Year 2 Pure Mathematics Examination Revision : Health Check N° 1

How well is your maths today ?

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

Solve the equation $sin\left(x + 30° \right) = 2\,cos\,x$, for $0 \leqslant x \leqslant 360°$
Give your answers as exact values.

[5 marks]

Question 2

$f(x) = 4x^3 + ax^2 + 20x + 4$ where a is an integer constant.

Given that $(4x + 1)$ is a factor of $f(x)$ find the value of a

[5 marks]

Question 3

(i) Given that, $x \in \mathbb{R}$ and $g(x) = x^3 - 12x^2 + 50x - 7$, find $g'(x)$

[2 marks]

(ii) By completing the square, show that $g(x)$ is an increasing function

[5 marks]

Question 4

Squirrels were introduced into a forest. The number of squirrels, S, in the forest T years after they were introduced is modelled by the equation

$$S = \frac{390}{1 + e^{4 - 1.4T}} \qquad T \in \mathbb{R},\ T \geqslant 0$$

(i) How many squirrels were initially introduced into the forest ?

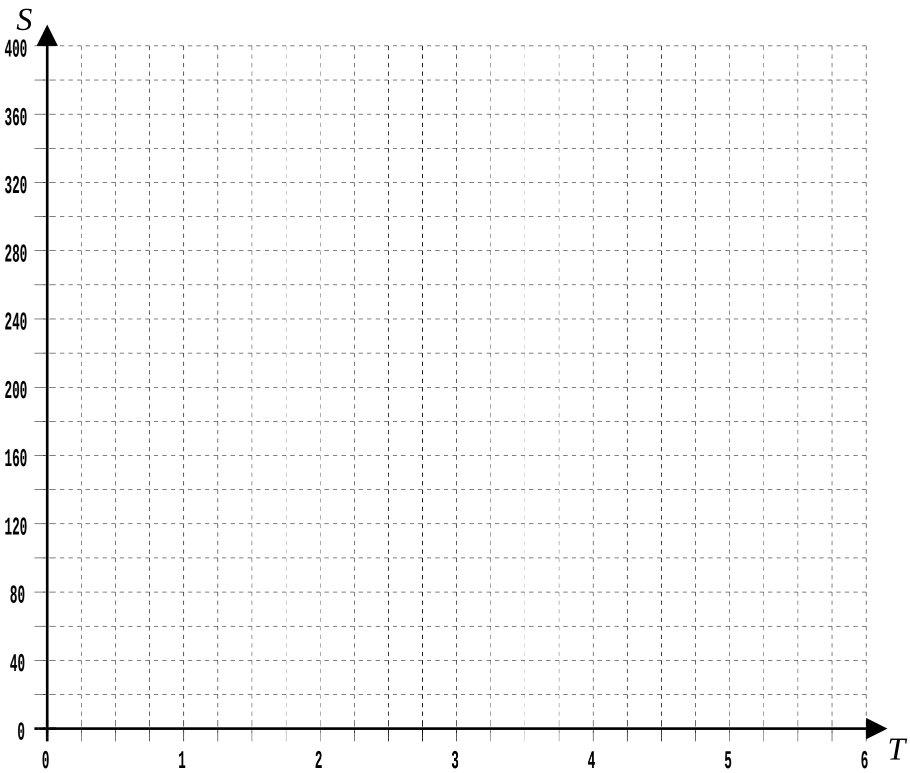
[1 mark]

(ii) Complete the following table to show how the model suggests the squirrel population will grow over the subsequent 6 years.

T (years)	0	1	2	3	4	5	6
S (N° of Squirrels)							

[2 marks]

(iii) On the graph below plot the squirrel's population growth curve



[2 marks]

(iv) George claims that, looking further into the future, the model predicts that the squirrel population will grow exponentially, because the equation contains an exponential.

State if you agree with George or not.

[1 mark]

Either way, give a mathematical reason for the future overall trend of the squirrel population, as predicted by the model.

[2 marks]

Question 5

Solve the following, giving your answer to 3 significant figures,

$$2^{2x} + 3\left(2^{x-1}\right) - 1 = 0$$

[5 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 1

Answer 1

$$\sin(x + 30) = 2\cos x$$
$$\sin x \cos 30 + \cos x \sin 30 = 2\cos x$$
$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = 2\cos x$$

multiply through by 2

$$\sqrt{3}\sin x + \cos x = 4\cos x$$
$$\sqrt{3}\sin x = 3\cos x$$
$$\frac{\sin x}{\cos x} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$\tan x = \sqrt{3}$$
$$x = 60°,\ 240°$$

[5 marks]

Answer 2

As $(4x + 1)$ is a factor $f\left(-\frac{1}{4}\right) = 0$ by the factor theorem

$$\therefore 4\left(-\frac{1}{4}\right)^3 + a\left(-\frac{1}{4}\right)^2 + 20\left(-\frac{1}{4}\right) + 4 = 0$$
$$-\frac{1}{16} + \frac{a}{16} - 5 + 4 = 0$$
$$\frac{a}{16} = \frac{1}{16} + 1$$

multiply through by 16

$$a = 1 + 16$$
$$a = 17$$

[5 marks]

Answer 3

(i)

$$g(x) = x^3 - 12x^2 + 50x - 7$$

$$g'(x) = 3x^2 - 24x + 50$$

[2 marks]

(ii)

$$\begin{aligned} g'(x) &= 3x^2 - 24x + 50 \\ &= \left[3x^2 - 24x \right] + 50 \\ &= 3\left[x^2 - 8x \right] + 50 \\ &= 3\left[(x - 4)^2 - 16 \right] + 50 \\ &= 3(x - 4)^2 - 48 + 50 \\ &= 3(x - 4)^2 + 2 \end{aligned}$$

- The smallest that $(x - 4)^2$ can be is zero.
- Even then, 2 is added on.
- Therefore, the gradient $g'(x)$ is always positive
- Therefore, $g(x)$ is an increasing function $\square$

[5 marks]

Answer 4

(i) When $T = 0$,

$$S = \frac{390}{1 + e^{4 - 1.4 \times 0}}$$
$$= \frac{390}{1 + e^{4}}$$
$$= 7 \text{ squirrels}$$

[1 mark]

(ii)

T (years)	0	1	2	3	4	5	6
S (N° of Squirrels)	7	27	90	214	324	372	385

[2 marks]

(iii)

[2 marks]

(iv) It is true that the equation contains an exponential but as T becomes large the power of $e^{4 - 1.4\,T}$ will be negative; it has exponential decay, not growth.

As $T \to \infty$, $e^{4 - 1.4\,T} \to 0$, and so $\dfrac{390}{1 + e^{4 - 1.4 \times T}} \to 390$

Thus the model predicts a stable, unchanging long term population in the forest of 390 squirrels.

[3 marks]

Answer 5

$$2^{2x} + 3\left(2^{x-1} \right) - 1 = 0$$

$$\left(2^{x} \right)^{2} + \frac{3\left(2^{x} \right)}{2} - 1 = 0$$

multiply through by 2

$$2\left(2^{x} \right)^{2} + 3\left(2^{x} \right) - 2 = 0$$

Let $z = 2^{x}$ in which case,

$$2z^{2} + 3z - 2 = 0$$

$$\left(2z - 1 \right)\left(z + 2 \right) = 0$$

Either $z = \frac{1}{2}$ or $z = -2$

But $2^{x} = -2$ has no solutions

so the only solution comes from $2^{x} = \frac{1}{2}$

That is, $x = -1$

[5 marks]

Year 2 Pure Mathematics Examination Revision : Health Check N° 2

Monitoring Your Math's Vibrant Signs

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

Given that $f(x) = \left(x + \frac{1}{x}\right)^2$ determine the exact value of $\int_1^3 f(x)\, dx$

[5 marks]

Question 2

In March 2020, is was reported that the WhatsApp mobile messaging software had 2 billion active users. It was in March 2014 that it had 1 billion active users.

Y (Year)	U (Billion Users)
2014	1
2020	2

The growth in the number of active users was described as being exponential.

(i) Assuming that the relationship between the number of active users, U (billions) and the Year Number, is of the form

$$U = a\,e^{b\,(\,Y - 2014\,)} \quad \text{where } a \text{ and } b \text{ are constants}$$

determine the value of a and the value of b

[2 marks]

(ii) Use your part **(i)** formula to predict the number of active WhatsApp users in March 2050.

[2 marks]

(iii) State with a reason if you think the prediction given by part (ii) is a realistic possibility or not.

[2 marks]

Question 3
Natalie is revising Calculus, and reads in her notes that;

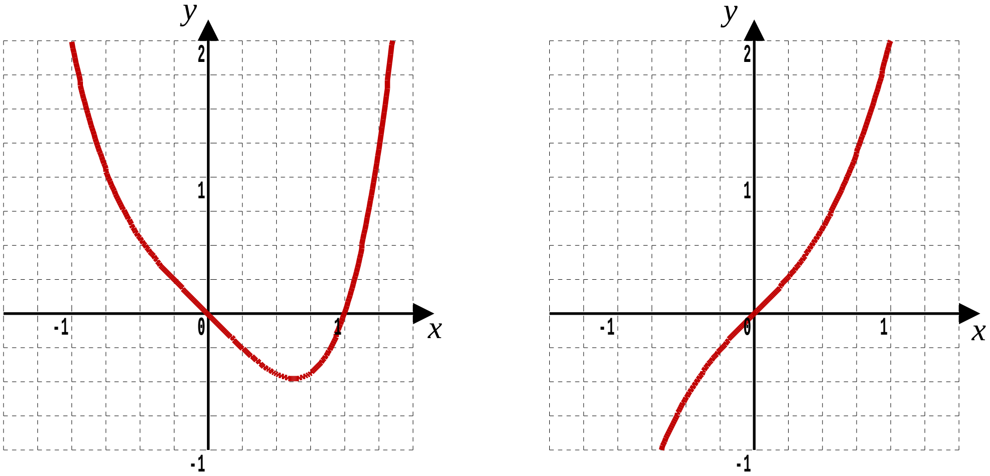
Point of Inflexion
At a point of inflexion:

1st Condition • $\dfrac{d^2y}{dx^2}$ must equal zero

2nd Condition • $\dfrac{d^2y}{dx^2}$ must have different signs either side of the point

---

Shown are the graphs of $y = x^4 - x$ (to the left) and $y = x^3 + x$ (to the right)



(i) Show that $y = x^4 - x$ does not have a point of inflection at $(0,\ 0)$

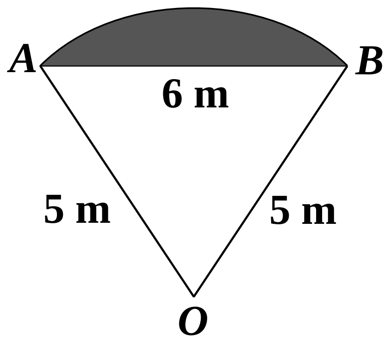
[3 marks]

(ii) Show that $y = x^3 + x$ has a point of inflection at $(0,\ 0)$

[3 marks]

Question 4

A-Level Examination Question from January 2006, Paper C2, Q5 (Edexcel)



OAB is a sector of a circle, radius 5 m
The chord *AB* is 6 m long

(a) Show that $\cos AOB = \frac{7}{25}$

[2 marks]

(b) Hence find the angle *AOB* in radians.
Give your answer to 3 decimal places.

[1 mark]

(c) Calculate the area of the sector *OAB*.

[2 marks]

(d) Hence calculate the shaded area.

[3 marks]

Question 5

A-Level Examination Question from January 2018, Paper C12, Q10(i) (Edexcel)

Use the laws of logarithms to solve the equation

$$3\log_8 2 + \log_8(7 - x) = 2 + \log_8 x$$

[5 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 2

Answer 1

$$f(x) = \left(x + \frac{1}{x}\right)^2$$
$$= x^2 + 2 + \frac{1}{x^2}$$
$$\int_1^3 f(x)\,dx = \int_1^3 x^2 + 2 + x^{-2}\,dx$$
$$= \left[\frac{x^3}{3} + 2x + \frac{x^{-1}}{(-1)}\right]_1^3$$
$$= \left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_1^3$$
$$= \left[9 + 6 - \frac{1}{3}\right] - \left[\frac{1}{3} + 2 - 1\right]$$
$$= 14\frac{2}{3} - 1\frac{1}{3}$$
$$= 13\frac{1}{3}$$

[5 marks]

Answer 2

(i) When $Y = 2014$, $U = 1$, gives $1 = a\,e^{b\,(2014 - 2014)} \Rightarrow a = 1$

When $Y = 2020$, $U = 2$, gives $2 = 1\,e^{b\,(2020 - 2014)} \Rightarrow b = \frac{1}{6}\,ln\,2$

(As an approximate decimal $b = 0.11552...$)

[2 marks]

(ii) When $y = 2050$, $U = e^{\frac{1}{6}\,ln\,2\,(2050 - 2014)}$

$$= e^{6\,ln\,2}$$
$$= e^{ln\,2^6}$$
$$= 2^6$$
$= 64$ billion active users

[2 marks]

(iii) The human population of Earth is predicted to be around 10 billion in 2050, placing enormous strain on the planet's ability to support life. (It's currently about 7.9 billion)
$\therefore$ The prediction of 64 billion active users is not possible in reality !

[2 marks]

Answer 3
(i)

$$y = x^4 - x$$
$$\frac{dy}{dx} = 4x^3 - 1$$
$$\frac{d^2y}{dx^2} = 12x^2$$

At $(0,\ 0)$, $\dfrac{d^2y}{dx^2} = 0$ satisfying the first condition.

As any number, other than zero, when squared is positive,

the sign of $\dfrac{d^2y}{dx^2}$ either side of $(0,\ 0)$ must be positive, and so

the second condition is not satisfied.

$\therefore\ y = x^4 - x$ does not have a point of inflection at $(0,\ 0)$ ☐

[3 marks]

(ii)

$$y = x^3 + x$$
$$\frac{dy}{dx} = 3x^2 + 1$$
$$\frac{d^2y}{dx^2} = 6x$$

At $(0,\ 0)$, $\dfrac{d^2y}{dx^2} = 0$ satisfying the first condition.

When $x < 0$, $\dfrac{d^2y}{dx^2}$ will be negative

When $x > 0$, $\dfrac{d^2y}{dx^2}$ will be positive

The sign of $\dfrac{d^2y}{dx^2}$ on one side of $(0,\ 0)$ is different to that on the other

and so the second condition is satisfied.

