Year FM

**Further Pure Mathematics Examination Revision**

**Health Check N° 10**

**Answer 1**

$r = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$ is perpendicular to $r_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \mu_3 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ and this is proven by showing that the scalar product of the two line directions is zero.

So the claim is that, $\begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = 0$

$$\begin{aligned} \text{LHS} &= \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \\ &= (-1)(4) + (-2)(3) + (5)(2) \\ &= -4 - 6 + 10 \\ &= 0 \\ &= \text{RHS} \qquad \square \end{aligned}$$

**[ 3 marks ]**

**Answer 2**

**( i )**

From $z_1 = r_1 ( \cos \theta_1 + \mathrm{i} \sin \theta_1 )$ we get that $| z_1 | = r_1$ and $\arg ( z_1 ) = \theta_1$

From $z_2 = r_2 ( \cos \theta_2 + \mathrm{i} \sin \theta_2 )$ we get that $| z_2 | = r_2$ and $\arg ( z_2 ) = \theta_2$

$$\frac{z_1}{z_2} = \frac{r_1 ( \cos \theta_1 + \mathrm{i} \sin \theta_1 )}{r_2 ( \cos \theta_2 + \mathrm{i} \sin \theta_2 )} \times \frac{r_2 ( \cos \theta_2 - \mathrm{i} \sin \theta_2 )}{r_2 ( \cos \theta_2 - \mathrm{i} \sin \theta_2 )}$$

$$= \frac{r_1 r_2}{r_2 r_2} \times \frac{\cos \theta_1 \cos \theta_2 - \mathrm{i} \cos \theta_1 \sin \theta_2 + \mathrm{i} \sin \theta_1 \cos \theta_2 - \mathrm{i}^2 \sin \theta_1 \sin \theta_2}{\cos^2 \theta_2 - \mathrm{i} \cos \theta_2 \sin \theta_2 + \mathrm{i} \cos \theta_2 \sin \theta_2 - \mathrm{i}^2 \sin^2 \theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + \mathrm{i} ( \sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2 )}{\cos^2 \theta_2 + \sin^2 \theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos ( \theta_1 - \theta_2 ) + \mathrm{i} \sin ( \theta_1 - \theta_2 )}{1}$$

$$= \frac{r_1}{r_2} \times ( \cos ( \theta_1 - \theta_2 ) + \mathrm{i} \sin ( \theta_1 - \theta_2 ) )$$

Now, referring back to the first two lines of our proof we can write,

$$\left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} \quad \text{and} \quad \arg\left( \frac{z_1}{z_2} \right) = \theta_1 - \theta_2$$

$$= \frac{|z_1|}{|z_2|} \quad \text{and} \quad = \arg(z_1) - \arg(z_2)$$

This answer tidies up the ragged ending to the similar proof in Core Book 1, page 25

**[ 5 marks ]**

**( ii )** **( a )** $\arg(z_1) = \arg( 1 - \sqrt{3}\,\mathrm{i} )$

$$= \arctan\left( \frac{-\sqrt{3}}{1} \right)$$

$$= -\frac{\pi}{3}$$

**( b )** $\arg(z_2) = \arg( \sqrt{3} + \mathrm{i} )$

$$= \arctan\left( \frac{1}{\sqrt{3}} \right)$$

$$= \frac{\pi}{6}$$

**( c )** $\arg\left( \frac{z_1}{z_2} \right) = \arg(z_1) - \arg(z_2)$

$$= -\frac{\pi}{3} - \frac{\pi}{6}$$

$$= -\frac{\pi}{2}$$

**[ 1, 1, 1 marks ]**

**Answer 3**

**( i )** $\quad$ LHS $= \tanh x$

$$= \frac{\sinh x}{\cosh x}$$

$$= \sinh x \div \cosh x$$

$$= \frac{e^x - e^{-x}}{2} \times \frac{2}{e^x + e^{-x}}$$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}} \times \frac{e^x}{e^x}$$

$$= \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$= \text{RHS} \qquad \square$$

**[ 3 marks ]**

**( ii )** As $x \to \infty$, $e^{2x} - 1 \to e^{2x}$ and $e^{2x} + 1 \to e^{2x}$

and so $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1} \to \dfrac{e^{2x}}{e^{2x}} = 1$

