int cosh^3 x \, dx$

[3 marks]

Question 4

Use the substitution $x = 6\,cosh\,u$ to find the exact value of $\int_{12}^{18} \frac{x^3}{\sqrt{x^2 - 36}}\,dx$

Give your answer in the form $s\left(p\sqrt{2} - q\sqrt{3}\right)$ where s is a square number and p and q are prime numbers.

[11 marks]

Question 5

Using the substitution $x = sec\,\theta$ find,

(i) $\displaystyle\int \frac{1}{x\sqrt{x^2 - 1}}\,dx$

[2 marks]

(ii) $\displaystyle\int \frac{\sqrt{x^2 - 1}}{x}\,dx$

[4 marks]

10.6 Answers

10.6.1 Solution to 10.2 Example #1 (By Parts)

$$\int x\,sinh\,3x\,dx \;=\; \frac{1}{3}\int x\,3 sinh\,3x$$

$$\begin{aligned}
& L \quad I \quad D \quad I \\
&= \frac{1}{3}\left[\, x\,cosh\,3x \;-\; \int 1\,cosh\,3x\,dx \,\right] \\
&= \frac{1}{3}\left[\, x\,cosh\,3x \;-\; \frac{1}{3}\int 3\,cosh\,3x\,dx \,\right] \\
&= \frac{1}{3}\left[\, x\,cosh\,3x \;-\; \frac{1}{3}\,sinh\,3x \,\right] \;+\; c \\
&= \frac{x\,cosh\,3x}{3} \;-\; \frac{sinh\,3x}{9} \;+\; c
\end{aligned}$$

Note that if the *sinh* $3x$ were replaced with exponentials integration by parts would still be needed on each of two pieces.

[3 marks]

10.6.2 Solution to 10.3 Example #2 (Trigonometric Odd Powers)

$$\begin{aligned}
\int sinh^3 x\,dx &= \int sinh\,x\,sinh^2 x\,dx \\
&= \int sinh\,x\left(\,cosh^2 x \;-\; 1\,\right)\,x\,dx \\
&= \int sinh\,x\,cosh^2 x \;-\; sinh\,x\;\;dx \\
&= \frac{1}{3}\,cosh^3 x \;-\; cosh\,x \;+\; c
\end{aligned}$$

[3 marks]

10.6.3 Solution to 10.4 Example #3 (Substitution)

Setting up the substitution...

Differentiating:

$$x = 3\,sinh\,u \;\Rightarrow\; \frac{dx}{du} = 3\,cosh\,u$$

Limits:

$$u = arsinh\left(\frac{x}{3}\right)$$

When $x = 0$ then $u = 0$,

When $x = 6$ then $u = arsinh\,2$

Making the substitution...

$$\int_{x=0}^{x=6} \frac{x^3}{\sqrt{x^2 + 9}}\,dx = \int_{u=0}^{u=arsinh\,2} \frac{27\,sinh^3 u}{\sqrt{9\,sinh^2 u + 9}}\,3\,cosh\,u\,du$$

$$= \int_0^{arsinh\,2} \frac{27\,sinh^3 u\,cosh\,u}{\sqrt{sinh^2 u + 1}}\,du$$

$$= \int_0^{arsinh\,2} \frac{27\,sinh^3 u\,cosh\,u}{\sqrt{cosh^2 u}}\,du$$

$$= 27\int_0^{arsinh\,2} sinh^3 u\,du$$

Using the result from Example #2...

$$= 27\left[\frac{1}{3}\,cosh^3 u - cosh\,u\right]_0^{u\;=\;arsinh\,2}$$

Some clever manipulation of the limits...

From $u = arsinh\,2$ we can extract that $sinh\,u = 2$

Identity: $cosh^2 u - sinh^2 u = 1$

$$cosh^2 u = 1 + sinh^2 u$$

$\therefore$ when $sinh\,u = 2$, $cosh\,u = \sqrt{5}$

Picking up the main thread...

$$\int_{x=0}^{x=6} \frac{x^3}{\sqrt{x^2 + 9}}\,dx = 27\left[\frac{5\sqrt{5}}{3} - \sqrt{5} - \frac{1}{3} + 1\right]$$

$$= 27 \times \frac{1}{3}\left[5\sqrt{5} - 3\sqrt{5} - 1 + 3\right]$$

$$= 9\left[2\sqrt{5} + 2\right]$$

$$= 18\left(\sqrt{5} + 1\right)$$

[11 marks]

