Year FM

Further Pure Mathematics Examination Revision

Health Check N° 6

Answer 1

(i)

$$f(x) = \ln \cos x$$

$$f'(x) = \frac{1}{\cos x} \times (-\sin x)$$

$$= -\tan x$$

[1 mark]

(ii) $f'(x) = -\tan x \Rightarrow f'(0) = 0$

$$f''(x) = -\sec^2 x \Rightarrow f''(0) = -\frac{1}{\cos^2(0)} = -1$$

$$f'''(x) = -2\sec^2 x \tan x \Rightarrow f'''(0) = 0$$

$$f'''(x) = -2\left(1 + \tan^2 x\right)\tan x$$

$$f''''(x) = -2\left(\tan x + \tan^3 x\right)$$

$$= -2\left(\sec^2 x + 3\tan^2 x \sec^2 x\right) \Rightarrow f''''(0) = -2\left(1 + 0\right)$$

$$= -2$$

[4 marks]

(iii) This is a Maclaurin series question without the words “Maclaurin series”

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \ldots$$

$$\therefore\ f(x) = 0 + 0\,x - \frac{1}{2!}x^2 + 0\,x^3 - \frac{2}{4!}x^4 + \ldots$$

$$= -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots$$

[2 marks]

Answer 2

(a) Writing the three planes as a matrix,

$$\begin{pmatrix} 1 & a & 2 \\ 1 & -1 & -1 \\ 1 & 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix}$$

Working out the determinant by expanding along the top row,

$$1\begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix} - a\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= 1(-4 + 4) - a(4 + 1) + 2(4 + 1)$$

$$= 0 - 5a + 10$$

For the planes to not meet at a single point the determinant must be zero.

$$\therefore a = 2$$

[4 marks]

(b) With $a = 2$, the equations of the planes are,

$$A : x + 2y + 2z = 2$$

$$B : x - y - z = 2$$

$$C : x + 4y + 4z = 0$$

$$A - B \text{ gives } 3y + 3z = 0 \Rightarrow y = -z$$

Replacing y with $(-z)$ in all three equations gives,

$$A : x - 2z + 2z = 2 \Rightarrow x = 2$$

$$B : x + z - z = 2 \Rightarrow x = 2$$

$$C : x - 4z + 4z = 0 \Rightarrow x = 0$$

So the equations are not forming a consistent system: The planes form a prism.

[4 marks]

Answer 3

(a)

$$f(n) = 2^n + 6^n$$

$$\begin{aligned} \text{LHS} &= f(k+1) \\ &= 2^{k+1} + 6^{k+1} \\ &= 2 \times 2^k + 6 \times 6^k \\ &= (6 - 4) \times 2^k + 6 \times 6^k \\ &= 6 \times 2^k + 6 \times 6^k - 4 \times 2^k \\ &= 6(2^k + 6^k) - 4(2^k) \\ &= 6f(k) - 4(2^k) \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(b)
Proof by Induction

When $n = 1$, $f(1) = 2^1 + 6^1$

$= 8$ which is divisible by 8

Assume that when $n = k$, $f(k) = 2^k + 6^k$ is divisible by 8
in which case...

$$f(k+1) = 2^{k+1} + 6^{k+1}$$

$$= 6f(k) - 4\left(2^k\right) \qquad \text{using part (a)}$$

$$f(k+1) - 6f(k) = 4\left(2^k\right)$$

$$= 8\left(2^{k-1}\right) \qquad \text{Valid for } k \geqslant 1,$$

$$\therefore f(k+1) \text{ is divisible by } 8$$

Therefore, if $f(n)$ is divisible by 8 when $n = k$,

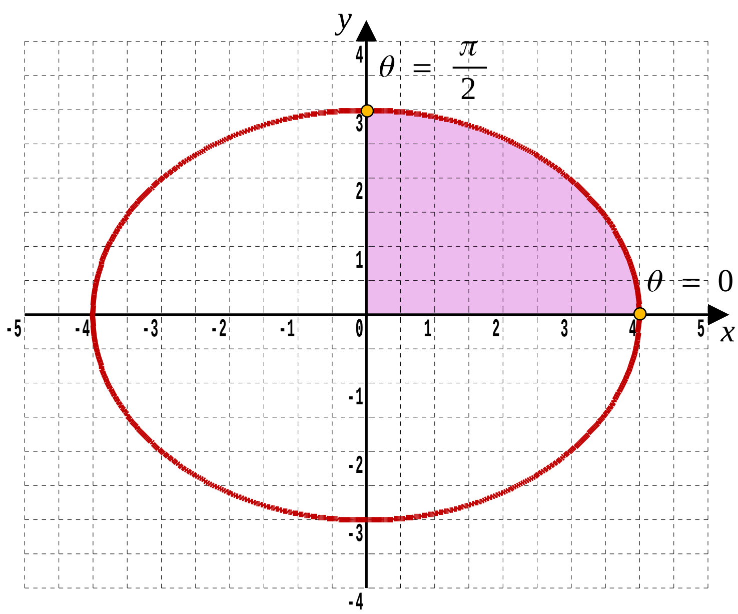
then $f(n)$ is also divisible by 8 when $n = k + 1$