$\therefore\ y = x^3 + x$ has a point of inflection at $(0,\ 0)$ ☐

[3 marks]

Answer 4

(a) From the cosine rule,

$$
\begin{aligned}
\cos AOB &= \frac{5^2 + 5^2 - 6^2}{2 \times 5 \times 5} \\
&= \frac{25 + 25 - 36}{50} \\
&= \frac{14}{50} \\
&= \frac{7}{25} \qquad \square
\end{aligned}
$$

[2 marks]

(b)

$$
\begin{aligned}
\therefore\ AOB &= \arccos\left(\frac{7}{25}\right) \\
&= 1.287 \text{ radians}
\end{aligned}
$$

[1 mark]

(c)

$$
\begin{aligned}
Area\ Sector &= \frac{1}{2} r^2 \theta \\
&= \frac{1}{2} \times 5^2 \times 1.287 \\
&= 16.1 \text{ m}^2
\end{aligned}
$$

[2 marks]

(d)

$$
\begin{aligned}
Area\ Shaded &= Area\ Sector - Area\ Triangle \\
&= 16.1 - \frac{1}{2} \times 5 \times 5 \times \sin 1.287 \\
&= 16.1 - 12.0 \\
&= 4.1 \text{ m}^2
\end{aligned}
$$

[3 marks]

Answer 5

$$
\begin{aligned}
3\,log_8 2 + log_8(7 - x) &= 2 + log_8 x \\
log_8 2^3 + log_8(7 - x) - log_8 x &= 2 \\
log_8\left(\frac{2^3(7 - x)}{x}\right) &= 2 \\
\therefore \left(\frac{2^3(7 - x)}{x}\right) &= 8^2 \\
8(7 - x) &= 64x \\
56 - 8x &= 64x \\
64x + 8x &= 56 \\
72x &= 56 \\
x &= \frac{56}{72} \\
&= \frac{7}{9}
\end{aligned}
$$

[5 marks]

Vaccinate against Maths Confusion

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30

Question 1

(i) Find the first four terms, in ascending powers of x, of the binomial expansion of $\dfrac{1}{1 + 2x}$ giving each coefficient in its simplest form.

[4 marks]

(ii) Write down the coefficient of x^8 were the part (i) expansion to be extended further.

[1 mark]

Question 2

$y = 5x^3 + ax^2 + 3x + 2$ where a is an integer constant.

Given that there is a point of inflection when $x = \dfrac{1}{5}$ determine the value of a

[4 marks]

Question 3

Two circles have equations;

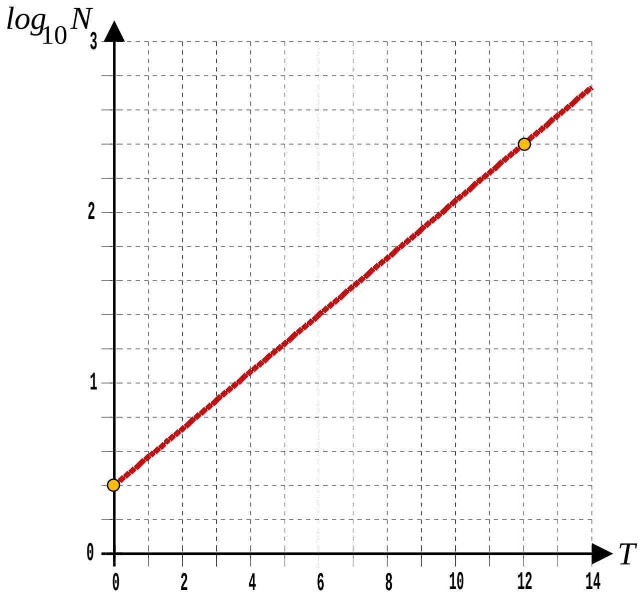
$$(x + 4)^2 + (y - 6)^2 = 25$$
$$(x - 7)^2 + (y + 3)^2 = 16$$

What is the minimum distance between the circles ?
Give your answer correct to 3 decimal places.

[4 marks]

Question 4

An epidemiologist is modelling the number of people, N, who have tested positive for a virus after T days. From looking at a graph of the results, he suggests that the number of people sick be modelled by the equation $N = a\,b^T$ where a and b are constants to be found. The graph passes through $(0,\ 0.4)$ and $(12,\ 2.4)$



(i) Write down the equation of the line.

[2 marks]

(ii) Hence, or otherwise, find the values of a and b.
Work to an accuracy of 4 significant figures.

[4 marks]

(iii) Interpret the meaning of the constant a in this model.

[1 mark]

(iv) Use your model to predict the number of sick people after 21 days.
Give one reason why this might be an overestimate.

[2 marks]

Question 5

The vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are given as,

$$\boldsymbol{a} = \begin{pmatrix} 8 \\ 23 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -15 \\ x \end{pmatrix} \text{ and } \boldsymbol{c} = \begin{pmatrix} -13 \\ 2 \end{pmatrix}$$

where x is an integer.

Given that $\boldsymbol{a} + \boldsymbol{b}$ is parallel to $\boldsymbol{b} - \boldsymbol{c}$, find the value of x

[4 marks]

Question 6

The functions p and q are defined by: $p(x) = x^2$, $q(x) = 5 - 2x$

(i) Given that $pq(x) = qp(x)$ show that $3x^2 - 10x + 10 = 0$

[3 marks]

(ii) Explain why $3x^2 - 10x + 10 = 0$ has no real solutions

[1 mark]

Year 2 Pure Mathematics Examination Revision
Health Check N° 3

Answer 1

$$\frac{1}{1 + 2x} = (1 + 2x)^{-1}$$

(i) The Binomial Theorem

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

... with $n = -1$ and x replaced with $2x$ yields ...

$$(1 + 2x)^{-1} = 1 + (-1)(2x) + \frac{(-1)(-2)}{2}(2x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2}(2x)^3 + \ldots$$

$$= 1 - 2x + 4x^2 - 8x^3 + \ldots$$

[4 marks]

(ii) The series could be written as,

$$2^0 x^0 - 2^1 x^1 + 2^2 x^2 - 2^3 x^3 + \ldots$$

So the coefficient of x^8 will be $+ 2^8 = 256$

[1 mark]

Answer 2

At a point of inflection $\dfrac{d^2y}{dx^2} = 0$

$$y = 5x^3 + ax^2 + 3x + 2$$

$$\frac{dy}{dx} = 15x^2 + 2ax + 3$$

$$\frac{d^2y}{dx^2} = 30x + 2a$$

We require $30x + 2a = 0$

$$2a = -30x$$

$$a = -15x$$

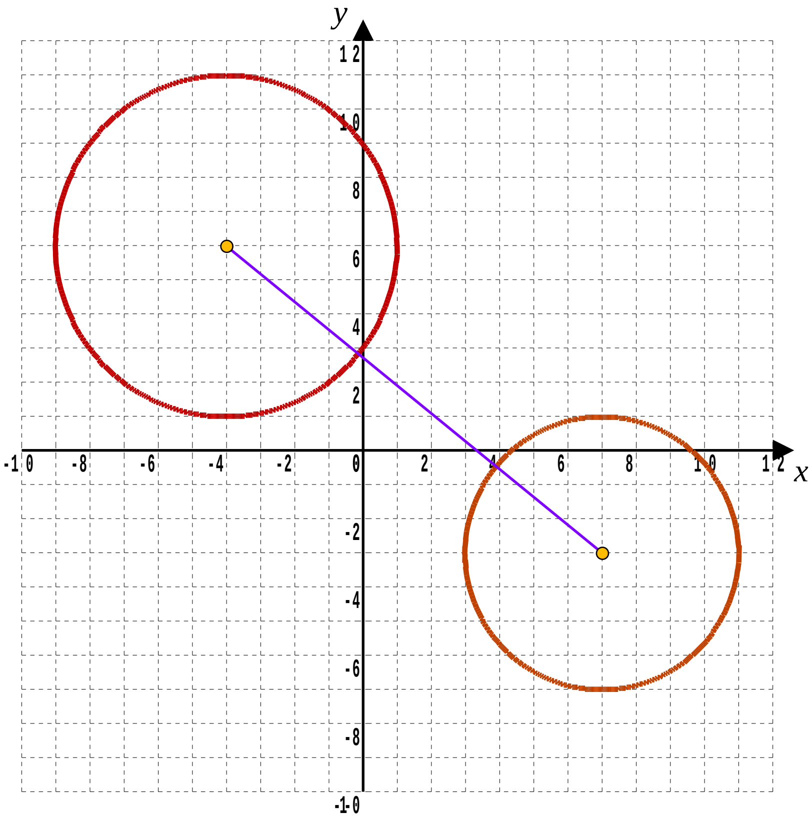
The point of inflection is at $x = \dfrac{1}{5}$ and so $a = -3$

[4 marks]

Answer 3

$$(x + 4)^2 + (y - 6)^2 = 25 \text{ has centre } (-4,\ 6),\ \text{radius } 5$$

$$(x - 7)^2 + (y + 3)^2 = 16 \text{ has centre } (7, -3),\ \text{radius } 4$$



The distance, D, between the two centres is,

$$\begin{aligned} D &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(-4 - 7)^2 + (6 - (-3))^2} \\ &= \sqrt{(-11)^2 + 9^2} \\ &= \sqrt{121 + 81} \\ &= \sqrt{202} \end{aligned}$$

The minimum distance between the two circles is thus,

$$\begin{aligned} \sqrt{202} - 5 - 4 &= \sqrt{202} - 9 \\ &= 5.213 \text{ (3 decimal places asked for)} \end{aligned}$$

[4 marks]

Answer 4

(i) $(0,\ 0.4)$ and $(12,\ 2.4)$

$$m = \frac{\Delta Y}{\Delta X} = \frac{2.4 - 0.4}{12 - 0} = \frac{1}{6}$$

$$log_{10} N = \frac{1}{6} \times T + 0.4$$

[2 marks]

(ii) You may use a remembered result or work it out from scratch. Here's the 'from scratch' method;

$$log_{10} N = \frac{1}{6} \times T + 0.4$$

$$10^{log_{10} N} = 10^{\frac{1}{6} T + 0.4}$$

$$N = 10^{\frac{1}{6} T}\ 10^{0.4}$$

$$N = 2.512 \times 1.468^{T}$$

That is, $a = 2.512$ and $b = 1.468$

[4 marks]

(iii) a is the (theoretical) initial number of sick people
That is, the value of N when $T = 0$

[1 mark]

(iv) When $T = 21$,

$$N = 2.512 \times 1.468^{21}$$

$$= 7966 \text{ people}$$

Once an outbreak is detected, mitigation measures are likely to be implemented, slowing the rate of transmission. (for example)

[2 marks]

Answer 5

$$\boldsymbol{a} + \boldsymbol{b} = k (\boldsymbol{b} - \boldsymbol{c})$$

$$\begin{pmatrix} 8 \\ 23 \end{pmatrix} + \begin{pmatrix} -15 \\ x \end{pmatrix} = k \left(\begin{pmatrix} -15 \\ x \end{pmatrix} - \begin{pmatrix} -13 \\ 2 \end{pmatrix} \right)$$

$$\begin{pmatrix} -7 \\ 23 + x \end{pmatrix} = k \begin{pmatrix} -2 \\ x - 2 \end{pmatrix}$$

$$\therefore k = 3.5$$

$$23 + x = 3.5x - 7$$

$$2.5x = 30$$

$$x = 12$$

[4 marks]

Answer 6

(i)

$$p(x) = x^2, \qquad q(x) = 5 - 2x$$

$$pq(x) = qp(x)$$

$$p\left(5 - 2x\right) = q\left(x^2\right)$$

$$(5 - 2x)^2 = 5 - 2x^2$$

$$25 - 20x + 4x^2 = 5 - 2x^2$$

$$6x^2 - 20x + 20 = 0$$

divide through by 2

$$3x^2 - 10x + 10 = 0 \qquad \square$$

[3 marks]

(ii) Looking at the discriminant,

$$(-10)^2 - 4 \times 3 \times 10 = 100 - 120$$

$$= -20$$

As the discriminant is less than zero the quadratic equation

$$3x^2 - 10x + 10 = 0$$

has no real solutions ☐

[1 mark]

Year 2 Pure Mathematics Examination Revision : Health Check N° 4

Fortify Your Maths

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

Question 1

Find the values of k for which $k\,x^2 + 8x + 5 = 0$ has distinct real roots, $k \in \mathbb{R}$

[3 marks]

Question 2

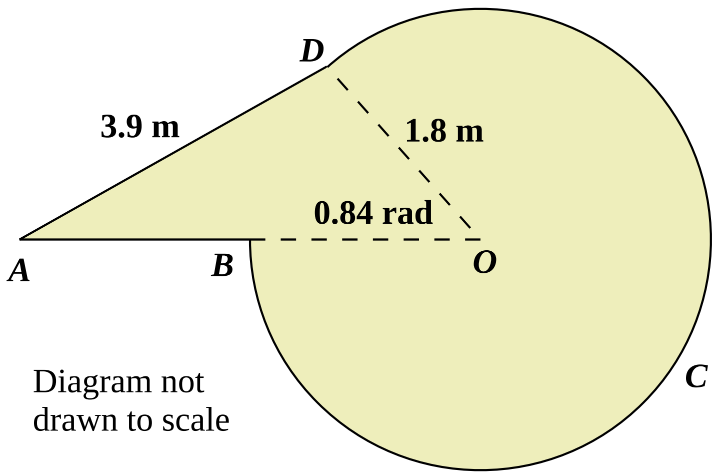
$$f\,(x) = p\,x^3 - 3p\,x^2 + x^2 - 4, \text{ where } p \text{ is a constant}$$

Given that, when $x = 2$, $f''(x) = -1$, find the value of p

[3 marks]

Question 3

A-Level Examination Question from January 2019, Paper C12, Q10b (Edexcel)



The diagram shows the design for a shop sign *ABCDA*.

The sign consists of a triangle *AOD* joined to a sector of a circle *DOBCD* with radius 1.8 m and centre *O*. The points *A*, *B* and *O* lie on a straight line.
It is given that *AD* = 3.9 m and angle *BOD* is 0.84 radians.

(a) Calculate the size of angle *DAO*, giving your answer in radians to 3 decimal places

[2 marks]

(b) Show that, to one decimal place, the length of *AO* is 4.9 metres.

[3 marks]

(c) Find, in m^2, the area of the shop sign, giving your answer to one decimal place.