10.6.4 Solutions to 10.5 Exercise

Answer 1

The integral for the shaded area will be,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} cos^3 x\, dx &= \int_0^{\frac{\pi}{2}} cos\, x\, cos^2\, x\, dx \\
&= \int_0^{\frac{\pi}{2}} cos\, x \left(1 - sin^2\, x\right) dx \\
&= \int_0^{\frac{\pi}{2}} cos\, x\, dx - \int_0^{\frac{\pi}{2}} cos\, x\, sin^2\, x\, dx \\
&= \left[sin\, x - \frac{1}{3} sin^3\, x \right]_0^{\frac{\pi}{2}} \\
&= sin\left(\frac{\pi}{2}\right) - \frac{1}{3} sin^3\left(\frac{\pi}{2}\right) - sin\, 0 + \frac{1}{3} sin^3\, 0 \\
&= \frac{2}{3}
\end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned}
\int 25x^2\, cosh\,(5x)\; dx &= 5 \int x^2\; (\, 5\, cosh\,(5x)\;)\; dx \\
&= 5\left[x^2\, sinh\,(5x) - \int 2x\, sinh\,(5x)\; dx \right] \\
&= 5x^2\, sinh\,(5x) - 2 \int x\; (\, 5\, sinh\,(5x)\;)\; dx \\
&= 5x^2\, sinh\,(5x) - 2\left[x\, cosh\,(5x) - \int 1\, cosh\,(5x)\; dx \right] \\
&= 5x^2\, sinh\,(5x) - 2x\, cosh\,(5x) + \frac{2}{5} \int 5\, cosh\,(5x)\; dx \\
&= 5x^2\, sinh\,(5x) - 2x\, cosh\,(5x) + \frac{2}{5}\, sinh\,(5x) + c
\end{aligned}$$

[5 marks]

Answer 3

$$\begin{aligned}
\int cosh^3\, x\, dx &= \int cosh\, x\, cosh^2\, x\, dx \\
&= \int cosh\, x \left(1 + sinh^2\, x\right) dx \\
&= \int cosh\, x\, dx + \int cosh\, x\, sinh^2\, x\, dx \\
&= sinh\, x + \frac{1}{3} sinh^3\, x + c
\end{aligned}$$

[3 marks]

Answer 4

Setting up the substitution...

Differentiating:

$$x = 6\cosh u \Rightarrow \frac{dx}{du} = 6\sinh u$$

Limits:

$$u = \operatorname{arcosh}\left(\frac{x}{6}\right)$$

When $x = 12$ then $u = \operatorname{arcosh} 2$

When $x = 18$ then $u = \operatorname{arcosh} 3$

Making the substitution...

$$\int_{x=12}^{x=18} \frac{x^3}{\sqrt{x^2 - 36}}\,dx = \int_{u=\operatorname{arcosh} 2}^{u=\operatorname{arcosh} 3} \frac{216\cosh^3 u}{\sqrt{36\cosh^2 u - 36}}\,6\sinh u\,du$$

$$= \int_{\operatorname{arcosh} 2}^{\operatorname{arcosh} 3} \frac{216\cosh^3 u\sinh u}{\sqrt{\cosh^2 u - 1}}\,du$$

$$= \int_{\operatorname{arcosh} 2}^{\operatorname{arcosh} 3} \frac{216\cosh^3 u\sinh u}{\sqrt{\sinh^2 u}}\,du$$

$$= 216\int_{\operatorname{arcosh} 2}^{\operatorname{arcosh} 3} \cosh^3 u\,du$$

Using the result from Question 2...

$$= 216\left[\sinh u + \frac{1}{3}\sinh^3 u\right]_{\operatorname{arcosh} 2}^{\operatorname{arcosh} 3}$$

Some clever manipulation of the limits...

From $u = \operatorname{arcosh} 3$ we can extract that $\cosh u = 3$

and from $u = \operatorname{arcosh} 2$ we can extract that $\cosh u = 2$

Identity: $\cosh^2 u - \sinh^2 u = 1$

$$\sinh^2 u = 1 + \sinh^2 u - 1$$

$\therefore$ when $\cosh u = 3$, $\sinh u = 2\sqrt{2}$ and when $\cosh u = 2$, $\sinh u = \sqrt{3}$

Picking up the main thread...

$$\int_{x=12}^{x=18} \frac{x^3}{\sqrt{x^2 - 36}}\,dx = 216\left[2\sqrt{2} + \frac{16\sqrt{2}}{3} - \sqrt{3} - \frac{3\sqrt{3}}{3}\right]$$

$$= 72\left[6\sqrt{2} + 16\sqrt{2} - 3\sqrt{3} - 3\sqrt{3}\right]$$

$$= 72\left[22\sqrt{2} - 6\sqrt{3}\right]$$

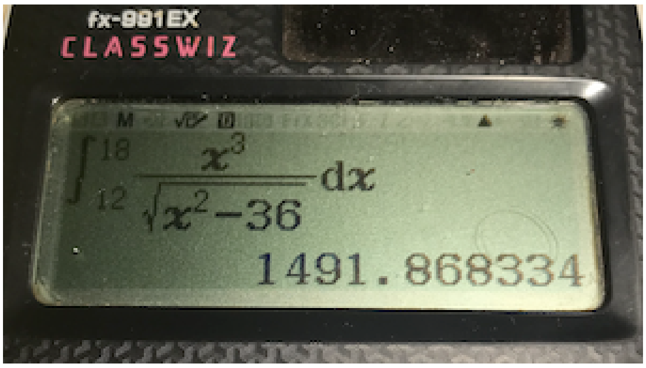
$$= 144\left(11\sqrt{2} - 3\sqrt{3}\right)$$

