Year 2 Pure Mathematics Examination Revision
Health Check N° 5

Answer 1

(a) To have no real roots, the discriminant must be negative.

$$D < 0$$

$$b^2 - 4ac < 0$$

$$(-4)^2 - 4 \times (k - 4) \times (k - 2) < 0$$

$$16 - 4\left(k^2 - 6k + 8\right) < 0$$

$$16 - 4k^2 + 24k - 32 < 0$$

$$4k^2 - 24k + 16 > 0$$

Divide through by 4

$$k^2 - 6k + 4 > 0 \qquad \square$$

[3 marks]

(b) To find the critical values, solve $k^2 - 6k + 4 = 0$

$$k = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 4}}{2 \times 1}$$

$$= \frac{6 \pm \sqrt{20}}{2}$$

$$= 3 \pm \sqrt{5}$$

$\therefore$ The possible range of values for k satisfy:

$$k < 3 - \sqrt{5} \quad \text{or} \quad k > 3 + \sqrt{5}$$

[4 marks]

Answer 2

(a) The situation is that of a Geometric Progression,

$$2100,\ 2100 \times 1.012,\ 2100 \times 1.012^2,\ \dots,\ 2100 \times 1.012^{13}$$

$$2017,\qquad 2018,\qquad 2019,\ \dots,\qquad 2030$$

$$n = 1,\qquad n = 2,\qquad n = 3,\qquad n = 14$$

$$Sum_{GP} = \frac{2100\left(1.012^{14} - 1\right)}{1.012 - 1}$$

$$= 31\,800 \text{ tonnes}$$

[3 marks]

(b) The total cost of harvesting the first 2000 tonnes of wheat each year

$$14 \times 2000 \times 5.15 = £144\,200$$

The amount of wheat harvested at £6.45 per tonne will be...

$$2100 - 2000$$

$$+\ 2100 \times 1.012 - 2000$$

$$+\ 2100 \times 1.012^2 - 2000$$

$$+\ \dots$$

But this is the sequence summed in part (a) less 14 lots of 2000

Total cost of harvesting the wheat at £6.45 will be...

$$(31\,800 - 14 \times 2000) \times 6.45 = 3800 \times 6.45$$

$$= £24\,510$$

Combined total is thus,

$$144\,200 + 24\,510 = £168\,710$$

$$= £169\,000 \text{ to the nearest } £1000$$

[3 marks]

Answer 3

(a)

$$f(x) = \frac{3x - 5}{x + 1}$$

$$y = \frac{3x - 5}{x + 1}$$

swap the inputs and the outputs

$$x = \frac{3y - 5}{y + 1}$$

$$x(y + 1) = 3y - 5$$

$$xy + x = 3y - 5$$

$$x + 5 = 3y - xy$$

$$x + 5 = y(3 - x)$$

$$y = \frac{x + 5}{3 - x}$$

$$f^{-1}(x) = \frac{x + 5}{3 - x} \qquad x \in \mathbb{R},\ x \neq 3$$

[3 marks]

(b)

$$ff(x) = f\left(\frac{3x - 5}{x + 1}\right)$$

$$= \frac{3 \times \frac{3x-5}{x+1} - 5}{\frac{3x-5}{x+1} + 1} \times \frac{x + 1}{x + 1}$$

$$= \frac{3(3x - 5) - 5(x + 1)}{3x - 5 + x + 1}$$

$$= \frac{9x - 15 - 5x - 5}{4x - 4}$$

$$= \frac{4x - 20}{4x - 4}$$

$$= \frac{x - 5}{x - 1}$$

$$\therefore\ a = -5$$

[4 marks]