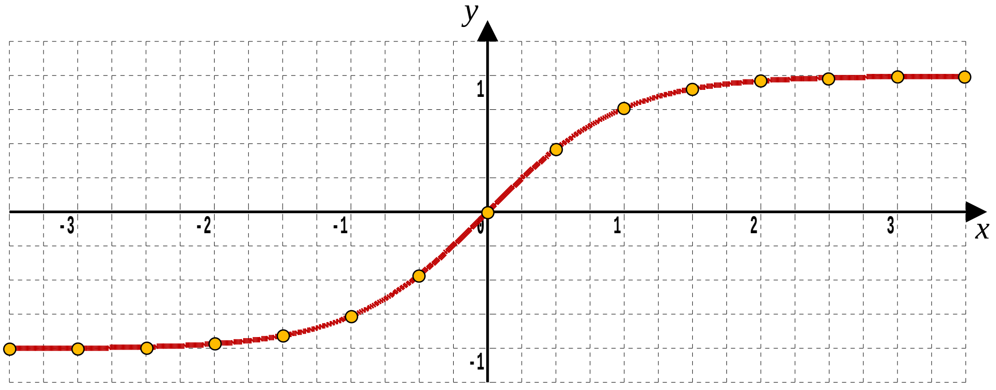
**[ 2 marks ]**

**( iii )** As $x \to -\infty$, $e^{2x} - 1 \to -1$ and $e^{2x} + 1 \to 1$

and so $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1} \to \dfrac{-1}{1} = -1$

**[ 2 marks ]**

**( iv )**




**[ 2 marks ]**

**( v )** From $3 \sinh x = p \cosh x$ we get that, $\tanh x = \dfrac{p}{3}$

From the graph, $-1 < \tanh x < 1$

$$-1 < \frac{p}{3} < 1$$

$$-3 < p < 3$$

**[ 2 marks ]**

**Answer 4**

$$\begin{aligned}
\text{Let } x &= r\cos\theta \\
&= (1 + \sin\theta)\cos\theta \\
&= \cos\theta + \sin\theta\cos\theta \\
&= \frac{1}{2}\sin(2\theta) + \cos\theta \\
\frac{dx}{d\theta} &= \cos(2\theta) - \sin\theta
\end{aligned}$$

$$\begin{aligned}
\text{For a vertical tangent, } \frac{dx}{d\theta} &= 0 \\
\cos(2\theta) - \sin\theta &= 0 \\
\cos^2\theta - \sin^2\theta - \sin\theta &= 0 \\
1 - 2\sin^2\theta - \sin\theta &= 0 \\
2\sin^2\theta + \sin\theta - 1 &= 0 \\
(2\sin\theta - 1)(\sin\theta + 1) &= 0
\end{aligned}$$

Either $\sin\theta = -1$ in which case $\theta = -\frac{\pi}{2}$ (in the specified interval)

Or $\sin\theta = \frac{1}{2}$ in which case $\theta = \frac{\pi}{6}$ (which is 30°)

When $\theta = \frac{\pi}{6}$, $r = 1 + \sin\left(\frac{\pi}{6}\right)$ giving $r = \frac{3}{2}$

Thus there could indeed be a vertical tangent to the curve that touches it at the point with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ □

**[ 6 marks ]**

**Answer 5**

**( i )** $y = arccos\, x \Rightarrow x = cos\, y$

$$\frac{dx}{dy} = -sin\, y$$

Now use the trigonometry identity $cos^2 y + sin^2 y = 1$ to get...

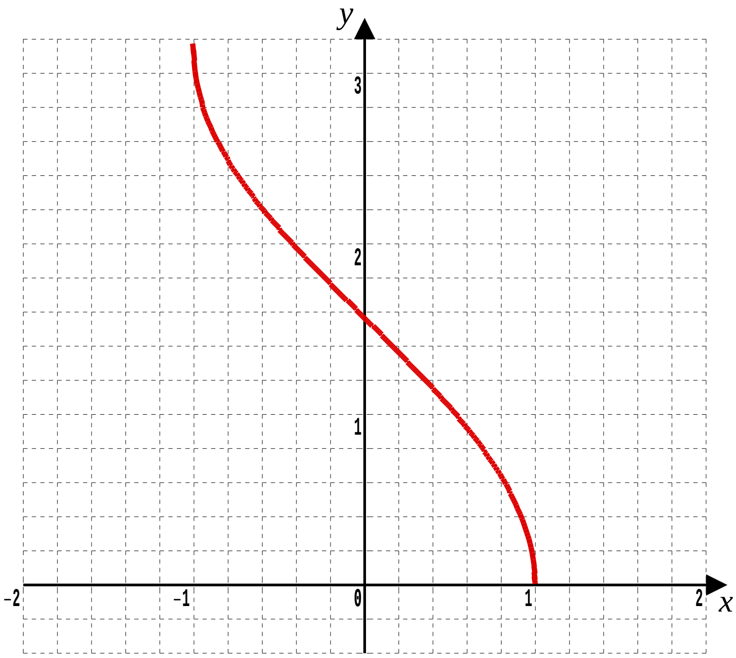
$$\frac{dx}{dy} = \pm\sqrt{1 - cos^2 y}$$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