[11 marks]

Calculator check...

As a decimal $144\left(11\sqrt{2} - 3\sqrt{3}\right) = 1491.868334...$

Getting the calculator to do the integral directly...



Answer 5

(i) $\quad x = \sec\theta \Rightarrow \dfrac{dx}{d\theta} = \sec\theta\tan\theta$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\sqrt{\sec^2\theta - 1}}\,d\theta$$

$$= \int \frac{\tan\theta}{\sqrt{\tan^2\theta}}\,d\theta$$

$$= \int 1\,d\theta$$

$$= \theta + c$$

$$= \operatorname{arcsec} x + c, \text{ where } c \text{ is a constant}$$

[2 marks]

(ii) $\quad x = \sec\theta \Rightarrow \dfrac{dx}{d\theta} = \sec\theta\tan\theta$

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \times \sec\theta\tan\theta\,d\theta$$

$$= \int \sqrt{\tan^2\theta}\,\tan\theta\,d\theta$$

$$= \int \tan^2\theta\,d\theta$$

$$= \int \sec^2\theta - 1\,d\theta$$

$$= \tan\theta - \theta + c$$

$$= \sqrt{x^2-1} - \operatorname{arcsec} x + c, \text{ where } c \text{ is a constant}$$

[4 marks]

Lesson 11

Further A-Level Pure Mathematics, Core 2

Hyperbolic Functions

11.1 Revision

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Provided : Examination Mathematical Formulae Book

Question 1

Find $\int sin^5 x\, dx$

[3 marks]

Question 2

Given that $y = artanh\,(cos\, x)$ find a simplified expression for $\frac{dy}{dx}$

[3 marks]

Question 3

Use the substitution $x = 5\,sinh\,u$ to show that, for some constant c,

$$\int \sqrt{x^2 + 25}\,dx = \frac{1}{2}x\sqrt{x^2 + 25} + \frac{25}{2}\,arsinh\left(\frac{x}{5}\right) + c$$

[5 marks]

Question 4

FM Examination Question from June 2021, Paper 1, Q4 (AQA)

Show that the solutions to the equation,

$$3\tanh^2 x - 2\operatorname{sech} x = 2$$

can be expressed in the form,

$$x = \pm \ln\left(a + \sqrt{b} \right)$$

where a and b are integers to be found.

You may use without proof the result $\operatorname{arcosh} y = \ln\left\{ y + \sqrt{y^2 - 1} \right\}$

[5 marks]

Question 5

Given that $y = arcosh\, x$, show that for $x > 1$,

$$\left(x^2 - 1 \right) \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

[7 marks]

Question 6

(i) Use the substitution $x = cosh^2 u$ to show that, for $x > 1$,

$$\int \sqrt{\frac{x}{x - 1}}\, dx = \sqrt{x - 1}\ \sqrt{x}\ + arcosh\left(\sqrt{x} \right) + c$$

for some unknown constant, c

[6 marks]

(ii) Hence, or otherwise, determine the exact value of $\int_2^4 \sqrt{\frac{x}{x - 1}}\, dx$

You may use without proof the result $arcosh\, y = ln\left\{ y + \sqrt{y^2 - 1} \right\}$

[4 marks]

Question 7

Show that $\int_1^3 \frac{1}{\sqrt{3x^2 - 6x + 7}}\, dx = \frac{1}{\sqrt{3}} \ln\left(2 + \sqrt{3} \right)$

[7 marks]

11.2 Answers

Answer 1

$$
\begin{aligned}
\int \sin^5 x \, dx &= \int \sin x \sin^4 x \, dx \\
&= \int \sin x \left(\sin^2 x \right)^2 \, dx \\
&= \int \sin x \left(1 - \cos^2 x \right)^2 \, dx \\
&= \int \sin x \left(1 - 2\cos^2 x + \cos^4 x \right) dx \\
&= \int \sin x + 2(-\sin x)\cos^2 x - (-\sin x)\cos^4 x \, dx \\
&= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + c
\end{aligned}
$$

[3 marks]

