Year FM
Further Pure Mathematics Examination Revision
Health Check N° 3

Answer 1
The key result needed, that can be quoted without proof, is,

- $$z^n + \frac{1}{z^n} = 2\cos(n\theta)$$

If forgotten it is easily proven...
Proof

$$\begin{aligned}
\text{LHS} &= z^n + \frac{1}{z^n} \\
&= z^n + z^{-n} \\
&= (\cos\theta + \mathrm{i}\sin\theta)^n + (\cos\theta + \mathrm{i}\sin\theta)^{-n} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta) && \text{de Moivre's theorem} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta) && \text{even \& odd functions} \\
&= 2\cos(n\theta) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

And so to the question...

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^3 &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \\
&= \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right) \\
(2\cos\theta)^3 &= 2\cos 3\theta + 3 \times 2\cos\theta \\
8\cos^3\theta &= 2\cos 3\theta + 6\cos\theta \\
\cos^3\theta &= \frac{1}{4}(\cos 3\theta + 3\cos\theta) \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

(i)

$$\text{LHS} = \int \frac{1}{a^2 + x^2}\, dx$$

$$= \int \frac{1}{a^2 + (a \tan\theta)^2}\ a \sec^2\theta\ d\theta$$

$$= \int \frac{a \sec^2\theta}{a^2 (1 + \tan^2\theta)}\, d\theta$$

$$= \int \frac{a \sec^2\theta}{a^2 \sec^2\theta}\, d\theta$$

$$= \int \frac{1}{a}\, d\theta$$

$$= \frac{\theta}{a} + c$$

$$= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$= \text{RHS} \qquad \square$$

[3 marks]

$$\int f(x)\ dx = \int \frac{8x - 3}{4 + x^2}\, dx$$

$$= 4 \int \frac{2x}{4 + x^2}\, dx - 3 \int \frac{1}{2^2 + x^2}\, dx$$

Setting up a chain rule backwards, and preparing to use the part (i) result

$$= 4\, ln\left(4 + x^2\right) - 3 \times \frac{1}{2} \arctan\left(\frac{x}{2}\right) + c$$

Which is of the form required with $A = 4$ and $B = -\frac{3}{2}$

[4 marks]

Answer 3

Step 1 : Find the determinant of of **A**, *det* **A**

There is a shortcut available by expanding along the middle row

$$\begin{vmatrix} 2p & p & 2 \\ 3 & 0 & 0 \\ -1 & 1 & -1 \end{vmatrix} = -3\begin{vmatrix} p & 2 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 2p & 2 \\ -1 & -1 \end{vmatrix} - 0\begin{vmatrix} 2p & p \\ -1 & 1 \end{vmatrix}$$

$$= -3\left(p \times (-1) - 1 \times 2\right)$$

$$= 3p + 6$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} p & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2p & 2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2p & p \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} p & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2p & 2 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 2p & p \\ 3 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -3 & 3 \\ -p-2 & -2p+2 & 2p+p \\ 0 & -6 & -3p \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

$$\mathbf{C} = \begin{pmatrix} 0 & 3 & 3 \\ p+2 & -2p+2 & -3p \\ 0 & 6 & -3p \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^T$, the transpose of the matrix of cofactors.

$$\mathbf{C}^T = \begin{pmatrix} 0 & p+2 & 0 \\ 3 & -2p+2 & 6 \\ 3 & -3p & -3p \end{pmatrix}$$

Step 5 : $\mathbf{A}^{-1} = \dfrac{1}{det\ \mathbf{A}}\mathbf{C}^T$

$$\mathbf{A}^{-1} = \frac{1}{3p+6}\begin{pmatrix} 0 & p+2 & 0 \\ 3 & -2p+2 & 6 \\ 3 & -3p & -3p \end{pmatrix}$$

[4 marks]

Answer 4

Statement

For all positive integers n, $f(n) = 2^{6n} + 3^{2n-2}$ is divisible by 5

Proof by Induction

When $n = 1$, $f(1) = 2^{6 \times 1} + 3^{2 \times 1 - 2}$

$= 65$ which is divisible by 5

Assume that when $n = k$, $f(k) = 2^{6k} + 3^{2k-2}$ is divisible by 5
in which case...

