Year FM
Further Pure Mathematics Examination Revision
Health Check N° 5

Answer 1
Using the quotable result that,

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\,arctan\left(\frac{x}{a}\right) + c,\ a > 0,\ |x| < a$$

$$\int \frac{1}{4 + 3x^2}\,dx = \frac{1}{3}\int \frac{1}{\left(\frac{4}{3}\right) + x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2 + x^2}\,dx$$

$$= \frac{1}{3} \times \frac{\sqrt{3}}{2}\,arctan\left(\frac{\sqrt{3}\,x}{2}\right) + c$$

$$= \frac{\sqrt{3}}{6}\,arctan\left(\frac{\sqrt{3}\,x}{2}\right) + c$$

[4 marks]

Answer 2

$$f(x) = 4\,cosh\,x - \frac{1}{4}\,cosh\,(2x)$$

$$f'(x) = 4\,sinh\,x - \frac{1}{2}\,sinh\,(2x)$$

For stationary points, $f'(x) = 0$

$$4\,sinh\,x - \frac{1}{2}\,sinh\,(2x) = 0$$

$$4\,sinh\,x - sinh\,x\,cosh\,x = 0$$

$$sinh\,x\,(4 - cosh\,x) = 0$$

Either $sinh\,x = 0$ in which case $x = 0$

Or $cosh\,x = 4$

$$\frac{e^x + e^{-x}}{2} = 4$$

$$e^x + e^{-x} = 8$$

Multiply through by e^x

$$(e^x)^2 - 8(e^x) + 1 = 0$$

$$e^x = \frac{8 \pm \sqrt{64 - 4 \times 1 \times 1}}{2}$$

$$= 4 \pm \sqrt{15} \qquad \Rightarrow\ x = ln\left(4 \pm \sqrt{15}\right)$$

[5 marks]

Answer 3

(i) A sheaf

[1 mark]

(ii) Putting the third equation, $z = 5$, into the first, $x + z = 4$, gives the same as putting it into the second, $2x + z = 3$. That is $x = -1$.

So, as they should be for a sheaf, the equations are consistent.

Thus, two point on the intersection line are $(-1,\ 0,\ 5)$ and $(-1,\ 1,\ 5)$

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} + \lambda\left(\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ Other descriptions of this same line are possible

[3 marks]

Answer 4

(a) $(x - 10)^2 + y^2 = 3^2$

[1 mark]

(b) By considering the radius of each circle, $lsf = \dfrac{9}{3} = 3$

Let the length of OP be x so it is required to show that x is 15

$$\therefore \frac{x}{x - 10} = 3$$

$$x = 3(x - 10)$$

$$x = 3x - 30$$

$$2x = 30$$

$$x = 15 \qquad \square$$

[3 marks]

(c) $y = mx + c \Rightarrow 0 = 15m + c \Rightarrow c = -15m$

$$\therefore y = mx - 15m$$

[1 mark]

(d)

$$x^2 + y^2 = 81$$

$$x^2 + (mx - 15m)^2 = 81$$

$$x^2 + m^2x^2 - 30m^2x + 225m^2 - 81 = 0$$

$$x^2\left(1 + m^2\right) - 30m^2x + \left(225m^2 - 81\right) = 0$$

[3 marks]

(e) The “two points” where the line cuts the circle C_1 will become “one point” at A as the line is a tangent to the circle at A.
So the discriminant of the part (d) equation will be zero.

$$\left(-30m^2\right)^2 - 4 \times \left(1 + m^2\right) \times \left(225m^2 - 81\right) = 0$$

$$900m^4 - 4\left(225m^2 - 81 + 225m^4 - 81m^2\right) = 0$$

Divide through by 4

$$225m^4 - 225m^2 + 81 - 225m^4 + 81m^2 = 0$$

$$-144m^2 + 81 = 0$$

$$144m^2 = 81$$

$$m^2 = \frac{9}{16}$$

$$m = -\frac{3}{4}$$

(As the gradient between A and P is negative)

Putting this back into the part (d) equation...

$$x^2\left(1 + m^2\right) - 30m^2x + \left(225m^2 - 81\right) = 0$$

$$x^2\left(1 + \frac{9}{16}\right) - 30 \times \frac{9}{16}x + \left(225 \times \frac{9}{16} - 81\right) = 0$$

$$\frac{25}{16}x^2 - \frac{270}{16}x + \frac{729}{16} = 0$$

$$25x^2 - 270x + 729 = 0$$

$$x = \frac{270 \pm 0}{50}$$

$$= \frac{27}{5}$$

$$y = -\frac{3}{4} \times \frac{27}{5} - 15 \times \left(-\frac{3}{4}\right)$$

$$= \frac{36}{5}$$

$\therefore$ The coordinates of A are $\left(\frac{27}{5}, \frac{36}{5}\right)$

[5 marks]

Answer 5

As a matrix equation,

$$\begin{pmatrix} 1 & -3 & -4 \\ 6 & 5 & -7 \\ 1 & 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 30 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{111}\begin{pmatrix} 58 & 2 & 41 \\ -43 & 10 & -17 \\ 19 & -7 & 23 \end{pmatrix}\begin{pmatrix} 3 \\ 30 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$\therefore$ Point of intersection is $(1,\ 2,\ -2)$

[4 marks]