Answer 2

$$y = \text{artanh}(\cos x)$$

Standard Result

$$\text{artanh}\, x \ \text{ differentiates to } \ \frac{1}{1 - x^2}$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1 - \cos^2 x} \times (-\sin x) \\
&= \frac{1}{\sin^2 x} \times (-\sin x) \\
&= -\frac{1}{\sin x} \\
&= -\csc x
\end{aligned}
$$

[3 marks]

Answer 3

$$\text{Preparation: } x = 5\,sinh\,u \iff \frac{x}{5} = sinh\,u$$

Use the identity

$$\begin{aligned}
cosh^2 u - sinh^2 u &= 1 \\
cosh^2 u &= 1 - sinh^2 u \\
cosh\,u &= \sqrt{1 + \left(\frac{x}{5}\right)^2} \quad \text{Positive as } cosh\,u \text{ is never negative}
\end{aligned}$$

$$x = 5\,sinh\,u \Rightarrow \frac{dx}{du} = 5\,cosh\,u$$

$$\begin{aligned}
\int \sqrt{x^2 + 25}\,dx &= \int \sqrt{25\,sinh^2 u + 25} \times 5\,cosh\,u\,\,du \\
&= \int 5\sqrt{sinh^2 u + 1} \times 5\,cosh\,u\,du \\
&= 25 \int \sqrt{cosh^2 u}\,\,cosh\,u\,du \\
&= 25 \int cosh^2 u\,du
\end{aligned}$$

$$\begin{aligned}
cosh^2 \theta + sinh^2 \theta &= cosh\,2\theta \\
\text{ADD}\quad cosh^2 \theta - sinh^2 \theta &= 1 \\
\hline
2\,cosh^2 \theta &= cosh\,2\theta + 1
\end{aligned}$$

$$\begin{aligned}
\int \sqrt{x^2 + 25}\,dx &= \frac{25}{2} \int cosh\,2u + 1\,du \\
&= \frac{25}{4} \int 2\,cosh\,2u + 2\,du \\
&= \frac{25}{4}\left(sinh\,2u + 2u\right) + c \\
&= \frac{25}{4}\left(2\,sinh\,u\,cosh\,u + 2u\right) + c \\
&= \frac{25}{4}\left(2\left(\frac{x}{5}\right)\sqrt{\left(\frac{x}{5}\right)^2 + 1} + 2\,arsinh\left(\frac{x}{5}\right)\right) + c \\
&= \frac{25}{4}\left(\frac{2x}{5}\sqrt{\frac{x^2 + 25}{25}} + 2\,arsinh\left(\frac{x}{5}\right)\right) + c \\
&= \frac{1}{2}x\sqrt{x^2 + 25} + \frac{25}{2}\,arsinh\left(\frac{x}{5}\right) + c
\end{aligned}$$

[5 marks]

Answer 4

First to establish a useful identity,

$$\cosh^2\theta - \sinh^2\theta = 1$$

Divide through by $\cosh^2\theta$

$$\frac{\cosh^2\theta}{\cosh^2\theta} - \frac{\sinh^2\theta}{\cosh^2\theta} = \frac{1}{\cosh^2\theta}$$

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta$$

$$\tanh^2\theta = 1 - \operatorname{sech}^2\theta$$

$$3\tanh^2 x - 2\operatorname{sech} x = 2$$

$$3\left(1 - \operatorname{sech}^2 x\right) - 2\operatorname{sech} x = 2$$

$$3 - 3\operatorname{sech}^2 x - 2\operatorname{sech} x = 2$$

$$3\operatorname{sech}^2 x + 2\operatorname{sech} x - 1 = 0$$

$$(3\operatorname{sech} x - 1)(\operatorname{sech} x + 1) = 0$$

Either $\operatorname{sech} x = \frac{1}{3}$ or $\operatorname{sech} x = -1$

$$\frac{1}{\cosh x} = \frac{1}{3} \text{ or } \frac{1}{\cosh x} = -1$$

$$\cosh x = 3 \text{ or } \cosh x = -1$$

Thinking of the graph of $y = \cosh x$ it is always positive and so the equation $\cosh x = -1$ has no solutions whereas $\cosh x = 3$ will have two solutions given by $x = \pm \operatorname{arcosh} 3$

$$x = \pm \ln\left(3 + \sqrt{3^2 - 1}\right)$$

$$= \pm \ln\left(3 + \sqrt{8}\right)$$

That is, $a = 3$ and $b = 8$

[5 marks]