[3 marks]

(d) Find, in metres, the perimeter of the shop sign, giving your answer to one decimal place

[3 marks]

Question 4

(i) Given that $y = \frac{1}{4x + 1}$ find the value of $\frac{dy}{dx}$ when $x = \frac{1}{4}$

[3 marks]

(ii) Hence find the equation of the tangent to the curve when $x = \frac{1}{4}$

[2 marks]

Question 5

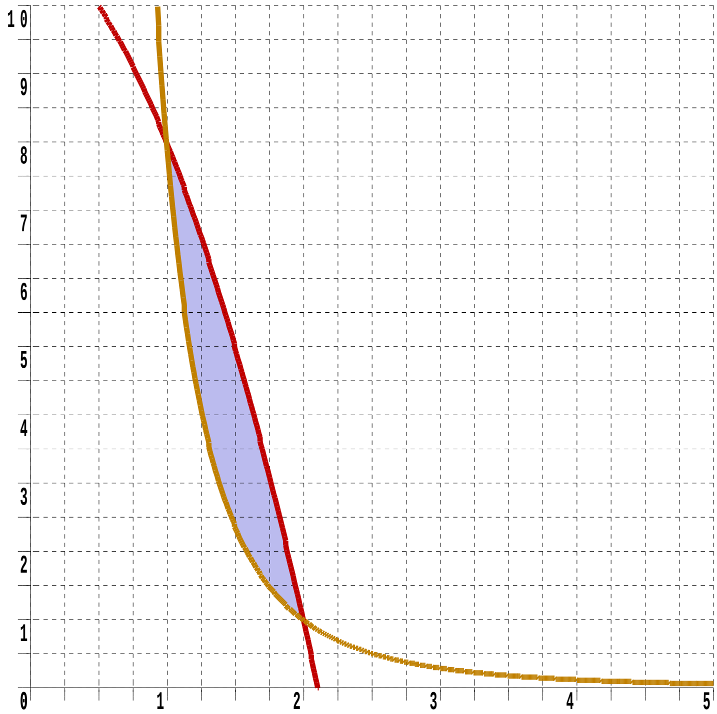
Given that $x = 3\,sin\,y$ $\quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

show that $\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}$

[4 marks]

Question 6

A-Level Examination Question from June 2017, Paper C2, Q6 (OCR)



The diagram shows parts of the curves $y = 11 - x - 2x^2$ and $y = \dfrac{8}{x^3}$

The curves intersect at (1, 8) and (2, 1)

Use integration to find the exact area of the shaded region enclosed between the two curves.

[4 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 4

Answer 1
To have distinct real roots, the discriminant must be positive.

$$\begin{aligned}
D &> 0 \\
b^2 - 4ac &> 0 \\
8^2 - 4 \times k \times 5 &> 0 \\
64 - 20k &> 0 \\
-20k &> -64 \\
20k &< 64 \\
k &< \frac{64}{20} \\
k &< 3.2
\end{aligned}$$

[3 marks]

Answer 2

$$\begin{aligned}
f(x) &= px^3 - 3px^2 + x^2 - 4 \\
f'(x) &= 3px^2 - 6px + 2x \\
f''(x) &= 6px - 6p + 2 \\
\text{We're told that, } f''(2) &= -1 \\
6p(2) - 6p + 2 &= -1 \\
12p - 6p &= -3 \\
6p &= -3 \\
p &= -\frac{1}{2}
\end{aligned}$$

[3 marks]

Answer 3

(a) Use the (inverted) sine rule,

$$\frac{\sin DAO}{1.8} = \frac{\sin 0.84}{3.9}$$

$$\sin DAO = 0.3437$$

$$DAO = 0.351 \text{ radians}$$

[2 marks]

(b) As the three angles in a triangle add up to π radians,

$$ODA = \pi - 0.351 - 0.84$$

$$= 1.951 \text{ radians}$$

Again, using the sine rule,

$$\frac{AO}{\sin 1.951} = \frac{3.9}{\sin 0.84}$$

$$= 4.86$$

$$= 4.9 \text{ to one decimal place} \qquad \square$$

[3 marks]

(c)

$$\mathit{Area\ Sector} = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \times 1.8^2 \times (2\pi - 0.84)$$

$$= 8.82 \text{ m}^2$$

$$\mathit{Area\ triangle} = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 1.8 \times 4.9 \times \sin 0.84$$

$$= 3.28 \text{ m}^2$$

$$\therefore \mathit{Area\ Shop\ Sign} = 8.82 + 3.28$$

$$= 12.1 \text{ m}^2$$

[3 marks]

(d)

$$\mathit{Perimeter} = arc\ BCD + DA + AB$$

$$= 1.8 \times (2\pi - 0.84) + 3.9 + (4.9 - 1.8)$$

$$= 9.8 + 3.9 + 3.1$$

$$= 16.8 \text{ m}$$

[3 marks]

Answer 4

(i) This could be done using the Quotient Rule but that's using a sledgehammer to crack a nut.
Instead, use the Chain Rule like this...

$$y = \frac{1}{4x + 1}$$
$$= (4x + 1)^{-1}$$
$$\frac{dy}{dx} = (-1)(4x + 1)^{-2} \times 4$$
$$= -\frac{4}{(4x + 1)^2}$$
$$\therefore \text{ When } x = \frac{1}{4}, \quad \frac{dy}{dx} = -1$$

[3 marks]

(ii) When $x = \frac{1}{4}$, $y = \frac{1}{2}$ and so the tangent will be the straight line through the point $\left(\frac{1}{4}, \frac{1}{2} \right)$ with gradient -1

$$y = mx + c$$
$$\frac{1}{2} = (-1) \times \frac{1}{4} + c$$
$$c = \frac{3}{4}$$

and the tangent's equation is thus $y = -x + \frac{3}{4}$

[2 marks]

Answer 5

$$x = 3\,sin\,y$$
$$\frac{dx}{dy} = 3\,cos\,y$$
$$\text{But } cos^2 y + sin^2 y = 1$$
$$\frac{dx}{dy} = 3\sqrt{1 - sin^2 y}$$
$$= \sqrt{9}\sqrt{1 - sin^2 y}$$
$$= \sqrt{9 - (3\,sin\,y)^2}$$
$$= \sqrt{9 - x^2}$$
$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}} \qquad \square$$

[4 marks]

Answer 6

$$\int_1^2 11 - x - 2x^2 - 8x^{-3}\ dx = \left[11x - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{8x^{-2}}{(-2)} \right]_1^2$$

$$= \left[11x - \frac{x^2}{2} - \frac{2x^3}{3} + \frac{4}{x^2} \right]_1^2$$

$$= \left[22 - 2 - \frac{16}{3} + 1 \right] - \left[11 - \frac{1}{2} - \frac{2}{3} + 4 \right]$$

$$= \left[\frac{47}{3} \right] - \left[\frac{83}{6} \right]$$

$$= \frac{11}{6}$$

[4 marks]

A Maths Tonic To Take

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

Question 1

A-Level Examination Question from June 2018, Paper C12, Q8 (Edexcel)

The equation $(k - 4)x^2 - 4x + k - 2 = 0$, where k is a constant, has no real roots.

(a) Show that k satisfies the inequality $k^2 - 6k + 4 > 0$

[3 marks]

(b) Find the exact range of possible values for k

[4 marks]

Question 2

A-Level Examination Question from 2018, Specimen Paper 1, Q8 (Edexcel)

There were 2100 tonnes of wheat harvested on a farm during 2017.

The mass of wheat harvested during each subsequent year is expected to increase by 1.2 % per year.

(a) Find the total mass of wheat expected to be harvested from 2017 to 2030 inclusive, giving your answer to 3 significant figures.

[3 marks]

Each year it costs

- £5.15 per tonne to harvest the first 2000 tonnes of wheat
- £6.45 per tonne to harvest wheat in excess of 2000 tonnes

(b) Use this information to find the expected cost of harvesting the wheat from 2017 to 2030 inclusive. Give your answer to nearest £1000.

[3 marks]

Question 3

A-Level Examination Question from 2018, Specimen Paper 1, Q10 (Edexcel)

The function *f* is defined by

$$f : x \to \frac{3x - 5}{x + 1}, \; x \in \mathbb{R}, \; x \neq -1$$

(a) Find $f^{-1}(x)$

[3 marks]

(b) Show that $ff(x) = \dfrac{x + a}{x - 1}, \; x \in \mathbb{R}, \; x \neq \pm 1$

where a is an integer to be found.

[4 marks]

The function g is defined by

$$g : x \to x^2 - 3x, \quad x \in \mathbb{R}, \quad 0 \leqslant x \leqslant 5$$

(c) Find the value of $fg(2)$

[2 marks]

(d) Find the range of g

[3 marks]

(e) Explain why the function g does not have an inverse.

[1 mark]

Question 4

The equation $|2x - 7| = x + k$ has exactly two distinct solutions.

Find the range of possible values of k

[4 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 5

Answer 1

(a) To have no real roots, the discriminant must be negative.

$$D < 0$$

$$b^2 - 4ac < 0$$

$$(-4)^2 - 4 \times (k - 4) \times (k - 2) < 0$$

$$16 - 4\left(k^2 - 6k + 8\right) < 0$$

$$16 - 4k^2 + 24k - 32 < 0$$

$$4k^2 - 24k + 16 > 0$$

Divide through by 4

$$k^2 - 6k + 4 > 0 \qquad \square$$

[3 marks]

(b) To find the critical values, solve $k^2 - 6k + 4 = 0$

$$k = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 4}}{2 \times 1}$$

$$= \frac{6 \pm \sqrt{20}}{2}$$

$$= 3 \pm \sqrt{5}$$

$\therefore$ The possible range of values for k satisfy:

$$k < 3 - \sqrt{5} \quad \text{or} \quad k > 3 + \sqrt{5}$$

[4 marks]

Answer 2

(a) The situation is that of a Geometric Progression,

$$2100,\ 2100 \times 1.012,\ 2100 \times 1.012^2,\ \ldots,\ 2100 \times 1.012^{13}$$

$$2017, \qquad 2018, \qquad 2019,\ \ldots, \qquad 2030$$

$$n = 1, \qquad n = 2, \qquad n = 3, \qquad n = 14$$

$$Sum_{GP} = \frac{2100\left(1.012^{14} - 1\right)}{1.012 - 1}$$

$$= 31\,800 \text{ tonnes}$$

[3 marks]

(b) The total cost of harvesting the first 2000 tonnes of wheat each year

$$14 \times 2000 \times 5.15 = £144\,200$$

The amount of wheat harvested at £6.45 per tonne will be...

$$2100 - 2000$$

$$+\ 2100 \times 1.012 - 2000$$

$$+\ 2100 \times 1.012^2 - 2000$$

$$+\ \ldots$$

But this is the sequence summed in part (a) less 14 lots of 2000

Total cost of harvesting the wheat at £6.45 will be...

$$(31\,800 - 14 \times 2000) \times 6.45 = 3800 \times 6.45$$

$$= £24\,510$$

Combined total is thus,

$$144\,200 + 24\,510 = £168\,710$$

$$= £169\,000 \text{ to the nearest } £1000$$

[3 marks]

Answer 3

(a)

$$f(x) = \frac{3x - 5}{x + 1}$$

$$y = \frac{3x - 5}{x + 1}$$

swap the inputs and the outputs

$$x = \frac{3y - 5}{y + 1}$$

$$x(y + 1) = 3y - 5$$

$$xy + x = 3y - 5$$

$$x + 5 = 3y - xy$$

$$x + 5 = y(3 - x)$$

$$y = \frac{x + 5}{3 - x}$$

$$f^{-1}(x) = \frac{x + 5}{3 - x} \qquad x \in \mathbb{R},\ x \neq 3$$

[3 marks]

(b)

$$ff(x) = f\left(\frac{3x - 5}{x + 1}\right)$$

$$= \frac{3 \times \frac{3x-5}{x+1} - 5}{\frac{3x-5}{x+1} + 1} \times \frac{x + 1}{x + 1}$$

$$= \frac{3(3x - 5) - 5(x + 1)}{3x - 5 + x + 1}$$

$$= \frac{9x - 15 - 5x - 5}{4x - 4}$$

$$= \frac{4x - 20}{4x - 4}$$

$$= \frac{x - 5}{x - 1}$$

$$\therefore\ a = -5$$

[4 marks]

(c)

$$fg(2) = f\left(2^2 - 3 \times 2\right)$$

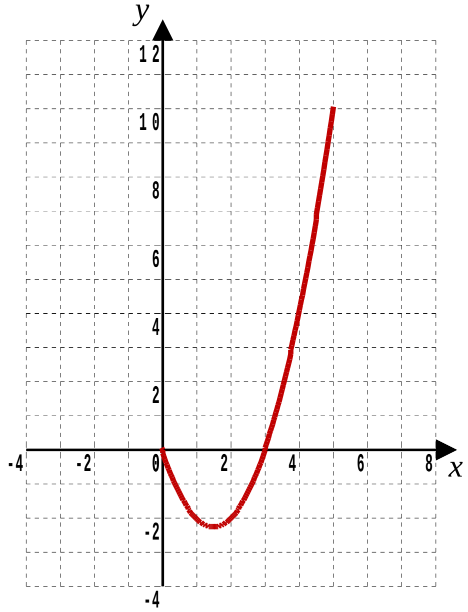
$$= f(-2)$$

$$= \frac{3(-2) - 5}{(-2) + 1}$$

$$= 11$$

[2 marks]

(d) Plotting the graph is the easiest way to get a feel for what's going on...



To find the minimum value you could complete the square or, as here, use differentiation;

$$g(x) = x^2 - 3x$$

$$g'(x) = 2x - 3$$

$$2x - 3 = 0 \text{ when } x = \frac{3}{2} \text{ in which case...}$$

$$g\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right)$$

$$= \frac{9}{4} - \frac{9}{2}$$

$$= -\frac{9}{4}$$

$$\therefore\ g \text{ has the range } -\frac{9}{4} \leqslant g(x) \leqslant 10$$

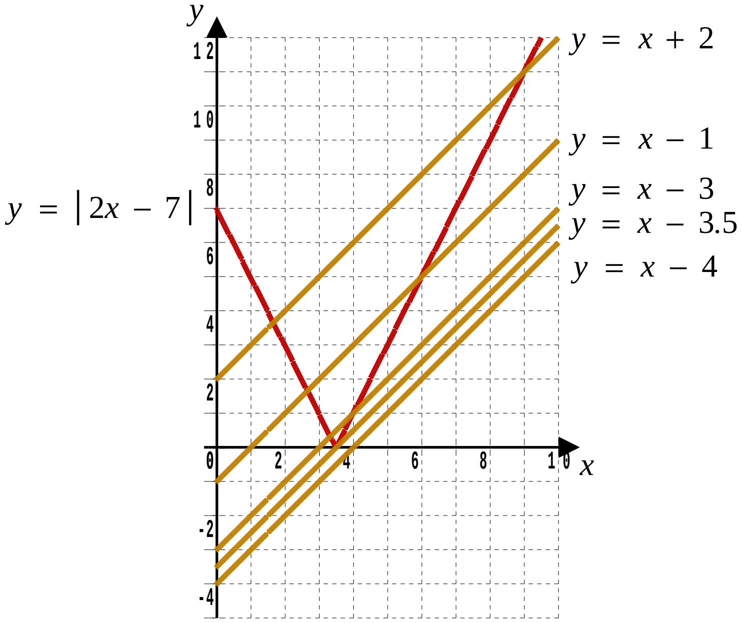
[3 marks]

(e) Only a function that is one-to-one has an inverse.
As g is not one-to-one it does not have an inverse

[1 mark]

Answer 4

The equation can be interpreted as being where the graphs of $y = |2x - 7|$ and $y = x + k$ intersect. Further, it can be noted that $y = x + k$ is the form of a straight line of gradient 1 that crosses the y-axis at $(0, k)$.