(c)

$$fg(2) = f\left(2^2 - 3 \times 2\right)$$

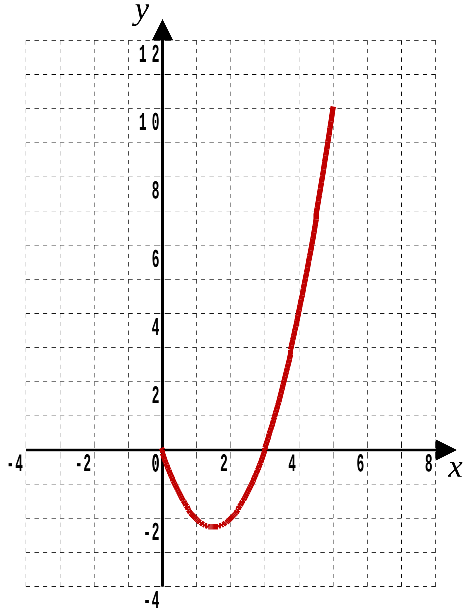
$$= f(-2)$$

$$= \frac{3(-2) - 5}{(-2) + 1}$$

$$= 11$$

[2 marks]

(d) Plotting the graph is the easiest way to get a feel for what's going on...



To find the minimum value you could complete the square or, as here, use differentiation;

$$g(x) = x^2 - 3x$$

$$g'(x) = 2x - 3$$

$$2x - 3 = 0 \text{ when } x = \frac{3}{2} \text{ in which case...}$$

$$g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right)$$

$$= \frac{9}{4} - \frac{9}{2}$$

$$= -\frac{9}{4}$$

$$\therefore\ g \text{ has the range } -\frac{9}{4} \leqslant g(x) \leqslant 10$$

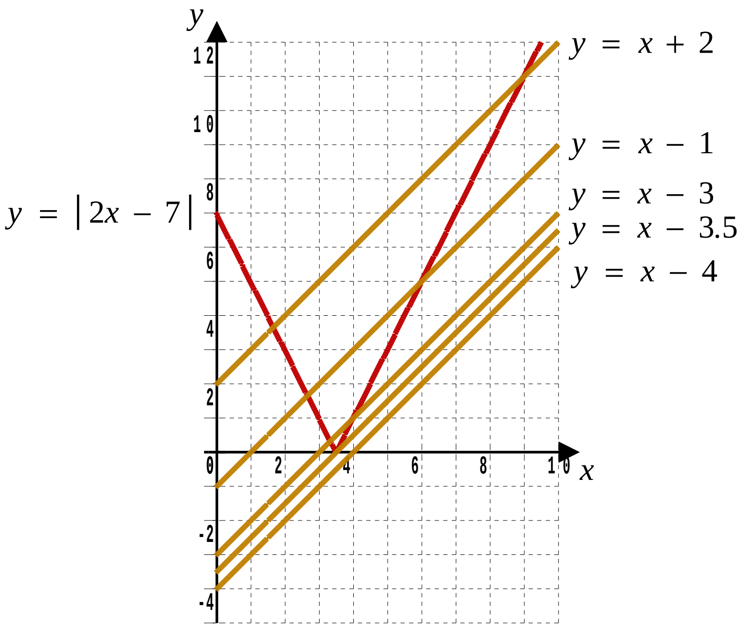
[3 marks]

(e) Only a function that is one-to-one has an inverse.
As g is not one-to-one it does not have an inverse

[1 mark]

Answer 4

The equation can be interpreted as being where the graphs of $y = |2x - 7|$ and $y = x + k$ intersect. Further, it can be noted that $y = x + k$ is the form of a straight line of gradient 1 that crosses the y-axis at $(0, k)$.



A key point is where the vertex of the graph of $y = |2x - 7|$ is on the x-axis. This will be where $y = 0$.

$$y = 0$$
$$|2x - 7| = 0$$
$$(2x - 7) = 0 \quad \text{or} \quad -(2x - 7) = 0$$
$$\therefore\ x = 3.5 \text{ (both 'branches' give the same answer)}$$

The straight line of the form $y = x + k$ through this key point $(3.5, 0)$will be...

$$0 = 3.5 + k \Rightarrow k = -3.5$$

$\therefore$ The critical line between there being two solutions, and there being no solutions is $y = x - 3.5$

So if $k > -3.5$ there will be two distinct points of intersection, corresponding to two distinct solutions to the equation $|2x - 7| = x + k$

[4 marks]