Answer 5

$$y = arcosh\, x$$

Standard Result

$$arcosh\, x \text{ differentiates to } \frac{1}{\sqrt{x^2 - 1}} \quad x > 1$$

$$\frac{dy}{dx} = \left(x^2 - 1\right)^{-\frac{1}{2}}$$

$$= \frac{1}{(x^2 - 1)^{\frac{1}{2}}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\left(x^2 - 1\right)^{-\frac{3}{2}} 2x$$

$$= -x\left(x^2 - 1\right)^{-\frac{3}{2}}$$

$$= -\frac{x}{(x^2 - 1)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = (-x) \times \left(-\frac{3}{2}\right)\left(x^2 - 1\right)^{-\frac{5}{2}} 2x + (-1)\left(x^2 - 1\right)^{-\frac{3}{2}}$$

$$= \frac{3x^2}{(x^2 - 1)^{\frac{5}{2}}} - \frac{1}{(x^2 - 1)^{\frac{3}{2}}}$$

$$\text{LHS} = \left(x^2 - 1\right)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx}$$

$$= \left(\frac{3x^2\left(x^2 - 1\right)}{(x^2 - 1)^{\frac{5}{2}}} - \frac{1 \times \left(x^2 - 1\right)}{(x^2 - 1)^{\frac{3}{2}}}\right) + \left(-\frac{3x^2}{(x^2 - 1)^{\frac{3}{2}}}\right) + \frac{1}{(x^2 - 1)^{\frac{1}{2}}}$$

$$= \frac{3x^2}{(x^2 - 1)^{\frac{3}{2}}} - \frac{1}{(x^2 - 1)^{\frac{1}{2}}} - \frac{3x^2}{(x^2 - 1)^{\frac{3}{2}}} + \frac{1}{(x^2 - 1)^{\frac{1}{2}}}$$

$$= 0$$

$$= \text{RHS} \qquad \square$$

[7 marks]

Answer 6

(i) $x = cosh^2 u \Rightarrow \dfrac{dx}{du} = 2\, cosh\, u\, sinh\, u$

$$\int \sqrt{\frac{x}{x-1}}\, dx = \int \sqrt{\frac{cosh^2 u}{cosh^2 u - 1}} \times 2\, cosh\, u\, sinh\, u\ du$$
$$= \int \sqrt{\frac{cosh^2 u}{sinh^2 u}} \times 2\, cosh\, u\, sinh\, u\ du$$
$$= \int \frac{cosh\, u}{sinh\, u} \times 2\, cosh\, u\, sinh\, u\ du$$
$$= \int 2\, cosh^2 u\, du$$

$$cosh^2 \theta + sinh^2 \theta = cosh\, 2\theta$$
$$\text{ADD} \quad cosh^2 \theta - sinh^2 \theta = 1$$

$$2\, cosh^2 \theta = cosh\, 2\theta + 1$$

$$\int \sqrt{\frac{x}{x-1}}\, dx = \int cosh\, 2u + 1\, du$$
$$= \frac{1}{2}\int 2\, cosh\, 2u\, du + \int 1\, du$$
$$= \frac{1}{2}\, sinh\, 2u + u + c$$
$$= sinh\, u\, cosh\, u + u + c$$
$$= \sqrt{cosh^2 u - 1}\ cosh\, u + u + c$$
$$= \sqrt{x-1}\,\sqrt{x} + arcosh\left(\sqrt{x}\right) + c$$

[6 marks]

(ii)
$$\int_2^4 \sqrt{\frac{x}{x-1}}\, dx = \left[\ \sqrt{x-1}\,\sqrt{x} + arcosh\left(\sqrt{x}\right)\ \right]_2^4$$
$$= \left[\ \sqrt{x-1}\,\sqrt{x} + ln\left\{\sqrt{x} + \sqrt{x-1}\right\}\ \right]$$
$$= \sqrt{4-1}\,\sqrt{4} + ln\left\{2 + \sqrt{4-1}\right\}$$
$$- \sqrt{2-1}\,\sqrt{2} - ln\left\{\sqrt{2} + \sqrt{2-1}\right\}$$
$$= 2\sqrt{3} + ln\left\{2 + \sqrt{3}\right\} - \sqrt{2} - ln\left\{\sqrt{2} + 1\right\}$$
$$= 2\sqrt{3} - \sqrt{2} + ln\left\{\frac{2+\sqrt{3}}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}\right\}$$
$$= 2\sqrt{3} - \sqrt{2} + ln\left\{\left(2+\sqrt{3}\right)\left(\sqrt{2}-1\right)\right\}$$

[4 marks]

Answer 7

$$\int_1^3 \frac{1}{\sqrt{3x^2 - 6x + 7}}\,dx = \int_1^3 \frac{1}{\sqrt{3\,(x^2 - 2x) + 7}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{3\,(\,(x-1)^2 - 1\,) + 7}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{3(x-1)^2 - 3 + 7}}\,dx$$

$$= \int_1^3 \frac{1}{\sqrt{(\sqrt{3})^2(x-1)^2 + 4}}\,dx$$

$$= \frac{1}{\sqrt{3}} \int_1^3 \frac{\sqrt{3}}{\sqrt{(\sqrt{3}\,x - \sqrt{3})^2 + 2^2}}\,dx$$