As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \in \mathbb{Z}^+$ by mathematical induction. $\square$

[4 marks]

Answer 4

$$x = 4\cos\theta, \qquad y = 3\sin\theta, \qquad 0 \leqslant \theta \leqslant 2\pi$$



(i) Of the various possible strategies here is one of the easiest which focuses on finding the area shaded which will be one quarter of the total area.

$$\begin{aligned}
Area &= 4\int_{\theta=\frac{\pi}{2}}^{\theta=0} y\ \frac{dx}{d\theta}\ d\theta \\
&= 4\int_{\theta=\frac{\pi}{2}}^{\theta=0} 3\sin\theta\ (-4\sin\theta)\ d\theta \\
&= -48\int_{\theta=\frac{\pi}{2}}^{\theta=0} \sin^2\theta\ d\theta \\
&= 48\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta\ d\theta \\
&= 24\int_{\theta=0}^{\theta=\frac{\pi}{2}} 1 - \cos(2\theta)\ d\theta \\
&= 24\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}} \\
&= 24\left[\frac{\pi}{2} - \frac{\sin(\pi)}{2} - 0 - \frac{\sin(0)}{2}\right] \\
&= 12\pi
\end{aligned}$$

[5 marks]

(ii) The strategy used here is to rotate the shaded area about the x-axis and double that result to get the total volume.

$$\begin{aligned}
V &= 2\pi\int_{\theta=\frac{\pi}{2}}^{\theta=0} y^2\ \frac{dx}{d\theta}\ d\theta \\
&= 2\pi\int_{\theta=\frac{\pi}{2}}^{\theta=0} 9\sin^2\theta\ (-4\sin\theta)\ d\theta \\
&= -72\pi\int_{\theta=\frac{\pi}{2}}^{\theta=0} \sin^2\theta\ \sin\theta\ d\theta \\
&= 72\pi\int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(1 - \cos^2\theta\right)\sin\theta\ d\theta \\
&= 72\pi\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta\ d\theta - 72\pi\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta\cos^2\theta\ d\theta \\
&= 72\pi\left[-\cos\theta\right]_0^{\frac{\pi}{2}} - 72\pi\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{2}} \\
&= 72\pi\left[-0 + \cos(0)\right] - 72\pi\left[0 - \frac{1}{3}\right] \\
&= 72\pi - 24\pi \\
&= 48\pi
\end{aligned}$$

[5 marks]

Answer 5

$$|w - 1 - i| = 3 \qquad arg(w - 2) = \frac{\pi}{4}$$

$$|x + yi - 1 - i| = 3 \qquad arg(x + yi - 2) = \frac{\pi}{4}$$

$$|(x-1) + (y-1)i| = 3 \qquad arg((x-2) + yi) = \frac{\pi}{4}$$

$$(x-1)^2 + (y-1)^2 = 9 \qquad \frac{y}{x-2} = tan\left(\frac{\pi}{4}\right)$$

$$y = x - 2$$

Substituting the straight line into the circle...

$$(x-1)^2 + (x-2-1)^2 = 9$$

$$x^2 - 2x + 1 + x^2 - 6x + 9 = 9$$

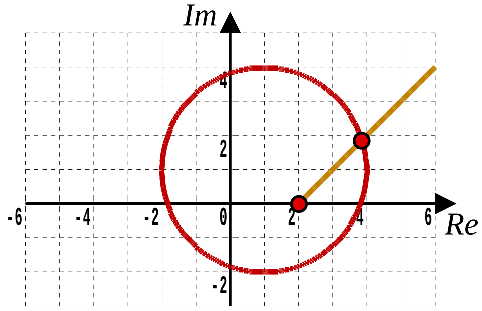
$$2x^2 - 8x + 1 = 0$$

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 2 \times 1}}{2 \times 2}$$

$$= \frac{8 \pm \sqrt{56}}{4}$$

$$= 2 \pm \frac{1}{2}\sqrt{14}$$

Only want positive value of x so the intersection is $\left(2 + \frac{1}{2}\sqrt{14}, \ \frac{1}{2}\sqrt{14}\right)$



$$|w|^2 = \left(\frac{4 + \sqrt{14}}{2}\right)^2 + \left(\frac{\sqrt{14}}{2}\right)^2$$

$$= \frac{16 + 8\sqrt{14} + 14 + 14}{4}$$

$$= 11 + 2\sqrt{14}$$

[8 marks]