Year 2 Pure Mathematics Examination Revision
Health Check N° 10

Answer 1

(i)

$$\begin{aligned} log_4 x + 2\, log_4 y - log_4 (xy) &= log_4 x + log_4 y^2 - log_4 (xy) \\ &= log_4 \left(\frac{x\, y^2}{xy} \right) \\ &= log_4 y \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} log_4 x + 2\, log_4 y - log_4 (xy) &= 7 \\ log_4 y &= 7 \\ \therefore\ y &= 4^7 \\ &= 16384 \end{aligned}$$

[2 marks]

Answer 2

(i)

$$\begin{aligned} f\,(ln\, 3) &= \frac{e^{ln\, 3} + 1}{e^{ln\, 3} - 1} \\ &= \frac{3 + 1}{3 - 1} \Rightarrow f\,(ln\, 3) = 2 \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} f\,(x) &= \frac{e^x + 1}{e^x - 1} \\ x &= \frac{e^y + 1}{e^y - 1} \\ x\,(\, e^y - 1\,) &= e^y + 1 \\ x\, e^y - x &= e^y + 1 \\ x\, e^y - e^y &= x + 1 \\ e^y\,(\, x - 1\,) &= x + 1 \\ e^y &= \frac{x + 1}{x - 1} \\ y &= ln\left(\frac{x + 1}{x - 1} \right) \\ f^{\,-1}(x) &= ln\left(\frac{x + 1}{x - 1} \right) \end{aligned}$$

[4 marks]

(iii) With help from the graph*, $f\,(x) \in \mathbb{R}, \quad x < -1, \quad x > 1$

[2 marks]

* Starting with $ln\left(\dfrac{x+1}{x-1}\right)$ and using the restriction imposed by the *ln* function, that $\dfrac{x+1}{x-1} > 0$, is tricky as "rational inequalities" are not covered by the A-Level course.
Essentially, you need to identify the critical values, which are the values of *x* that make the denominator equal to zero, as you would expect, and **the values of *x* that make the numerator equal zero**.
This gives ± 1 as the critical values.
Each of the possible regions are then tested with a sample value of *x*.

Sample $x = -2 \Rightarrow \left(\dfrac{-2+1}{-2-1}\right) = \dfrac{1}{3} > 0 \;\therefore$ TRUE

Sample $x = 0 \Rightarrow \left(\dfrac{0+1}{0-1}\right) = -1 < 0 \;\therefore$ FALSE

Sample $x = 2 \Rightarrow \left(\dfrac{2+1}{2-1}\right) = 3 > 0 \;\therefore$ TRUE

$$\therefore \; f(x) \in \mathbb{R},\; f(x) < -1,\; f(x) > 1$$

Answer 3

(i) From the graph, the number of solutions, given the domain of *x*, is 4

[1 mark]

(ii)

$$\begin{aligned} g(x) &= f(x) \\ sin(2x - 30) &= sin\,x \\ arcsin(sin(2x-30)) &= arcsin(sin\,x) \\ 2x - 30 &= x,\; 180 - x, \\ &\quad x + 360,\; 180 - x + 360, \\ &\quad x + 360 + 360,\; 180 - x + 360 + 360 \\ &= x,\; 180 - x, \qquad \text{case 1, case 2} \\ &\quad x + 360,\; 540 - x \qquad \text{case 3, case 4} \\ &\quad x + 720,\; 900 - x \qquad \text{case 5, case 6} \end{aligned}$$

Case 1 : $2x - 30 = x \Rightarrow x = 30°$

Case 2 : $2x - 30 = 180 - x \Rightarrow 3x = 210 \Rightarrow x = 70°$

Case 3 : $2x - 30 = x + 360 \Rightarrow x = 390°$ Too big - REJECT !

Case 4 : $2x - 30 = 540 - x \Rightarrow 3x = 570 \Rightarrow x = 190°$

Case 5 : $2x - 30 = x + 720 \Rightarrow x = 750°$ Too big - REJECT !

Case 6 : $2x - 30 = 900 - x \Rightarrow 3x = 930 \Rightarrow x = 310°$

$$\therefore \; x \in \{30°,\; 70°,\; 190°,\; 310°\}$$

[5 marks]

Answer 4

(i)

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

$$= \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -10 \\ 8 \end{pmatrix} \qquad (\text{i.e. } 6\boldsymbol{i} - 10\boldsymbol{j} + 8\boldsymbol{k})$$

[2 marks]

(ii) Part (i) is a subtle hint that $\overrightarrow{AB}$ is one of the parallel sides, if a kind person made up the question !

Examining $\overrightarrow{DC}$ $\left(\text{or } \overrightarrow{CD}\right)$ is thus a tentative next step.

$$\overrightarrow{DC} = \boldsymbol{c} - \boldsymbol{d}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix} \qquad (\text{i.e. } -3\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k})$$

Observe that $\overrightarrow{AB} = (-2)\ \overrightarrow{DC}$

From the wording of the question, A, B, C and D are not collinear.

The inescapable conclusion is that $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

$\therefore$ $ABCDA$ is a trapezium ☐

[3 marks]

(iii) Strategy: show that $\overrightarrow{BC}$ is not parallel to $\overrightarrow{DA}$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} \qquad \overrightarrow{DA} = \boldsymbol{a} - \boldsymbol{d}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix} \qquad = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 3 \\ -8 \end{pmatrix} \qquad = \begin{pmatrix} -5 \\ 12 \\ -4 \end{pmatrix}$$

Clearly, there is no value of λ such that $\overrightarrow{BC} = \lambda\left(\overrightarrow{DA}\right)$

$\therefore$ $ABCDA$ is not a parallelogram ☐

[3 marks]

Answer 5

Firstly, when $y = \frac{\pi}{3}$, we have $x^2 \cos\left(\frac{\pi}{3}\right) = 1$

$$x^2 \times \frac{1}{2} = 1$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

From the graph, the point at which the tangent is to be found is $\left(\sqrt{2}, \frac{\pi}{3}\right)$

$$x^2 \cos y = 1$$

Differentiate implicitly with respect to x

$$x^2 \left(-\sin y\right) \frac{dy}{dx} + 2x \cos y = 0$$

$$x^2 \sin y \frac{dy}{dx} = 2x \cos y$$

$$\frac{dy}{dx} = \frac{2x \cos y}{x^2 \sin y} \qquad \text{as } x \neq 0$$

$$= \frac{2}{x \tan y}$$

$$= \frac{2}{\sqrt{2} \tan\left(\frac{\pi}{3}\right)}$$

$$= \frac{2}{\sqrt{2}\sqrt{3}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{2\sqrt{6}}{6}$$

$$= \frac{\sqrt{6}}{3}$$

$$y = mx + c \Rightarrow \frac{\pi}{3} = \frac{\sqrt{6}}{3} \times \sqrt{2} + c$$

$$c = \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

$$\therefore \text{ Tangent is } y = \frac{\sqrt{6}}{3} x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

[8 marks]