Standard Result

$$\frac{1}{\sqrt{x^2 + a^2}} \text{ integrates to } arsinh\left(\frac{x}{a}\right),\; ln\left\{ x + \sqrt{x^2 + a^2} \right\}$$

$$= \frac{1}{\sqrt{3}}\left[ln\left\{ (\sqrt{3}\,x - \sqrt{3}) + \sqrt{(\sqrt{3}\,x - \sqrt{3})^2 + 2^2} \right\} \right]_1^3$$

$$= \frac{1}{\sqrt{3}}\left[ln\left\{ 2\sqrt{3} + \sqrt{12 + 4} \right\} - ln\left\{ 0 + \sqrt{0^2 + 4} \right\} \right]$$

$$= \frac{1}{\sqrt{3}}\left[ln\left\{ 2\sqrt{3} + 4 \right\} - ln\left\{ 2 \right\} \right]$$

$$= \frac{1}{\sqrt{3}}\left[ln\left\{ 2 + \sqrt{3} \right\} \right]$$

[7 marks]

Lesson 12

Further A-Level Pure Mathematics, Core 2
Hyperbolic Functions

12.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Provided : Examination Mathematical Formulae Book

Question 1

Find $\int cos^5 x \, dx$

[3 marks]

Question 2

Given that $y = artanh\,(sin\,x)$ find a simplified expression for $\frac{dy}{dx}$

[3 marks]

Question 3

Use the substitution $x = 2\cosh u$ to show that, for some constant c,

$$\int \sqrt{x^2 - 4}\,dx = \frac{1}{2}x\sqrt{x^2 - 4} - 2\,\mathrm{arcosh}\left(\frac{x}{2}\right) + c$$

[5 marks]

Question 4

Show that the solutions to the equation,

$$2\,tanh^2 x \;-\; sech\,x \;=\; 1$$

can be expressed in the form,

$$x \;=\; \pm\, ln\left(a \;+\; \sqrt{b}\,\right)$$

where *a* and *b* are integers to be found.

You may use without proof the result $arcosh\,y \;=\; ln\left\{ y \;+\; \sqrt{y^2 \;-\; 1}\,\right\}$

[5 marks]

Question 5

Given that $y = arsinh\, x$, show that,

$$\left(x^2 + 1 \right) \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + 1 = 0$$

[7 marks]

Question 6

(i) Use the substitution $x = sinh^2 u$ to show that, for $x \geqslant 0$,

$$\int \sqrt{\frac{x}{x+1}}\, dx = \sqrt{x}\ \sqrt{x+1} - arsinh(\sqrt{x}) + c$$

for some unknown constant, c

[6 marks]

(ii) Hence, or otherwise, determine the exact value of $\int_0^1 \sqrt{\frac{x}{x+1}}\, dx$

[4 marks]

Question 7

FM A-Level Examination Question from June 2015, Paper FP3, Q4 (Edexcel)

The curve C has equation $y = \dfrac{1}{\sqrt{x^2 + 2x - 3}}, \ x > 1$

(a) Find $\int y \, dx$

[3 marks]

The region R is bounded by the curve C, the x-axis and the lines with equations $x = 2$ and $x = 3$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. Give your answer in the form $p\pi \ln q$, where p and q are rational numbers to be found.

[4 marks]

12.2 Answers

Answer 1

$$\begin{aligned}
\int \cos^5 x \, dx &= \int \cos x \cos^4 x \, dx \\
&= \int \cos x \left(\cos^2 x \right)^2 \, dx \\
&= \int \cos x \left(1 - \sin^2 x \right)^2 \, dx \\
&= \int \cos x \left(1 - 2 \sin^2 x + \sin^4 x \right) \, dx \\
&= \int \cos x - 2 \cos x \sin^2 x + \cos x \sin^4 x \, dx \\
&= \sin x + \frac{2 \sin^3 x}{3} + \frac{\sin^5 x}{5} + c
\end{aligned}$$

[3 marks]

Answer 2

$$y = \operatorname{artanh}(\sin x)$$

Standard Result

$$\operatorname{artanh} x \text{ differentiates to } \frac{1}{1 - x^2}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{1 - \sin^2 x} \times (\cos x) \\
&= \frac{1}{\cos^2 x} \times (\cos x) \\
&= \frac{1}{\cos x} \\
&= \sec x
\end{aligned}$$

[3 marks]