$$
\begin{aligned}
f(k+1) &= 2^{6k+6} + 3^{2k} \\
&= 2^{6k} \times 2^{6} + 3^{2k-2+2} \\
&= 64 \times 2^{6k} + 9 \times 3^{2k-2} \\
&= 9\left(2^{6k} + 3^{2k-2}\right) + 55 \times 2^{6k} \\
&= 9f(k) + 55 \times 2^{6k} \\
f(k+1) - 9f(k) &= 55 \times 2^{6k}
\end{aligned}
$$

$\therefore f(k+1)$ is divisible by 5

Therefore, if $f(n)$ is divisible by 5 when $n = k$,

then $f(n)$ is also divisible by 5 when $n = k + 1$

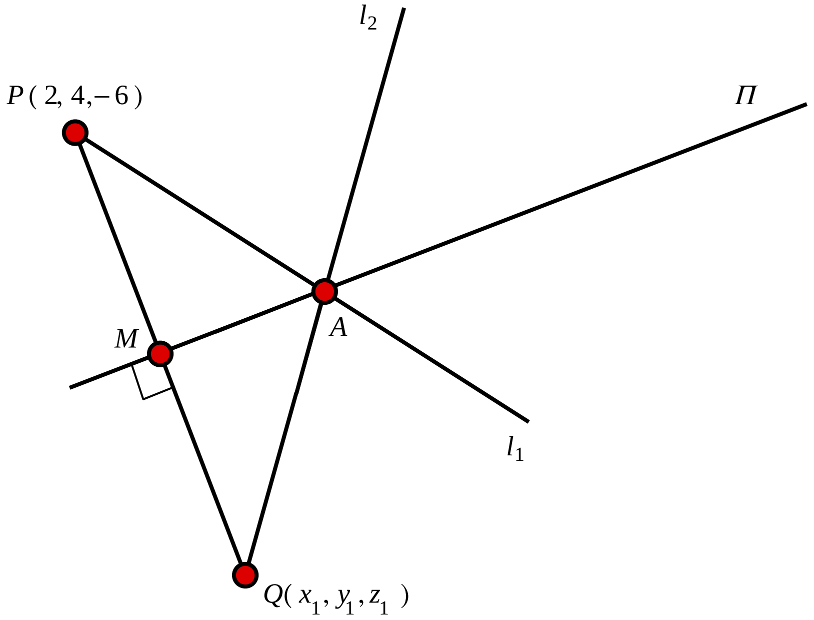
As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \in \mathbb{Z}^{+}$ by mathematical induction. □

[6 marks]

Answer 5

A vector equation of l_1 is $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ so $P(2,\ 4,\ -6)$ is a point on l_1

Let A be the point of intersection of l_1 and Π.



A lies on l_1 and on $2x - 3y + z = 8$

A has position vector $\begin{pmatrix} 2 + 2\lambda \\ 4 - 2\lambda \\ -6 + \lambda \end{pmatrix}$ and satisfies

$$2(2 + 2\lambda) - 3(4 - 2\lambda) - 6 + \lambda = 8$$

$$4 + 4\lambda - 12 + 6\lambda - 6 + \lambda = 8$$

$$11\lambda = 22$$

$$\lambda = 2$$

So A has coordinates $(6,\ 0,\ -4)$

A perpendicular direction vector to Π is $\boldsymbol{n} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

A vector equation of the line through P perpendicular to Π is,

$$r = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Let $Q(x_1,\ y_1,\ z_1)$ be the point of intersection of this line and l_2

Let M be the midpoint of PQ

M lies on line and on $2x - 3y + z = 8$ so satisfies

$$\begin{aligned} 2(2 + 2\mu) - 3(4 - 3\mu) - 6 + \mu &= 8 \\ 4 + 4\mu - 12 + 9\mu - 6 + \mu &= 8 \\ 14\mu &= 22 \\ \mu &= \frac{11}{7} \end{aligned}$$

So M has position vector $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \frac{11}{7}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

and Q has position vector $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + 2 \times \frac{11}{7}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{58}{7} \\ -\frac{38}{7} \\ -\frac{20}{7} \end{pmatrix}$

So Q has coordinates $\left(\frac{58}{7}, -\frac{38}{7}, -\frac{20}{7} \right)$

l_2 is the line through Q and A, so has direction,

$$\overrightarrow{AQ} = \begin{pmatrix} \frac{16}{7} \\ -\frac{38}{7} \\ \frac{8}{7} \end{pmatrix}$$

A vector equation of l_2 is $r = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix} + t\begin{pmatrix} 8 \\ -19 \\ 4 \end{pmatrix}$

[9 marks]