A key point is where the vertex of the graph of $y = |2x - 7|$ is on the x-axis. This will be where $y = 0$.

$$y = 0$$

$$|2x - 7| = 0$$

$$(2x - 7) = 0 \quad \text{or} \quad -(2x - 7) = 0$$

$$\therefore\ x = 3.5 \ \text{(both 'branches' give the same answer)}$$

The straight line of the form $y = x + k$ through this key point $(3.5, 0)$ will be...

$$0 = 3.5 + k \Rightarrow k = -3.5$$

$\therefore$ The critical line between there being two solutions, and there being no solutions is $y = x - 3.5$

So if $k > -3.5$ there will be two distinct points of intersection, corresponding to two distinct solutions to the equation $|2x - 7| = x + k$

[4 marks]

Year 2 **Pure Mathematics Examination Revision : Health Check N° 6**

Your Prescription Is Attached

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

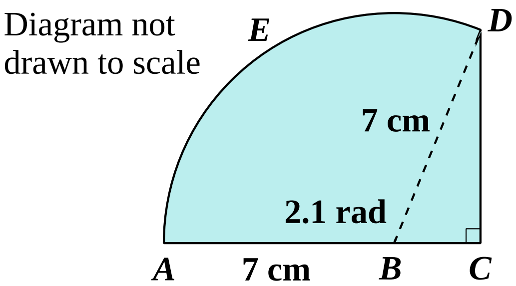
Question 1

Differentiate $\dfrac{x^4}{\cos 3x}$ with respect to x

[4 marks]

Question 2

A-Level Examination Question from May 2014, Paper C2, Q5 (Edexcel)



The diagram shows the shape *ABCDEA* which consists of a right-angled triangle *BCD* joined to a sector *ABDEA* of a circle with radius 7 cm and centre .
A, *B* and *C* lie on a straight line with *AB* = 7 cm.
Given that the size of angle *ABD* is exactly 2.1 radians,

(a) find, in cm, the length of the arc *DEA*

[2 marks]

(b) find, in cm, the perimeter of the shape *ABCDEA*, giving your answer to 1 decimal place

[4 marks]

Question 3

(a) Express $12\,sin\,x + 5\,cos\,x$ in the form $R\,sin(x + \alpha)$, where R and α are constants $R > 0$ and $0 < \alpha < 90°$. Round α to 1 decimal place.

[4 marks]

A runner's speed, v in m/s, in an endurance race can be modelled by the equation,

$$v(x) = \frac{50}{12\,sin\left(\frac{2x}{5}\right)^{\circ} + 5\,cos\left(\frac{2x}{5}\right)^{\circ}}, \quad 0 \leqslant x \leqslant 300$$

where x is the time in minutes since the beginning of the race.

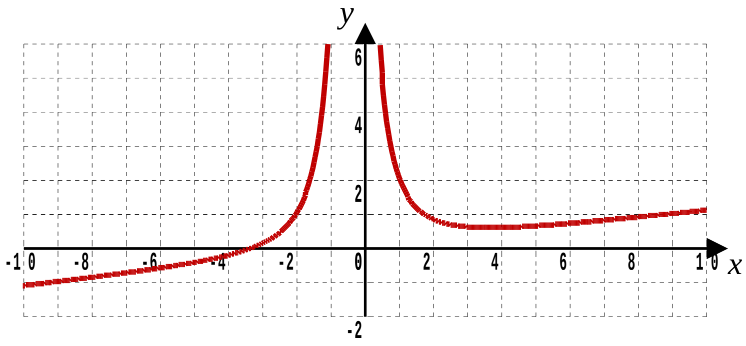
(b) Find the minimum value of v

[2 marks]

(c) Find the time into the race when this speed occurs.

[1 mark]

Question 4



The graph is of the function,

$$f(x) = \frac{32}{(3x + 1)^2} + \frac{x}{9}$$

Find the exact value of x at the minimum point.

[4 marks]

Question 5

A-Level Examination Question from June 2017, Paper C3, Q9 (Edexcel)

(a) Prove that $\sin 2x - \tan x = \tan x \cos 2x, \quad x \neq (2n + 1)\,90°, \; n \in \mathbb{Z}$

[4 marks]

(b) Given that $x \neq 90°$ and $x \neq 270°$, solve, for $0 \leqslant x < 360°$

$$\sin 2x - \tan x = 3 \tan x \sin x$$

Give your answers in degrees to one decimal place where appropriate.

[5 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 6

Answer 1

$$y = \frac{x^4}{\cos 3x}$$

Using the Quotient Rule...

$$\frac{dy}{dx} = \frac{(\cos 3x)\ 4x^3 - (-\sin 3x)(3)\ x^4}{(\cos 3x)^2}$$

$$= \frac{x^3(4\cos 3x + 3x\sin 3x)}{\cos^2 3x}$$

[4 marks]

Answer 2

(a)

$$Arc\ Length = r\theta$$

$$Arc\ DEA = 7 \times 2.1$$

$$= 14.7\ \text{cm}$$

[2 marks]

(b)

$$Angle\ CBD = \pi - 2.1$$

$$= 1.042\ \text{radians}$$

$$Perimeter\ ABCDEA = 7 + 7\cos 1.042 + 7\sin 1.042 + 14.7$$

$$= 31.3\ \text{cm} \quad \text{(to 1 decimal place asked for)}$$

[4 marks]

Answer 3

(a)

$$R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$$

Compare this with the expression given of $12\sin x + 5\cos x$

$$\Rightarrow \frac{R\sin\alpha}{R\cos\alpha} = \frac{5}{12}$$

$$\alpha = \arctan\left(\frac{5}{12}\right)$$

$$= 22.6°$$

$$R = \sqrt{5^2 + 12^2}$$

$$= 13$$

$$\therefore\ 12\sin x + 5\cos x = 13\sin(x + 22.6)$$

[4 marks]

(b) Notice that v is speed and not velocity (i.e. Can't have a negative speed)

$$v(x) = \frac{50}{12\,sin\left(\frac{2x}{5}\right)^{\circ} + 5\,cos\left(\frac{2x}{5}\right)^{\circ}}$$

$$= \frac{50}{13\,sin\left(\frac{2x}{5} + 22.6\right)} \qquad \text{using part (a)}$$

v will have a minimum value when the *sine* function, on the denominator, has it's maximum value of 1

$$\therefore\ v_{MIN} = \frac{50}{13} \qquad \text{(About } 3.85\text{ m s}^{-1}\text{)}$$

[2 marks]

(c)

$$sin\left(\frac{2x}{5} + 22.6\right) = 1 \qquad 0 \leqslant x \leqslant 300$$

$$\frac{2x}{5} + 22.6 = 90$$

$$\frac{2x}{5} = 67.4$$

$$x = 168.5\text{ minutes}$$

[1 mark]

Answer 4

$$f(x) = \frac{32}{(3x + 1)^2} + \frac{x}{9}$$

$$= 32\,(3x + 1)^{-2} + \frac{x}{9}$$

(to avoid the unnecessarily complicated Quotient Rule method)

$$f'(x) = -64\,(3x + 1)^{-3} \times 3 + \frac{1}{9}$$

$$= -\frac{192}{(3x + 1)^3} + \frac{1}{9}$$

For the minimum, $f'(x) = 0$

$$\therefore \frac{1}{9} = \frac{192}{(3x + 1)^3}$$

$$(3x + 1)^3 = 1728$$

$$3x + 1 = 12$$

$$x = \frac{11}{3}$$

[4 marks]

Answer 5

(a)

$$
\begin{aligned}
\text{LHS} &= \sin 2x - \tan x \\
&= 2\sin x \cos x - \frac{\sin x}{\cos x} \\
&= \frac{2\sin x \cos^2 x}{\cos x} - \frac{\sin x}{\cos x} \\
&= \frac{\sin x\left(2\cos^2 x - (1) \right)}{\cos x} \\
&= \frac{\sin x\left(2\cos^2 x - \left(\cos^2 x + \sin^2 x \right) \right)}{\cos x} \\
&= \frac{\sin x \left(\cos^2 x - \sin^2 x \right)}{\cos x} \\
&= \tan x \cos 2x \\
&= \text{RHS} \qquad \square
\end{aligned}
$$

[4 marks]

(b)

$$\sin 2x - \tan x = 3\tan x \sin x$$

$$\tan x \cos 2x = 3\tan x \sin x$$

do not divide by $\tan x$ as this would be a division by zero

$$\tan x \cos 2x - 3\tan x \sin x = 0$$

$$\tan x\left(\cos 2x - 3\sin x \right) = 0$$

Either $\tan x = 0$ in which case $x = 0°, 180°$

Or $\cos 2x - 3\sin x = 0$

$$\cos^2 x - \sin^2 x - 3\sin x = 0$$

$$1 - 2\sin^2 x - 3\sin x = 0$$

$$2\sin^2 x + 3\sin x - 1 = 0$$

$$\sin x = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times (-1)}}{2 \times 2}$$

$$= \frac{-3 \pm \sqrt{17}}{4}$$

Either $x = 16.3°$ or $x = 163.7°$

$\therefore$ Solution Set is $x \in \left\{ 0°, 16.3°, 163.7°, 180° \right\}$

[5 marks]

Kill or Cure ?

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Simplify $\dfrac{x^2 + 4x + 4}{y^2 - 6y + 9} \div \dfrac{x^2 - 4}{y^2 - 9}$

[4 marks]

Question 2

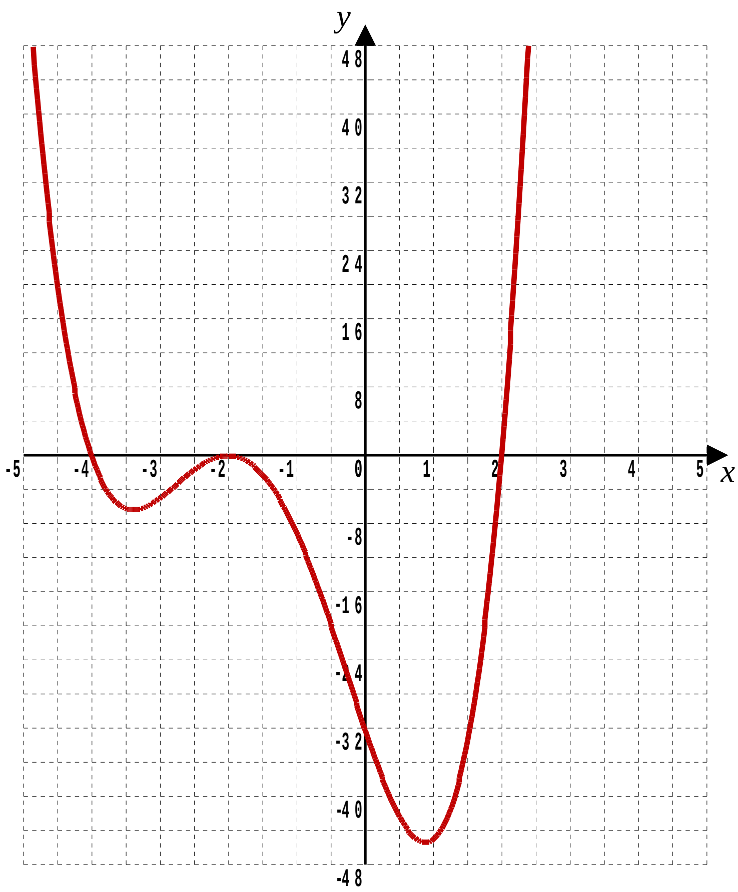
Find the exact value of

$$\sum_{r=3}^{7} 3\left(-\frac{1}{2}\right)^{r}$$

[4 marks]

Question 3

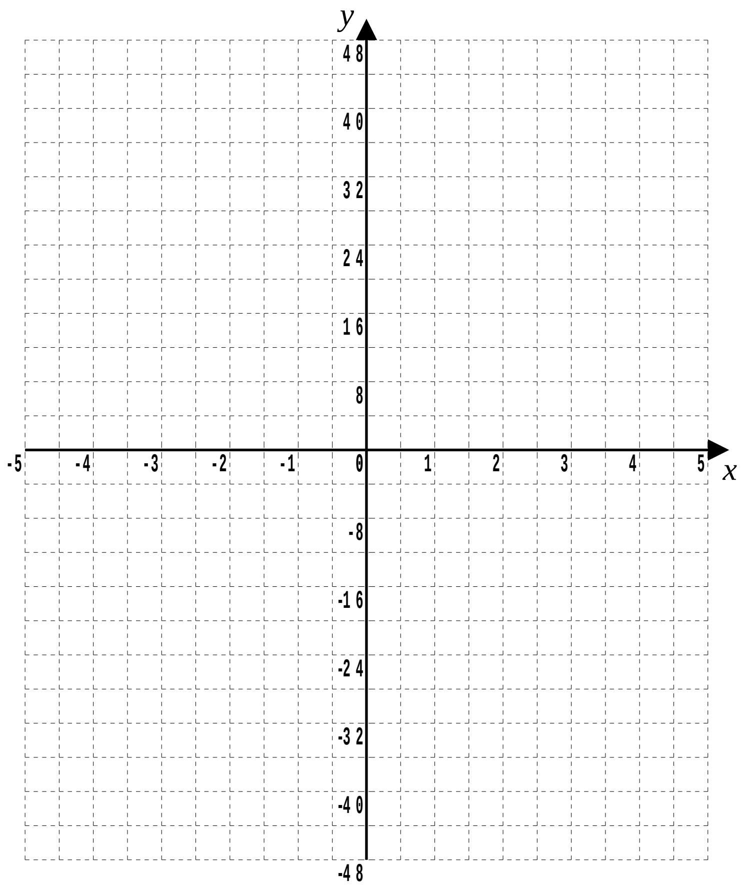
The diagram shows a sketch of part of the curve with equation $y = g(x)$, $x \in \mathbb{R}$