Answer 3

$$x = 2\cosh u \Rightarrow \frac{dx}{du} = 2\sinh u$$

$$\begin{aligned}
\int \sqrt{x^2 - 4}\, dx &= \int \sqrt{4\cosh^2 u - 4} \times 2\sinh u \; du \\
&= \int 2\sqrt{\cosh^2 u - 1} \times 2\sinh u\, du \\
&= 4 \int \sqrt{\sinh^2 u}\; \sinh u\, du \\
&= 4 \int \sinh^2 u\, du
\end{aligned}$$

$$\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$$

SUBTRACT $\cosh^2\theta - \sinh^2\theta = 1$

$$2\sinh^2\theta = \cosh 2\theta - 1$$

$$\begin{aligned}
\int \sqrt{x^2 - 4}\, dx &= 2\int \cosh 2u - 1\, du \\
&= \int 2\cosh 2u - 2\, du \\
&= \sinh 2u - 2u + c \\
&= 2\sinh u \cosh u - 2u + c \\
&= 2\sqrt{\left(\frac{x}{2}\right)^2 - 1}\left(\frac{x}{2}\right) - 2\,\mathrm{arcosh}\left(\frac{x}{2}\right) + c \\
&= \frac{1}{2}x\sqrt{x^2 - 4} - 2\,\mathrm{arcosh}\left(\frac{x}{2}\right) + c
\end{aligned}$$

[5 marks]

Answer 4

First to establish a useful identity,

$$\cosh^2 \theta - \sinh^2 \theta = 1$$

Divide through by $\cosh^2 \theta$

$$\frac{\cosh^2 \theta}{\cosh^2 \theta} - \frac{\sinh^2 \theta}{\cosh^2 \theta} = \frac{1}{\cosh^2 \theta}$$

$$1 - \tanh^2 \theta = \operatorname{sech}^2 \theta$$

$$\tanh^2 \theta = 1 - \operatorname{sech}^2 \theta$$

$$2 \tanh^2 x - \operatorname{sech} x = 1$$

$$2\left(1 - \operatorname{sech}^2 x \right) - \operatorname{sech} x = 1$$

$$2 - 2 \operatorname{sech}^2 x - \operatorname{sech} x = 1$$

$$2 \operatorname{sech}^2 x + \operatorname{sech} x - 1 = 0$$

$$(2 \operatorname{sech} x - 1)(\operatorname{sech} x + 1) = 0$$

Either $\operatorname{sech} x = \frac{1}{2}$ or $\operatorname{sech} x = -1$

$$\frac{1}{\cosh x} = \frac{1}{2} \quad \text{or} \quad \frac{1}{\cosh x} = -1$$

$$\cosh x = 2 \quad \text{or} \quad \cosh x = -1$$

Thinking of the graph of $y = \cosh x$, it is always positive and so the equation $\cosh x = -1$ has no solutions whereas $\cosh x = 2$ will have two solutions given by $x = \pm \operatorname{arcosh} 2$

$$x = \pm \ln\left(2 + \sqrt{2^2 - 1} \right)$$

$$= \pm \ln\left(2 + \sqrt{3} \right)$$

That is, $a = 2$ and $b = 3$

[5 marks]

Answer 5

$$y = arsinh\, x$$

Standard Result

$$arsinh\, x \text{ differentiates to } \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{dy}{dx} = \left(x^2 + 1\right)^{-\frac{1}{2}}$$

$$= \frac{1}{(x^2 + 1)^{\frac{1}{2}}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\left(x^2 + 1\right)^{-\frac{3}{2}} 2x$$

$$= -x\left(x^2 + 1\right)^{-\frac{3}{2}}$$

$$= -\frac{x}{(x^2 + 1)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = (-x) \times \left(-\frac{3}{2}\right)\left(x^2 + 1\right)^{-\frac{5}{2}} 2x + (-1)\left(x^2 + 1\right)^{-\frac{3}{2}}$$

$$= \frac{3x^2}{(x^2 + 1)^{\frac{5}{2}}} - \frac{1}{(x^2 + 1)^{\frac{3}{2}}}$$

$$\text{LHS} = \left(x^2 + 1\right)\frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx}$$

$$= \left(\frac{3x^2\left(x^2 + 1\right)}{(x^2 + 1)^{\frac{5}{2}}} - \frac{1 \times \left(x^2 + 1\right)}{(x^2 + 1)^{\frac{3}{2}}}\right) + \left(-\frac{3x^2}{(x^2 + 1)^{\frac{3}{2}}}\right) + \frac{1}{(x^2 + 1)^{\frac{1}{2}}}$$

$$= \frac{3x^2}{(x^2 + 1)^{\frac{3}{2}}} - \frac{1}{(x^2 + 1)^{\frac{1}{2}}} - \frac{3x^2}{(x^2 + 1)^{\frac{3}{2}}} + \frac{1}{(x^2 + 1)^{\frac{1}{2}}}$$

$$= 0$$

$$= \text{RHS} \qquad \square$$

[7 marks]