(Need to remove $\pm 1$ to avoid a division by zero)

From the graph of $y = arccos\, x$ it can be seen that the gradient is always negative

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1 \qquad \square$$




The graph of $y = arccos\, x$

It is the inverse of a one-to-one piece of the $cos\, x$ function

**[ 4 marks ]**

**( ii )** **( a )** The function $g(x) = arccos\, x$ is only defined on domain $-1 \leqslant x \leqslant 1$ so the related function $f(x) = arccos\left(e^x\right)$ requires that $e^x$ be constrained such that $-1 \leqslant e^x \leqslant 1$. As $e^x$ is always positive (and an increasing function) there is no issue at the left end of this interval. To prevent $e^x$ exceeding the maximum value of +1 requires the constraint $x \leqslant 0$.

**[ 2 marks ]**

**( b )**

$$f(x) = arccos\left(e^x\right) \qquad x \leqslant 0$$

$$f'(x) = -\frac{1}{\sqrt{1-(e^x)^2}} \times e^x \text{ by the chain rule, } x < 0$$

At a stationary point $f'(x) = 0$.

However, we have $x < 0$ or, more usefully, $0 < e^x < 1$

from which it is clear $f'(x)$ cannot equal zero.

$\therefore$ There can be no stationary points. $\square$

**[ 2 marks ]**

**Answer 6**

**( i )**

$$\frac{4x^2 + 3x + 14}{[\,(x+2)(x^2+4)\,]} = \frac{A}{(x+2)} + \frac{Bx + C}{(x^2+4)}$$

Multiply through by the red bracket...

$$4x^2 + 3x + 14 = A(x^2 + 4) + (Bx + C)(x + 2)$$

Let $x = -2$ in which case, $24 = 8A \Rightarrow A = 3$

Let $x = 0$ in which case, $14 = 12 + 2C \Rightarrow C = 1$

Let $x = 1$ in which case, $21 = 15 + (B + 1)(3)$

$$6 = 3(B + 1) \Rightarrow B = 1$$

Conclusion:

$$\frac{4x^2 + 3x + 14}{(x+2)(x^2+4)} = \frac{3}{(x+2)} + \frac{x + 1}{(x^2+4)}$$

**[ 3 marks ]**

**( ii )** $\displaystyle\int_0^1 \frac{4x^2 + 3x + 14}{(x+2)(x^2+4)}\,dx$

$$= 3\int_0^1 \frac{1}{x+2}\,dx + \frac{1}{2}\int_0^1 \frac{2x}{(x^2+4)}\,dx + \int_0^1 \frac{1}{(x^2+2^2)}\,dx$$

$$= \left[\, 3\,ln|x+2| + \frac{1}{2}\,ln|x^2+4| + \frac{1}{2}\,arctan\left(\frac{x}{2}\right) \right]_0^1$$

$$= 3\,ln\,3 + \frac{1}{2}\,ln\,5 + \frac{1}{2}arctan\left(\frac{1}{2}\right) - 3\,ln\,2 - \frac{1}{2}ln\,4$$

$$= ln\,27 + ln\sqrt{5} - ln\,8 - ln\,2 + \frac{1}{2}arctan\left(\frac{1}{2}\right)$$

$$= ln\left(\frac{27\sqrt{5}}{16}\right) + \frac{1}{2}arctan\left(\frac{1}{2}\right)$$

**[ 5 marks ]**

**Answer 7**

**( i )**

$$
\begin{aligned}
|z_1 - z_2| &= |(4 + 2i) - (-3 + i)| \\
&= |4 + 2i + 3 - i| \\
&= |7 + i| \\
&= \sqrt{7^2 + 1^2} \\
&= \sqrt{50} \\
&= 5\sqrt{2}
\end{aligned}
$$

**[ 2 marks ]**

**( ii )**

$$
\begin{aligned}
w &= \frac{z_1}{z_2} \\
&= \frac{4 + 2i}{-3 + i} \times \frac{-3 - i}{-3 - i} \\
&= \frac{-12 - 4i - 6i - 2i^2}{9 - i^2} \\
&= \frac{-10 - 10i}{10} \\
&= -1 - i
\end{aligned}
$$

**[ 2 marks ]**

**( iii )**

$$
\begin{aligned}
arg\, w &= arg(-1 - i) \\
&= arctan\left(\frac{-1}{-1}\right) \\
&= arctan(1) \\
&= -\pi + \frac{\pi}{4} \text{ or } \frac{\pi}{4} \text{ (Reject)} \\
&= -\frac{3\pi}{4}
\end{aligned}
$$

**[ 2 marks ]**