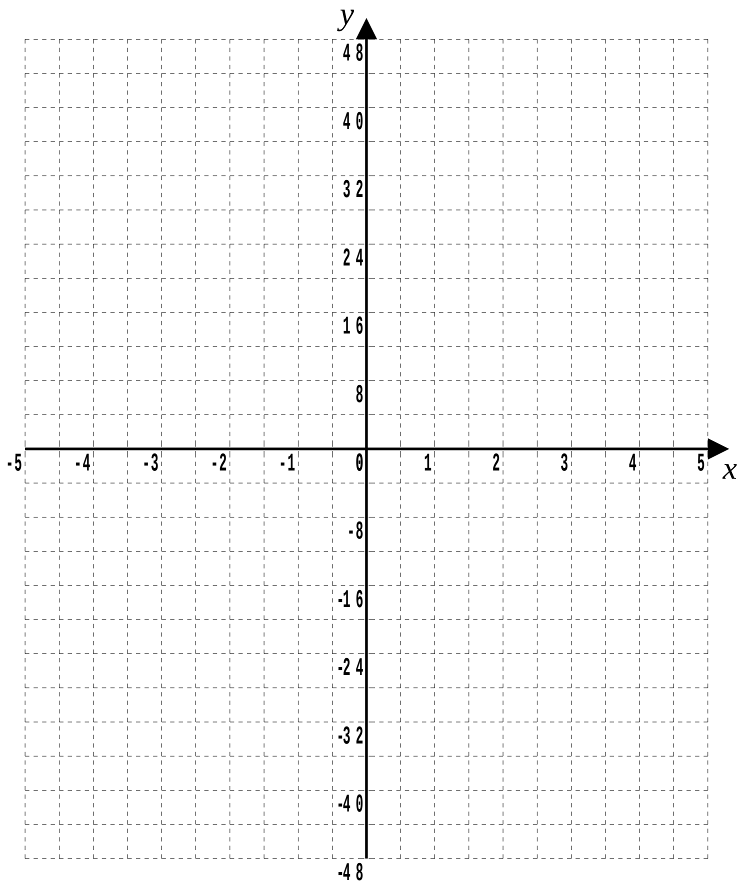
On separate diagrams sketch the curves with equations,

(i) $y = |g(x)|$

(ii) $y = g(|x|)$



[3 marks]

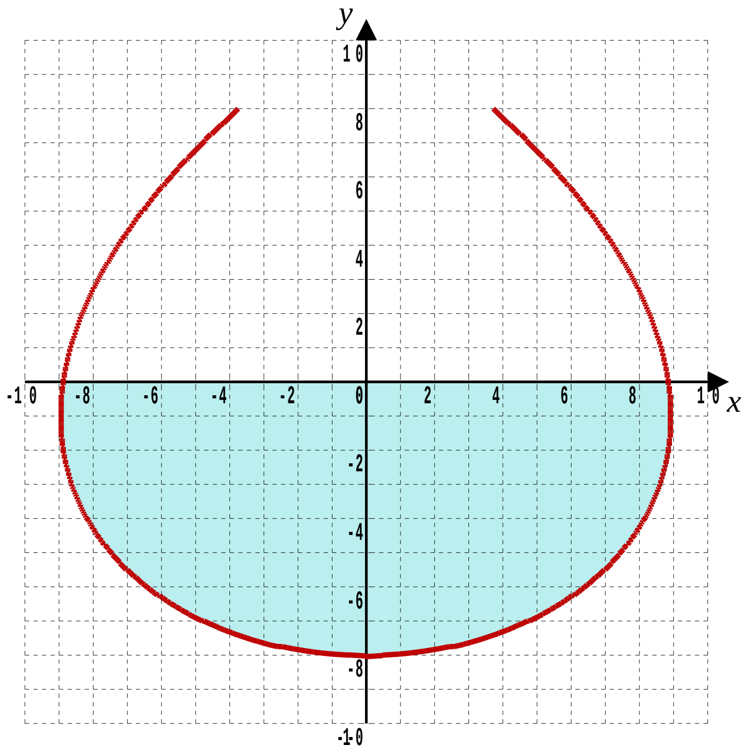


[3 marks]

Question 4

The cross-section of a vase design is given by the parametric equations

$$x = (18t - 32\sin t)\text{ cm},\ y = (8 - 16\cos t)\text{ cm}\quad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$



(i) Find, to three significant figures the width of the opening of the vase

[4 marks]

(ii) The vase is filled with water up to the level of the *x*-axis.
Find the radius of the vase at the surface of the water.

[3 marks]

Question 5

$$f(x) = x^2(2 - 5x)^5$$

(i) Where does the graph of $f(x)$ cross the x-axis ?

[2 marks]

(ii) Use the product rule to obtain an expression for $f'(x)$

[3 marks]

(iii) Show that the gradient is zero at each of the x-axis crossing points of part (i) and, for each of those points deduce if it is a local minimum, a local maximum, or a point of inflection.

[4 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 7

Answer 1

$$\frac{x^2 + 4x + 4}{y^2 - 6y + 9} \div \frac{x^2 - 4}{y^2 - 9} = \frac{(x + 2)(x + 2)}{(y - 3)(y - 3)} \times \frac{(y - 3)(y + 3)}{(x - 2)(x + 2)}$$

$$= \frac{(x + 2)(y + 3)}{(y - 3)(x - 2)}$$

[4 marks]

Answer 2

$$\sum_{r=3}^{7} 3\left(-\frac{1}{2}\right)^r = 3 \sum_{r=3}^{7} \left(-\frac{1}{2}\right)^r$$

$$= 3\left(-\frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \frac{1}{2^6} - \frac{1}{2^7}\right)$$

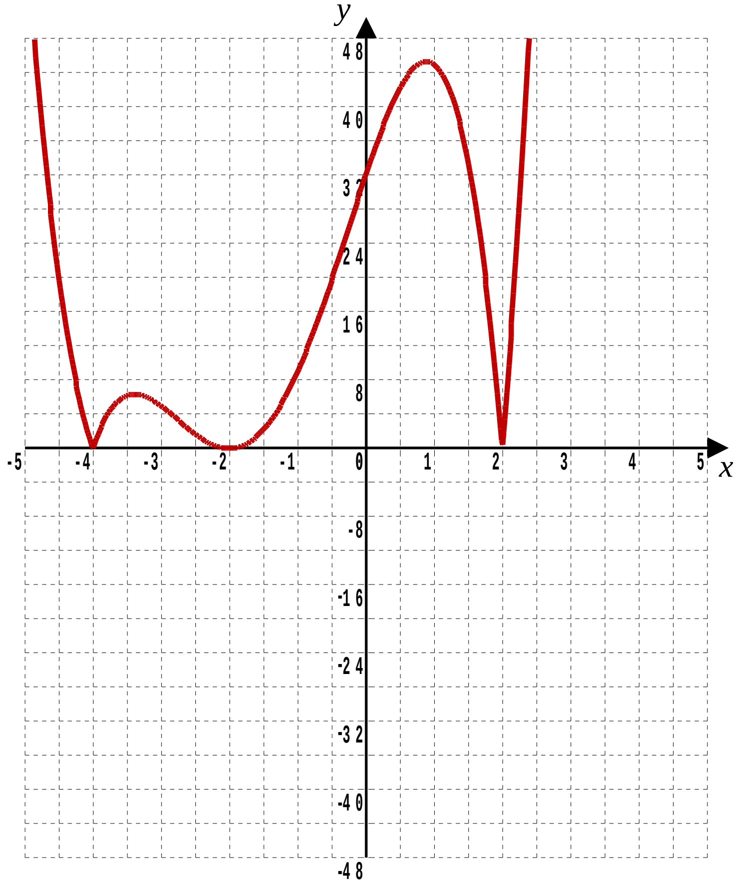
$$= 3\left(-\frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128}\right)$$

$$= 3\left(-\frac{11}{128}\right) = -\frac{33}{128}$$

[4 marks]

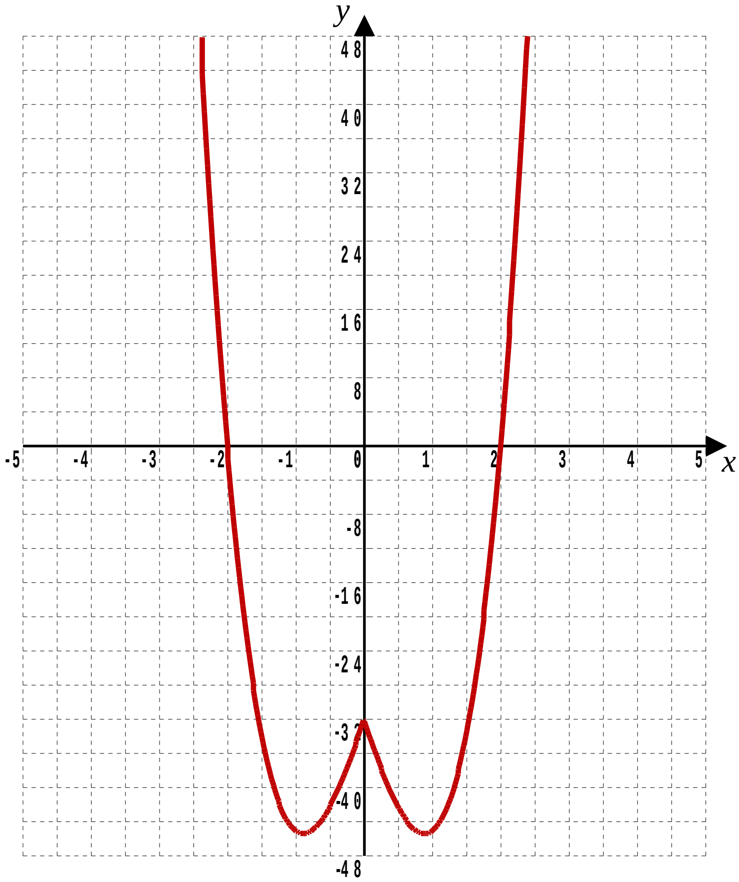
Answer 3

(i)



[3 marks]

(ii)



[3 marks]

Answer 4

The endpoints are likely to correspond to $t = \pm\dfrac{\pi}{2}$

(i) When $t = -\dfrac{\pi}{2}$, $x = 18 \times \left(-\dfrac{\pi}{2}\right) - 32\, sin\left(-\dfrac{\pi}{2}\right)$

$= -9\pi + 32$ (About 3.726)

and $y = 8 - 16\, cos\left(-\dfrac{\pi}{2}\right)$

$= 8$

When $t = \dfrac{\pi}{2}$, $x = 18 \times \left(\dfrac{\pi}{2}\right) - 32\, sin\left(\dfrac{\pi}{2}\right)$

$= 9\pi - 32$ (About – 3.726)

and $y = 8 - 16\, cos\left(-\dfrac{\pi}{2}\right)$

$= 8$

Thus, the width of the opening of the vase is given by,

$-9\pi + 32 - 9\pi + 32 = 64 - 18\pi$

$= 7.45$ cm (to 3 significant figures)

[4 marks]

(ii) $y = 0$ when $8 - 16 \cos t = 0$

$$\cos t = \frac{8}{16}$$

$$t = \pm \frac{\pi}{3}$$

When $t = -\frac{\pi}{3}$, $x = 18 \times \left(-\frac{\pi}{3}\right) - 32 \sin\left(-\frac{\pi}{3}\right)$

$= -6\pi + 16\sqrt{3}$ (About 8.863 cm)

and when $t = \frac{\pi}{3}$, $x = 18 \times \left(\frac{\pi}{3}\right) - 32 \sin\left(\frac{\pi}{3}\right)$

$= 6\pi - 16\sqrt{3}$

Thus the radius of the vase at the surface of the water is 8.863 cm

[3 marks]

Answer 5

(i) The graph of $f(x)$ will cross the x-axis when $x^2 (2 - 5x)^5 = 0$

That is, at $(0, 0)$ and $\left(\frac{2}{5}, 0\right)$

[2 marks]

(ii) By the product rule, $f'(x) = x^2 (5)(2 - 5x)^4 (-5) + 2x(2 - 5x)^5$

$= -25x^2 (2 - 5x)^4 + 2x (2 - 5x)^5$

$= x(2 - 5x)^4 (-25x + 4 - 10x)$

$= x (2 - 5x)^4 (4 - 35x)$

[3 marks]

(iii) When $x = 0$, $f'(0) = 0 \times 2^4 \times 4$ $\Rightarrow f'(0) = 0$ ☐

When $x = \frac{2}{5}$, $f'\left(\frac{2}{5}\right) = \left(\frac{2}{5}\right) \times 0^4 \times (-10) \Rightarrow f'\left(\frac{2}{5}\right) = 0$ ☐

The function is a polynomial of degree 7 with a negative coefficient for the x^7 term. So its graph must “enter” top left and “exit” bottom right.

Looking at $f(x) = x^2 (2 - 5x)^5$ the square indicates a tangent to the x-axis at the origin, so (0, 0) must be a minimum.

The power of 5 indicates a proper crossing of the x-axis but, as the point $\left(\frac{2}{5}, 0\right)$ has a gradient of zero that must be a point of inflection.

[4 marks]

Year 2 Pure Mathematics Examination Revision : Health Check N° 8

Just what the Doctor ordered

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

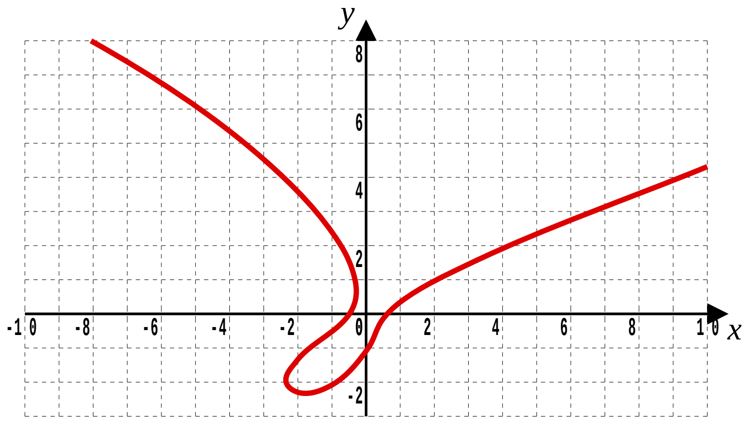
Solve the equation,

$$2^{2x+1} - 17 \times 2^{x} + 2^{3} = 0$$

[4 marks]

Question 2

The curve shown has equation $3x^2 - y^3 - 5xy = 1$



Find the numerical value of the gradient of the curve at the point $(2,\ 1)$

[5 marks]

Question 3

The number of daylight hours, h, in England, d days after the spring equinox (the day in spring when the number of daylight hours is 12) is modelled by,

$$h = 12 + \frac{9}{2}\,sin\left(\frac{2\pi}{365}\,d\right)$$

(i) Find the number of daylight hours 25 days after the spring equinox. Give your answer in hours and minutes.

[2 marks]

(ii) What is the maximum possible number of daylight hours, according to the model ? Give your answer in hours and minutes.