Answer 6

(i) $x = sinh^2 u \Rightarrow \dfrac{dx}{du} = 2\, sinh\, u\, cosh\, u$

$$\int \sqrt{\frac{x}{x+1}}\, dx = \int \sqrt{\frac{sinh^2 u}{sinh^2 u + 1}} \times 2\, sinh\, u\, cosh\, u \;\; du$$
$$= \int \sqrt{\frac{sinh^2 u}{cosh^2 u}} \times 2\, sinh\, u\, cosh\, u \;\; du$$
$$= \int \frac{sinh\, u}{cosh\, u} \times 2\, cosh\, u\, sinh\, u \;\; du$$
$$= \int 2\, sinh^2 u\, du$$

$$cosh^2 \theta + sinh^2 \theta = cosh\, 2\theta$$

SUBTRACT $\quad cosh^2 \theta - sinh^2 \theta = 1$

$$2\, sinh^2 \theta = cosh\, 2\theta - 1$$

$$\int \sqrt{\frac{x}{x+1}}\, dx = \int cosh\, 2u - 1\, du$$
$$= \frac{1}{2} \int 2\, cosh\, 2u\, du - \int 1\, du$$
$$= \frac{1}{2}\, sinh\, 2u - u + c$$
$$= sinh\, u\, cosh\, u - u + c$$
$$= sinh\, u \sqrt{sinh^2 u + 1} - u + c$$
$$= \sqrt{x}\sqrt{x+1} - arsinh\left(\sqrt{x}\right) + c$$

[6 marks]

(ii) $$\int_0^1 \sqrt{\frac{x}{x+1}}\, dx = \left[\sqrt{x}\sqrt{x+1} - arsinh\left(\sqrt{x}\right) \right]_0^1$$
$$= \left[\sqrt{x}\sqrt{x+1}\sqrt{x} - ln\left\{ \sqrt{x} + \sqrt{x+1} \right\} \right]$$
$$= \sqrt{1}\sqrt{1+1} - ln\left\{ \sqrt{1} + \sqrt{1+1} \right\}$$
$$- \sqrt{0}\sqrt{0+1} + ln\left\{ \sqrt{0} + \sqrt{0+1} \right\}$$
$$= \sqrt{2} - ln\left\{ 1 + \sqrt{2} \right\} - 0 + ln\left\{ 1 \right\}$$
$$= \sqrt{2} - ln\left\{ 1 + \sqrt{2} \right\}$$

[4 marks]

Answer 7

(i) $\int y\,dx = \int \frac{1}{\sqrt{x^2 + 2x - 3}}\,dx$

$$= \int \frac{1}{\sqrt{(x^2 + 2x) - 3}}\,dx$$

$$= \int \frac{1}{\sqrt{((x+1)^2 - 1) - 3}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2 - 2^2}}\,dx$$

Standard Result

$\frac{1}{\sqrt{x^2 - a^2}}$ integrates to $arcosh\left(\frac{x}{a}\right),\ x > a$

$= arcosh\left(\frac{x+1}{2}\right),\quad x > 2$ $\Leftarrow$ Full marks from here on

$$= ln\left\{x + 1 + \sqrt{(x+1)^2 - 2^2}\right\}$$

$$= ln\left\{x + 1 + \sqrt{x^2 + 2x - 3}\right\},\ x > 2$$

[3 marks]

(ii) $$V = \pi \int_2^3 y^2 \, dx$$

$$= \pi \int_2^3 \frac{1}{x^2 + 2x - 3} \, dx$$

$$= \pi \int_2^3 \frac{1}{(x + 1)^2 - 2^2} \, dx$$

Standard Result

$$\frac{1}{x^2 - a^2} \text{ integrates to } \frac{1}{2a} \, ln \left| \frac{x - a}{x + a} \right|$$

$$= \frac{\pi}{4} \, ln \left| \frac{x + 1 - 2}{x + 1 + 2} \right|_2^3$$

$$= \frac{\pi}{4} \, ln \left| \frac{x - 1}{x + 3} \right|_2^3$$

$$= \frac{\pi}{4} \left[\, ln \left| \frac{2}{6} \right| - ln \left| \frac{1}{5} \right| \, \right]$$

$$= \frac{\pi}{4} \, ln \left| \frac{1 \times 5}{3 \times 1} \right|$$

$$= \frac{\pi}{4} \, ln \left(\frac{5}{3} \right)$$

$$\therefore p = \frac{1}{4} \text{ and } q = \frac{5}{3}$$

[4 marks]