[1 mark]

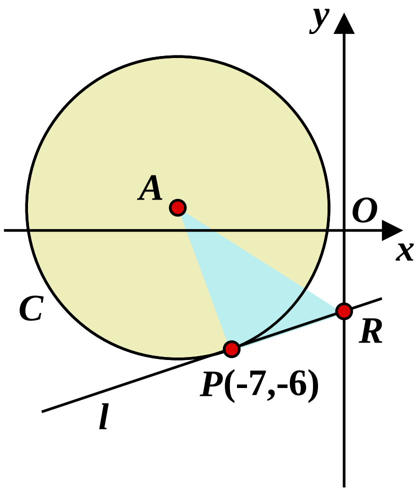
(iii) How many days after the spring equinox, does this maximum number of daylight hours occur ?

[1 mark]

(iv) For how many days of the year does the model suggest the number of daylight hours exceeds 15 hours.
Give your answer as a whole number of days.

[4 marks]

Question 4



The circle C has equation $x^2 + 18x + y^2 - 2y + 29 = 0$

(i) Verify the point $P(-7, -6)$ lies on C.

[2 marks]

(ii) Find an equation for the tangent to C at the point P, giving your answer in the form $y = mx + c$

[4 marks]

(iii) Find the area of the triangle APR

[4 marks]

Question 5

The second term of a geometric sequence is 4, and the sixteenth term of the same sequence is 9. The common ratio is r, where $r > 0$.

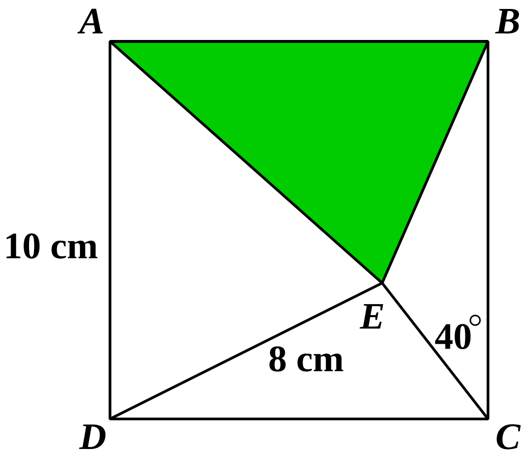
(i) Show that r satisfies the equation, $14\, ln\, r + ln\left(\frac{4}{9}\right) = 0$

[3 marks]

(ii) Hence, or otherwise, find the value of r correct to 3 significant figures.

[3 marks]

Question 6



ABCD is a square.
Angle *CED* is obtuse.
Find the area of the shaded triangle.

[7 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 8

Answer 1

$$2^{2x+1} - 17 \times 2^{x} + 2^{3} = 0$$

$$2\left(2^{x}\right)^{2} - 17\left(2^{x}\right) + 8 = 0$$

Let $y = 2^{x}$ in which case,

$$2y^{2} - 17y + 8 = 0$$

$$(2y - 1)(y - 8) = 0$$

Either $y = \frac{1}{2}$ or $y = 8$

That is, either $2^{x} = \frac{1}{2}$ or $2^{x} = 8$

$$\therefore x = -1,\ 3$$

[4 marks]

Answer 2

$$3x^{2} - y^{3} - 5xy = 1$$

Differentiating implicitly gives,

$$6x - 3y^{2}\frac{dy}{dx} - 5x\frac{dy}{dx} - 5y = 0$$

$$6x - 5y = \frac{dy}{dx}\left(3y^{2} + 5x\right)$$

$$\frac{dy}{dx} = \frac{6x - 5y}{3y^{2} + 5x}$$

At the point $(2,\ 1)$, $\frac{dy}{dx} = \frac{6 \times 2 - 5 \times 1}{3 \times 1^{2} + 5 \times 2}$

$$= \frac{7}{13}$$

[5 marks]

Answer 3

(i)

$$h = 12 + \frac{9}{2} sin\left(\frac{2\pi}{365} \times 25\right)$$

$$= 13.877$$

$$= 13 \text{ hours, } 53 \text{ minutes}$$

[2 marks]

(ii) 16 hours, 30 minutes

[1 mark]

(iii) $$\sin\left(\frac{2\pi}{365} \times d\right) = 1$$

$$\left(\frac{2\pi}{365} \times d\right) = \frac{\pi}{2}$$

$$d = \frac{\pi}{2} \times \frac{365}{2\pi}$$

$$= 91 \text{ days} \quad \text{That is, one season later}$$

[1 mark]

(iv) $$12 + \frac{9}{2}\sin\left(\frac{2\pi}{365} \times d\right) = 15$$

$$\frac{9}{2}\sin\left(\frac{2\pi}{365} \times d\right) = 3$$

$$\sin\left(\frac{2\pi}{365} \times d\right) = \frac{2}{3}$$

$$\left(\frac{2\pi}{365} \times d\right) = 0.729,\ \ 2.412$$

$$d = 42.35,\ \ 140.11$$

$$\therefore \text{ Days exceeding 15 hours of daylight } = 140.11 - 42.35$$

$$= 97.7$$

$$= 97 \text{ days}$$

[4 marks]

Answer 4

(i) $$\text{LHS} = x^2 + 18x + y^2 - 2y + 29$$

$$= (-7)^2 + 18(-7) + (-6)^2 - 2(-6) + 29$$

$$= 49 - 126 + 36 + 12 + 29$$

$$= 0$$

$$= \text{RHS}$$

$\therefore$ The point $(-7, -6)$ lies on C ☐

[2 marks]

(ii) First, find the coordinates of the centre of the circle, A

$$x^2 + 18x + y^2 - 2y + 29 = 0$$

$$\left[x^2 + 18x\right] + \left[y^2 - 2y\right] + 29 = 0$$

$$\left[(x + 9)^2 - 81\right] + \left[(y - 1)^2 - 1\right] + 29 = 0$$

$$(x + 9)^2 + (y - 1)^2 = 53$$

$\therefore$ The circle has centre $A(-9,\ 1)$ and radius $\sqrt{53}$

Secondly, get the gradient at P, either by using the fact that the tangent at P is perpendicular to the radius to P, or by implicitly differentiating the equation of the circle.

$$x^2 + 18x + y^2 - 2y + 29 = 0$$

$$2x + 18 + 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0$$

Divide through by 2

$$x + 9 + y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$x + 9 = \frac{dy}{dx}(1 - y)$$

$$\frac{dy}{dx} = \frac{x + 9}{1 - y}$$

$$= \frac{-7 + 9}{1 - (-6)} \quad \text{at } P(-7, -6)$$

$$= \frac{2}{7}$$

$$\therefore\ y = mx + c \Rightarrow -6 = \frac{2}{7}(-7) + c \Rightarrow c = -4$$

$$\therefore\ \text{Tangent to } C \text{ at } P \text{ is } y = \frac{2}{7}x - 4$$

(iii) We know that $A(-9,\ 1)$, $P(-7, -6)$ and $R(0, -4)$.

Also, there is a right angle at P in triangle APR where tangent meets radius.

Also, the length of AP is the radius of the circle, $\sqrt{53}$.

If the length of PR is found, the triangles area will be "half base times height".

$$|PR| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(0 - 7)^2 + (-4 - (-6))^2}$$

$$= \sqrt{49 + 4}$$

$$= \sqrt{53}$$

$$\therefore\ Area\ \Delta APR = \frac{1}{2} \times \sqrt{53} \times \sqrt{53}$$

$$= \frac{53}{2} \qquad \text{(i.e. 26.5)}$$

[4 marks]

Answer 5

(i) The question is telling us that,

$$ar = 4 \text{ and } a\,r^{15} = 9$$

As there is a $\frac{4}{9}$ in the result being aimed for, and no a,

$$\frac{ar}{a\,r^{15}} = \frac{4}{9}$$

$$\frac{1}{r^{14}} = \frac{4}{9}$$

$$r^{-14} = \frac{4}{9}$$

$$ln\left(r^{-14}\right) = ln\left(\frac{4}{9}\right)$$

$$-14\,ln\,r - ln\left(\frac{4}{9}\right) = 0$$

$$14\,ln\,r + ln\left(\frac{4}{9}\right) = 0 \qquad \square$$

[3 marks]

(ii)

$$14\,ln\,r + ln\left(\frac{4}{9}\right) = 0$$

$$ln\,r = -\frac{1}{14}ln\left(\frac{4}{9}\right)$$

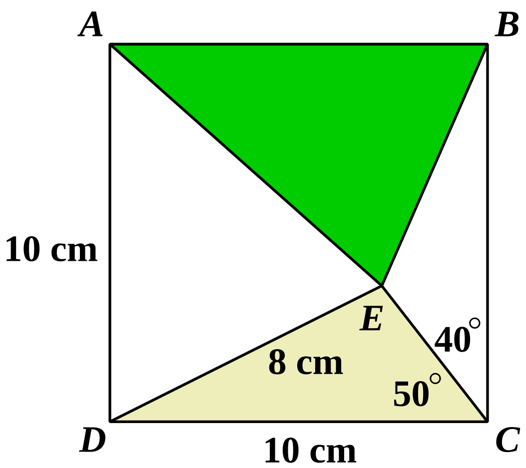
$$= 0.0579236$$

$$\therefore\ r = e^{0.0579236}$$

$$= 1.06 \text{ (3 sig fig asked for)}$$

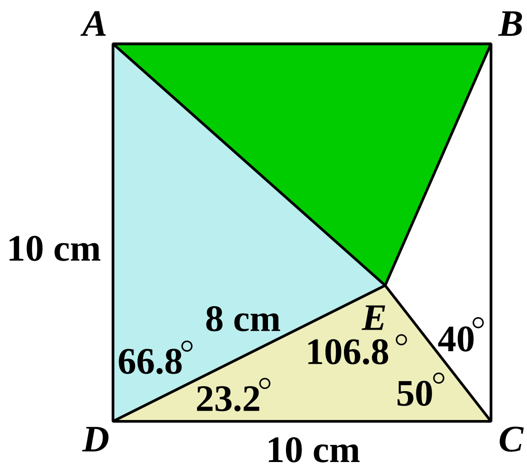
[3 marks]

Answer 6

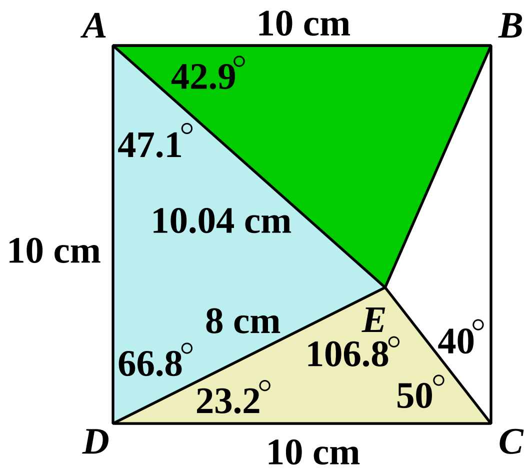


In ΔCDE $\quad \frac{sin\,E}{10} = \frac{sin\,50}{8} \Rightarrow sin\,E = 0.9576$

$\therefore\ E = 106.753°$ (Reject 73.247° as told it's obtuse)



$$\text{In } \triangle AED \qquad d^2 = 10^2 + 8^2 - 2 \times 10 \times 8 \cos 66.753°$$
$$= 100.85$$
$$d = 10.04 \text{ cm}$$
$$\cos A = \frac{10^2 + 10.04^2 - 8^2}{2 \times 10 \times 10.04}$$
$$= 0.6813$$
$$A = 47.1°$$



$$\text{In } \triangle ABE \qquad Area = \frac{1}{2} \times 10 \times 10.04 \times \sin 42.9°$$
$$= 34.2 \text{ cm}^2$$

[7 marks]

How did the doctor cure the invisible man ?
The doctor took him to the ICU !

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

Question 1

Find the exact solutions to the equation

$$e^{x} + 12\,e^{-x} = 7$$

[4 marks]

Question 2

Given that $|x| < \frac{1}{6}$ find the binomial expansion of $\sqrt{1 - 6x}$ in ascending powers of x up to and including the term in x^3, simplifying each term.

[4 marks]

Question 3

Prove that $\csc\theta \sec^2\theta = \csc\theta + \tan\theta \sec\theta$

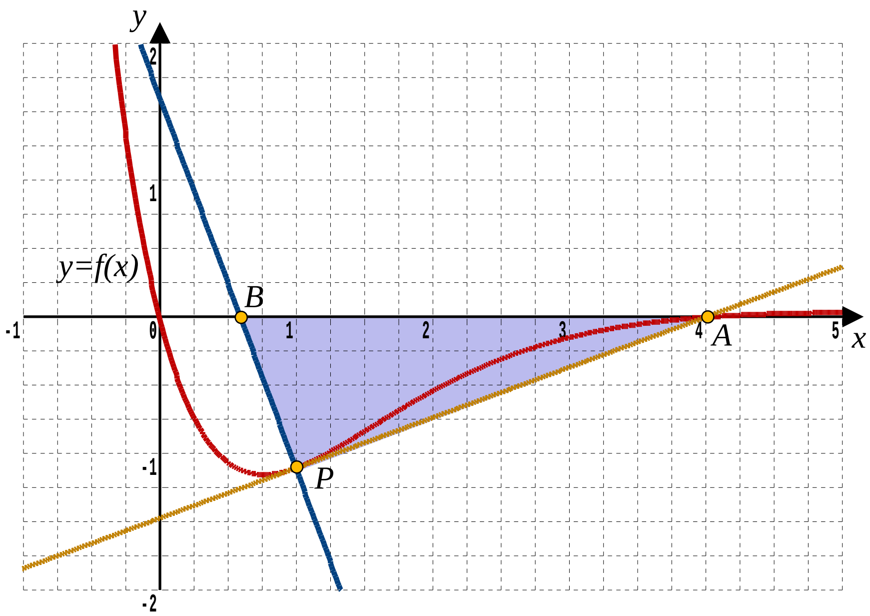
[4 marks]

Question 4

The graph is of the curve $y = f(x)$, where $f(x) = \left(x^2 - 4x\right) e^{-x}$

The point P has coordinates $\left(1, -\dfrac{3}{e}\right)$

The tangent to the curve at P intersects the x-axis at the point A.



(i) Find the (exact) equation of the tangent.

[5 marks]

The normal to the curve at P intersects the x-axis at the point B.

(ii) Show that the area of the triangle ABP is $\dfrac{9\left(e^2 + 1\right)}{2e^3}$

[8 marks]

Question 5

Sheldon's grandfather gives him £10 on his first birthday.
He says that he will give him £20 on his second birthday, £40 on his third, and so on, doubling his gift each year until Sheldon is 18.
If he goes through with his plan,

(i) How much will he have to give to Sheldon on his eighteenth birthday ?

[2 marks]

(ii) How much in total will he have given Sheldon by (and including) his eighteenth birthday ?

[3 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 9

Answer 1

$$e^{x} + 12\,e^{-x} = 7$$

Multiply through by e^{x}

$$(e^{x})^{2} + 12 = 7(e^{x})$$

$$(e^{x})^{2} - 7(e^{x}) + 12 = 0$$

$$(e^{x} - 3)(e^{x} - 4) = 0$$

Either $e^{x} = 3$ or $e^{x} = 4$

$x = ln\,3$ or $x = ln\,4$

[4 marks]

Answer 2

The Binomial Theorem states that,

$$(1 + x)^{n} = 1 + nx + \frac{n(n-1)}{2\,!}x^{2} + \ \ldots$$

... with $n = \dfrac{1}{2}$ and x replaced with $-6x$ yields ...

$$(1 - 6x)^{\frac{1}{2}} =$$

$$1 + \frac{1}{2}(-6x) + \frac{1}{2!} \times \frac{1}{2} \times \left(-\frac{1}{2}\right)(-6x)^{2} + \frac{1}{3!} \times \frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right)(-6x)^{3}$$

$$= 1 - 3x - \frac{9}{2}x^{2} - \frac{27}{2}x^{3}$$

[4 marks]

Answer 3

$$\begin{aligned}
\text{LHS} &= csc\,\theta\, sec^{2}\,\theta \\
&= \left(\frac{1}{\sin\theta}\right)(1 + tan^{2}\,\theta) \\
&= \left(\frac{1}{\sin\theta}\right)\left(1 + \frac{sin^{2}\,\theta}{cos^{2}\,\theta}\right) \\
&= \left(\frac{1}{\sin\theta}\right) + \left(\frac{\sin\theta}{cos^{2}\,\theta}\right) \\
&= csc\,\theta + \frac{sin\,\theta}{cos\,\theta} \times \frac{1}{cos\,\theta} \\
&= csc\,\theta + tan\,\theta\, sec\,\theta \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

Answer 4

(i)

$$f(x) = \left(x^2 - 4x\right)\left(e^{-x}\right)$$

$$f'(x) = \left(x^2 - 4x\right)\left(-e^{-x}\right) + \left(2x - 4\right)\left(e^{-x}\right)$$

$$f'(1) = (-3)\left(-e^{-1}\right) + (-2)\left(e^{-1}\right)$$

$$= \frac{3}{e} - \frac{2}{e}$$

$$= \frac{1}{e}$$

For the tangent, $y = mx + c$

$$-\frac{3}{e} = \frac{1}{e} \times 1 + c \Rightarrow c = -\frac{4}{e}$$

$$\therefore y = \frac{x}{e} - \frac{4}{e}$$

[5 marks]

(ii) The tangent will intersect the x-axis when $y = 0$,

$$x - 4 = 0 \Rightarrow x = 4 \Rightarrow A(4,\ 0)$$

For the normal, $y = mx + c$

$$-\frac{3}{e} = -e \times 1 + c \Rightarrow c = e - \frac{3}{e}$$

$$\therefore y = -ex + e - \frac{3}{e}$$

The normal will intersect the x-axis when $y = 0$,

$$ex = e - \frac{3}{e} \Rightarrow x = 1 - \frac{3}{e^2} \Rightarrow B\left(1 - \frac{3}{e^2},\ 0\right)$$

For $\triangle APB$,

$$Area = \frac{1}{2}\left(4 - 1 + \frac{3}{e^2}\right)\left(\frac{3}{e}\right)$$

$$= \frac{1}{2}\left(\frac{3e^2 + 3}{e^2}\right)\left(\frac{3}{e}\right)$$

$$= \frac{9\left(e^2 + 1\right)}{2e^3} \qquad \square$$

[8 marks]

Answer 5

(i) $a, \quad ar, \; a\,r^2, \; a\,r^3, \; a\,r^4, \; \ldots a\,r^{17}$

$1^{st}, \; 2^{nd}, \; 3^{rd}, \; 4^{th}, \; 5^{th}, \; \ldots 18^{th}$

The given sequence is in Geometric Progression with $a = 10, \; r = 2$

$$u_{18} = 10 \times 2^{17}$$
$$= £\ 1\ 310\ 720$$

[2 marks]

(ii)
$$SUM_{GP} = \frac{a\,(\,r^n - 1\,)}{r - 1}$$
$$= \frac{10\,(\,2^{18} - 1\,)}{2 - 1}$$
$$= £\ 2\ 621\ 430$$

[3 marks]

Year 2 Pure Mathematics Examination Revision : Health Check N° 10

Why did the banana go to the doctor ?
It wasn't peeling well !

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 34

Question 1

(i) Simplify,

$$log_4 x \;+\; 2\, log_4 y \;-\; log_4 (xy)$$

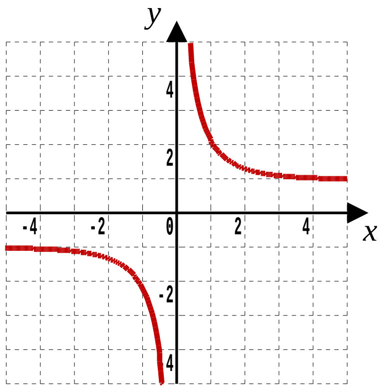
[2 marks]

(ii) Hence, or otherwise, solve the following, finding the exact value of y;

$$log_4 x \;+\; 2\, log_4 y \;-\; log_4 (xy) \;=\; 7$$

[2 marks]

Question 2



The graph is of the function, $f(x) = \dfrac{e^x + 1}{e^x - 1} \quad x \in \mathbb{R},\ x \neq 0$

(i) Determine the exact value of $f(ln\,3)$

[2 marks]

(ii) Find the inverse function, $f^{-1}(x)$

[4 marks]

(iii) With the help of the graph, state the domain of the inverse function.

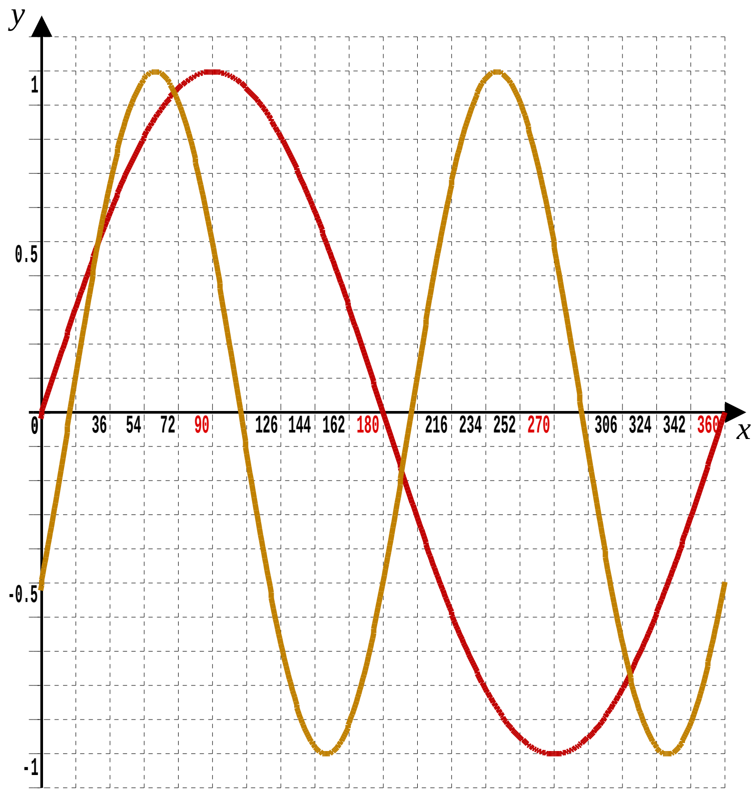
[2 marks]

Question 3

The two functions that graphed below are,

In red: $f(x) = \sin x \qquad 0 \leqslant x \leqslant 360°$

In gold: $g(x) = \sin(2x - 30°) \qquad 0 \leqslant x \leqslant 360°$



(i) From the graph, how many solutions are to be expected from solving the equation, $g(x) = f(x)$, $0 \leqslant x \leqslant 360°$

[1 mark]

(ii) Solve the equation $g(x) = f(x)$, $0 \leqslant x \leqslant 360°$

[5 marks]

Question 4

The following four points for a quadrilateral *ABCDA* in 3 dimensional space;

- $A(\ 2,\quad 7,\ -3\ )$
- $B(\ 8,\ -3,\quad 5\ )$
- $C(\ 4,\quad 0,\ -3\ )$
- $D(\ 7,\ -5,\quad 1\ )$

(i) Find $\overrightarrow{AB}$

[2 marks]

(ii) Show that quadrilateral *ABCDA* is a trapezium.
Give reasons for your answer.

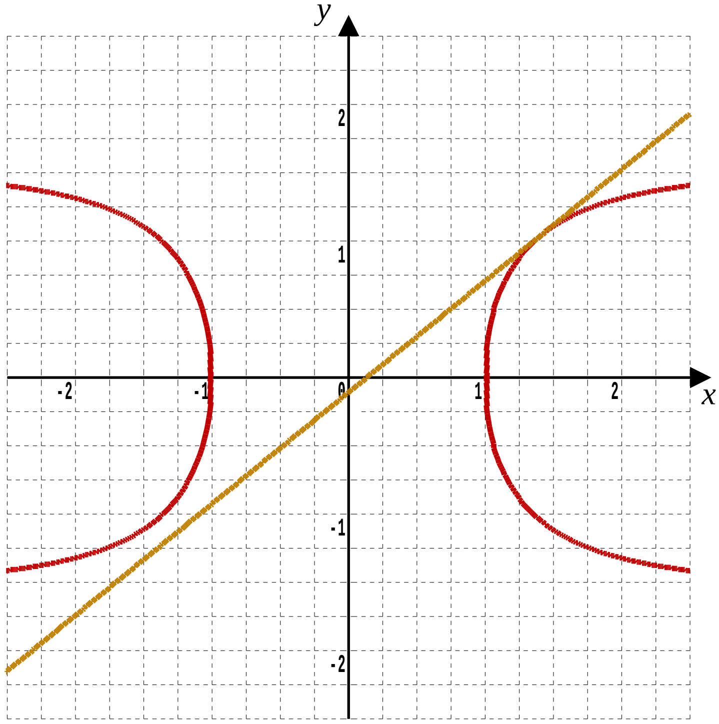
[3 marks]

(iii) Show that quadrilateral *ABCD* is not a parallelogram.
Give reasons for your answer.

[3 marks]

Question 5

The red curve in the graph below is of the equation, $x^2 \cos y = 1$



Find the (exact) equation of the tangent (shown in gold) to the curve when $y = \frac{\pi}{3}$

[8 marks]

Year 2 Pure Mathematics Examination Revision
Health Check N° 10

Answer 1

(i)

$$\begin{aligned} log_4 x + 2\,log_4 y - log_4(xy) &= log_4 x + log_4 y^2 - log_4(xy) \\ &= log_4\left(\frac{x\,y^2}{xy}\right) \\ &= log_4 y \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} log_4 x + 2\,log_4 y - log_4(xy) &= 7 \\ log_4 y &= 7 \\ \therefore\ y &= 4^7 \\ &= 16384 \end{aligned}$$

[2 marks]

Answer 2

(i)

$$\begin{aligned} f(ln\,3) &= \frac{e^{ln\,3} + 1}{e^{ln\,3} - 1} \\ &= \frac{3 + 1}{3 - 1} \Rightarrow f(ln\,3) = 2 \end{aligned}$$

[2 marks]

(ii)

$$\begin{aligned} f(x) &= \frac{e^x + 1}{e^x - 1} \\ x &= \frac{e^y + 1}{e^y - 1} \\ x(e^y - 1) &= e^y + 1 \\ x\,e^y - x &= e^y + 1 \\ x\,e^y - e^y &= x + 1 \\ e^y(x - 1) &= x + 1 \\ e^y &= \frac{x + 1}{x - 1} \\ y &= ln\left(\frac{x + 1}{x - 1}\right) \\ f^{-1}(x) &= ln\left(\frac{x + 1}{x - 1}\right) \end{aligned}$$

[4 marks]

(iii) With help from the graph*, $f(x) \in \mathbb{R}, \quad x < -1, \quad x > 1$

[2 marks]

* Starting with $ln\left(\dfrac{x+1}{x-1}\right)$ and using the restriction imposed by the ln function, that $\dfrac{x+1}{x-1} > 0$, is tricky as "rational inequalities" are not covered by the A-Level course.
Essentially, you need to identify the critical values, which are the values of *x* that make the denominator equal to zero, as you would expect, and **the values of *x* that make the numerator equal zero**.
This gives ± 1 as the critical values.
Each of the possible regions are then tested with a sample value of *x*.

Sample $x = -2 \Rightarrow \left(\dfrac{-2+1}{-2-1}\right) = \dfrac{1}{3} > 0 \ \therefore$ TRUE

Sample $x = 0 \Rightarrow \left(\dfrac{0+1}{0-1}\right) = -1 < 0 \ \therefore$ FALSE

Sample $x = 2 \Rightarrow \left(\dfrac{2+1}{2-1}\right) = 3 > 0 \ \therefore$ TRUE

$$\therefore \ f(x) \in \mathbb{R}, \ f(x) < -1, \ f(x) > 1$$

Answer 3

(i) From the graph, the number of solutions, given the domain of *x*, is 4

[1 mark]

(ii)

$$\begin{aligned} g(x) &= f(x) \\ sin(2x - 30) &= sin\,x \\ arcsin(sin(2x - 30)) &= arcsin(sin\,x) \\ 2x - 30 &= x,\ 180 - x, \\ &\quad x + 360,\ 180 - x + 360, \\ &\quad x + 360 + 360,\ 180 - x + 360 + 360 \\ &= x,\ 180 - x, \quad \text{case 1, case 2} \\ &\quad x + 360,\ 540 - x \quad \text{case 3, case 4} \\ &\quad x + 720,\ 900 - x \quad \text{case 5, case 6} \end{aligned}$$

Case 1 : $2x - 30 = x \Rightarrow x = 30°$

Case 2 : $2x - 30 = 180 - x \Rightarrow 3x = 210 \Rightarrow x = 70°$

Case 3 : $2x - 30 = x + 360 \Rightarrow x = 390°$ Too big - REJECT !

Case 4 : $2x - 30 = 540 - x \Rightarrow 3x = 570 \Rightarrow x = 190°$

Case 5 : $2x - 30 = x + 720 \Rightarrow x = 750°$ Too big - REJECT !

Case 6 : $2x - 30 = 900 - x \Rightarrow 3x = 930 \Rightarrow x = 310°$

$$\therefore \ x \in \{30°,\ 70°,\ 190°,\ 310°\}$$

[5 marks]

Answer 4

(i)

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$$

$$= \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -10 \\ 8 \end{pmatrix} \qquad (\text{i.e. } 6\boldsymbol{i} - 10\boldsymbol{j} + 8\boldsymbol{k})$$

[2 marks]

(ii) Part (i) is a subtle hint that $\overrightarrow{AB}$ is one of the parallel sides, if a kind person made up the question !

Examining $\overrightarrow{DC}$ $\left(\text{or } \overrightarrow{CD}\right)$ is thus a tentative next step.

$$\overrightarrow{DC} = \boldsymbol{c} - \boldsymbol{d}$$

$$= \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix} \qquad (\text{i.e. } -3\boldsymbol{i} + 5\boldsymbol{j} - 4\boldsymbol{k})$$

Observe that $\overrightarrow{AB} = (-2)\,\overrightarrow{DC}$

From the wording of the question, A, B, C and D are not collinear.

The inescapable conclusion is that $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$

$\therefore$ $ABCDA$ is a trapezium ☐

[3 marks]

(iii) Strategy: show that $\overrightarrow{BC}$ is not parallel to $\overrightarrow{DA}$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ -8 \end{pmatrix}$$

$$\overrightarrow{DA} = \boldsymbol{a} - \boldsymbol{d} = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 12 \\ -4 \end{pmatrix}$$

Clearly, there is no value of λ such that $\overrightarrow{BC} = \lambda\left(\overrightarrow{DA}\right)$

$\therefore$ $ABCDA$ is not a parallelogram ☐

[3 marks]

Answer 5

Firstly, when $y = \frac{\pi}{3}$, we have $x^2 \cos\left(\frac{\pi}{3}\right) = 1$

$$x^2 \times \frac{1}{2} = 1$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

From the graph, the point at which the tangent is to be found is $\left(\sqrt{2}, \frac{\pi}{3}\right)$

$$x^2 \cos y = 1$$

Differentiate implicitly with respect to x

$$x^2(-\sin y)\frac{dy}{dx} + 2x\cos y = 0$$

$$x^2 \sin y \frac{dy}{dx} = 2x\cos y$$

$$\frac{dy}{dx} = \frac{2x\cos y}{x^2 \sin y} \qquad \text{as } x \neq 0$$

$$= \frac{2}{x\tan y}$$

$$= \frac{2}{\sqrt{2}\tan\left(\frac{\pi}{3}\right)}$$

$$= \frac{2}{\sqrt{2}\sqrt{3}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{2\sqrt{6}}{6}$$

$$= \frac{\sqrt{6}}{3}$$

$$y = mx + c \Rightarrow \frac{\pi}{3} = \frac{\sqrt{6}}{3} \times \sqrt{2} + c$$

$$c = \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

$$\therefore \text{ Tangent is } y = \frac{\sqrt{6}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

[8 marks]

Year 2 Pure Mathematics Examination Revision : Health Check N° 11

Who is the coolest doctor in the hospital ?
The hip consultant !

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 47

Question 1

Given $y = x(2x+1)^4$, show that $\dfrac{dy}{dx} = (2x+1)^n(Ax+B)$

where n, A and B are constants to be found.

[4 marks]

Question 2

(i) The fourth term of a geometric series is 16 and the seventh term of the series is 250. Find both the common ratio and the first term.

[4 marks]

(ii) The third, fourth and fifth terms of an arithmetic sequence are $3k$, $4k + 3$ and $6k - 9$ respectively, where k is a constant.
Show that the sum of the first n terms of the sequence is a square number.

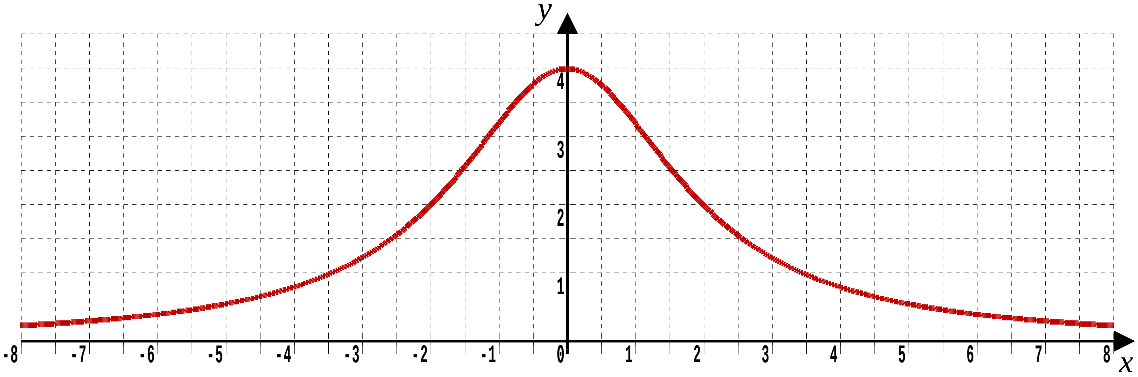
[5 marks]

(iii) The first term of a geometric progression is 4 and the sum to infinity is 12.
Find the common ratio.

[3 marks]

Question 3

The graph is of the function, $f(x) = \dfrac{16}{4 + x^2} \quad x \in \mathbb{R}$



(a) Find the exact coordinates of the two points of inflection

[6 marks]

The function m and the function n are defined by

$$m(x) = \frac{1}{4} f(x) \qquad x \in \mathbb{R}$$

$$n(x) = \frac{1}{3} f(x) + 4 \qquad x \in \mathbb{R},\ x \geqslant \frac{2\sqrt{3}}{3}$$

(b) Find **(i)** the range of m

[1 mark]

(ii) the range of n

[2 marks]

Question 4

(a) Express $5\,cos\,\theta\ -\ 8\,sin\,\theta$ in the form $R\,cos\,(\,\theta\ +\ \alpha\,)$, where $R\ >\ 0$ and $0 < \alpha < \pi$. Write R in surd form and give α to 4 decimal places.

[4 marks]

The temperature of a kiln, T °C, used to fire pottery, can be modelled by,

$$T\ =\ 1100\ +\ 5\,cos\left(\frac{x}{3}\right)\ -\ 8\,sin\left(\frac{x}{3}\right),\qquad 0\ \leqslant\ x\ \leqslant\ 72$$

where x is the time in hours since the pottery was placed in the kiln.

(b) Calculate the maximum value of T predicted by this model and the value of x, to 2 decimal places, when this maximum first occurs.

[4 marks]

(c) Calculate the times during the first 24 hours when the temperature is predicted, by this model, to be exactly 1097 °C

[4 marks]

Question 5

The value, £V, of a vintage car t years after it was first valued on 1st January 2001 is modelled by the equation,

$$V = A p^t \qquad \text{where } A \text{ and } p \text{ are constants}$$

Given that the value of the car was £32 000 on 1st January 2005
and £50 000 on 1st January 2012

(a) **(i)** find p to 4 decimal places

(ii) show that A is approximately 24 800

[4 marks]

(b) With reference to the model, interpret

(i) the value of the constant A

(ii) the value of the constant p

[2 marks]

Using the model.

(c) find the year during which the value of the car first exceeds £100 000

[4 marks]

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Answer 1

$$y = x(2x+1)^4$$
$$\frac{dy}{dx} = x \times 4(2x+1)^3 \times 2 + (2x+1)^4$$
$$= (2x+1)^3(8x+2x+1)$$
$$= (2x+1)^3(10x+1)$$
$$\therefore\ n = 3,\ A = 10 \text{ and } B = 1$$

[4 marks]

Answer 2

(i) The question is telling us that, $a r^3 = 16$ and $a r^6 = 250$

$$\therefore \frac{a r^6}{a r^3} = \frac{250}{16}$$
$$r^3 = \frac{125}{8}$$
$$r = \frac{5}{2}$$

As $a r^3 = 16 \Rightarrow a\left(\frac{5}{2}\right)^3 = 16$

$$a = \frac{16 \times 8}{125}$$
$$= \frac{128}{125}$$

[4 marks]

(ii) Equating the two ways that the common difference can be found,

$$(6k-9)-(4k+3) = (4k+3)-(3k)$$
$$2k-12 = k+3$$
$$k = 15$$

So the arithmetic progression is, 9, 27, 45, 63, 81, 99, 117, ...

That is, a is 9 and d is 18

$$Sum = \frac{n}{2}\{2a+(n-1)d\}$$
$$= \frac{n}{2}\{18+(n-1)18\}$$
$$= 9n^2$$
$$= (3n)^2 \quad \text{which is a square number} \quad \square$$

[5 marks]

(iii) The question is telling us that $a = 4$ and $\dfrac{a}{1 - r} = 12$

$$\therefore \frac{4}{1 - r} = 12$$

$$4 = 12(1 - r)$$

$$4 = 12 - 12r$$

$$12r = 8$$

$$r = \frac{2}{3}$$

[3 marks]

Question 3

(a) By the quotient rule,

$$f'(x) = \frac{(4 + x^2) \times 0 - (2x) \times 16}{(4 + x^2)^2}$$

$$= \frac{(-32x)}{(4 + x^2)^2}$$

$$f''(x) = \frac{(4 + x^2)^2 \ (-32) - 2(4 + x^2)^1 \ 2x(-32x)}{(4 + x^2)^4}$$

$$= \frac{(4 + x^2)(-32) - 4x(-32x)}{(4 + x^2)^3}$$

$$= \frac{-128 - 32x^2 + 128x^2}{(4 + x^2)^3}$$

$$= \frac{96x^2 - 128}{(4 + x^2)^3}$$

At the points of inflection, $f''(x) = 0$

$$\frac{96x^2 - 128}{(4 + x^2)^3} = 0$$

$$\therefore 96x^2 - 128 = 0$$

$$x^2 = \frac{128}{96}$$

$$x = \pm\frac{2\sqrt{3}}{3}$$

$$\Rightarrow\ y = 16 \div \left(4 + \left(\frac{2\sqrt{3}}{3}\right)^2\right)$$
$$= 16 \div \left(\frac{36 + 12}{9}\right)$$
$$= 16 \times \frac{9}{48}$$
$$= 3$$

So the two points of inflection are, $\left(\pm\frac{2\sqrt{3}}{3},\ 3\right)$

[6 marks]

(b) **(i)** The range of the original function is $f(x) \in \mathbb{R},\ 0 < f(x) \leqslant 4$

For $m(x)$ the range will be $m(x) \in \mathbb{R},\ 0 < m(x) \leqslant 1$

[1 mark]

(ii) In order to get $n(x)$, x first has to go into $f(x)$ but what can come out of $f(x)$ is now restricted by what is allowed in.

The range of $f(x)$ this time is $f(x) \in \mathbb{R},\ 0 < f(x) \leqslant 3$

For $n(x)$ the range will be $n(x) \in \mathbb{R},\ 4 < n(x) \leqslant 5$

[2 marks]

Answer 4

(a) $R\cos(x + \alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

Compare this with the expression given of $5\cos x - 8\sin x$

$$\Rightarrow\ \frac{R\sin\alpha}{R\cos\alpha} = \frac{8}{5}$$
$$\alpha = \arctan\left(\frac{8}{5}\right)$$
$$= 1.0122 \text{ radians}$$
$$R = \sqrt{5^2 + 8^2}$$
$$= \sqrt{89}$$
$$\therefore\ 5\cos x - 8\sin x = \sqrt{89}\cos(x + 1.0122)$$

[4 marks]

(b)

$$T = 1100 + 5\,cos\left(\frac{x}{3}\right) - 8\,sin\left(\frac{x}{3}\right), \qquad 0 \leqslant x \leqslant 72$$

$$= 1100 + \sqrt{89}\,cos\left(\frac{x}{3} + 1.0122\right) \qquad \text{using part (a)}$$

T will have a maximum value when the *cosine* function

has it's maximum value of 1

$$\therefore\ T_{MAX} = 1100 + \sqrt{89} \qquad \text{(About } 1109.4\,°\text{C)}$$

In more detail, this will occur when,

$$cos\left(\frac{x}{3} + 1.0122\right) = 1$$

$$\frac{x}{3} + 1.0122 = 0,\ 2\pi,\ 4\pi,\ \ldots$$

$$\frac{x}{3} = 5.271 \qquad \text{(As only after when first occurs)}$$

$$x = 15.81 \text{ hours} \qquad \text{(15 hours 49 minutes)}$$

[4 marks]

(c)

$$T = 1097$$

$$1100 + 5\,cos\left(\frac{x}{3}\right) - 8\,sin\left(\frac{x}{3}\right) = 1097$$

$$1100 + \sqrt{89}\,cos\left(\frac{x}{3} + 1.0122\right) = 1097$$

$$\sqrt{89}\,cos\left(\frac{x}{3} + 1.0122\right) = -3$$

$$cos\left(\frac{x}{3} + 1.0122\right) = -\frac{3}{\sqrt{89}} \qquad (WA \text{ is } 1.2471)$$

$$\frac{x}{3} + 1.0122 = 1.8944,\ 4.3887,\ 8.1776,\ 10.6719,\ \ldots$$

$$\frac{x}{3} = 0.8822,\ 3.3765,\ 7.1654,\ 9.6597,\ \ldots$$

$$x = 2.6466,\ 10.1295,\ 21.4962,\ 28.9791,\ \ldots$$

x = 2 hours 39 minutes,

10 hours 8 minutes,

21 hours 30 minutes

[4 marks]

Answer 5

(a) As 1st January 2005 is four years after 1st January 2001,

$$32000 = A p^4 \Rightarrow A = \frac{32000}{p^4}$$

Similarly, as 1st January 2012 is eleven years after 1st January 2001,

$$50000 = A p^{11} \Rightarrow A = \frac{50000}{p^{11}}$$

$$\therefore \frac{32000}{p^4} = \frac{50000}{p^{11}}$$

$$\frac{p^{11}}{p^4} = \frac{50000}{32000}$$

$$p^7 = \frac{25}{16}$$

$$p = 1.0658$$

$$A = \frac{32000}{p^4}$$

$$= \frac{32000}{1.0658^4}$$

$$= 24799.7$$

Which is approximately 24 800 ☐

[4 marks]

(b) **(i)** A is the value of the car on 1st January 2001

(ii) p is a measure of the growth rate of the value.
1.0658 corresponds to an annual increase of 6.58%

[2 marks]

(c) To find the critical value, $v = 100000$

$$24799.7 \times 1.0658^n = 100000$$

$$1.0658^n = 4.0323$$

$$ln\, 1.0658^n = ln\, 4.0323$$

$$n = \frac{ln\, 4.0323}{ln\, 1.0658}$$

$$= 21.88$$

$\therefore$ The value first exceeds £100 000 in 2022

[4 marks]