Year FM

Further Pure Mathematics Examination Revision

Health Check N° 7

Answer 1

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ a & 1 & 1 \end{pmatrix}\begin{pmatrix} -6 & 3 & -4 \\ -1 & 6 & -4 \\ 8 & -8 & -1 \end{pmatrix} = \begin{pmatrix} -12 & -1 & 5 \\ -1 & -20 & 3 \\ -6a+7 & 3a-2 & -4a-5 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} -6 & 3 & -4 \\ -1 & 6 & -4 \\ 8 & -8 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ a & 1 & 1 \end{pmatrix} = \begin{pmatrix} -4a & -1 & 5 \\ 11-4a & -20 & 3 \\ -8-a & 7 & -17 \end{pmatrix}$$

$$a = 3 \text{ gives } \mathbf{AB} = \mathbf{BA} = \begin{pmatrix} -12 & -1 & 5 \\ -1 & -20 & 3 \\ -3 & 7 & -17 \end{pmatrix}$$

[3 marks]

Answer 2

Let the sought after cubic be in terms of the variable w in which case,

$$w = \frac{1}{2}z - 1$$

$$w + 1 = \frac{1}{2}z$$

$$z = 2w + 2$$

$$z^3 + 2z^2 - 5z - 3 = 0$$

$$\Downarrow$$

$$(2w+2)^3 + 2(2w+2)^2 - 5(2w+2) - 3 = 0$$

$$\begin{aligned} 8w^3 + 24w^2 + 24w &+ 8 \\ + 8w^2 + 16w &+ 8 \\ - 10w &- 10 \\ &- 3 = 0 \end{aligned}$$

$$8w^3 + 32w^2 + 30w + 3 = 0$$

[3 marks]

Answer 3

(a) $y = ln\left(\frac{1}{1 - 2x}\right), \qquad |x| < \frac{1}{2}$

$$= ln(1) - ln(1 - 2x)$$

$$\frac{dy}{dx} = 0 - \frac{1}{1 - 2x} \times (-2)$$

$$= 2(1 - 2x)^{-1}$$

$$\frac{d^2y}{dx^2} = 2(-1)(1 - 2x)^{-2}(-2)$$

$$= 4(1 - 2x)^{-2}$$

$$\frac{d^3y}{dx^3} = 4(-2)(1 - 2x)^{-3}(-2)$$

$$= 16(1 - 2x)^{-3}$$

[4 marks]

(b) $f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \ldots$

$$\therefore f(x) = 0 + 2x + \frac{4}{2!}x^2 + \frac{16}{3!}x^3 + \ldots$$

$$= 2x + 2x^2 + \frac{8}{3}x^3 + \ldots$$

[3 marks]

(c) We require,

$$\frac{3}{2} = \frac{1}{1 - 2x}$$

$$\frac{2}{3} = 1 - 2x$$

$$2x = 1 - \frac{2}{3}$$

$$x = \frac{1}{6}$$

$$ln\left(\frac{3}{2}\right) \approx 2 \times \frac{1}{6} + 2 \times \left(\frac{1}{6}\right)^2 + \frac{8}{3} \times \left(\frac{1}{6}\right)^3 + \ldots$$

$$\approx \frac{1}{3} + \frac{1}{18} + \frac{1}{81} + \ldots$$

$$\approx \frac{65}{162} \qquad \text{(About 0.401)}$$

[3 marks]

Answer 4

(a) $f(z) = 4z^4 - 12z^3 + 41z^2 - 128z + 185$

As the coefficients of the polynomial are all real, another root will be $2 - \mathrm{i}$

$$\begin{aligned}(z - 2 - \mathrm{i})(z - 2 + \mathrm{i}) &= ((z - 2) - \mathrm{i})((z - 2) + \mathrm{i}) \\ &= (z - 2)^2 - \mathrm{i}^2 \\ &= z^2 - 4z + 5\end{aligned}$$

$$\begin{array}{r|rrrrr}
 & & & 4z^2 & +\ 4z & +\ 37 \\ \hline
z^2 - 4z + 5 & 4z^4 & -\ 12z^3 & +\ 41z^2 & -\ 128z & +\ 185 \\
 & 4z^4 & -\ 16z^3 & +\ 20z^2 & & \\ \cline{2-4}
 & & 4z^3 & +\ 21z^2 & -\ 128z & \\
 & & 4z^3 & -\ 16z^2 & +\ 20z & \\ \cline{3-5}
 & & & 37z^2 & -\ 148z & +\ 185 \\
 & & & 37z^2 & -\ 148z & +\ 185 \\ \cline{4-6}
 & & & & & 0
\end{array}$$

[5 marks]

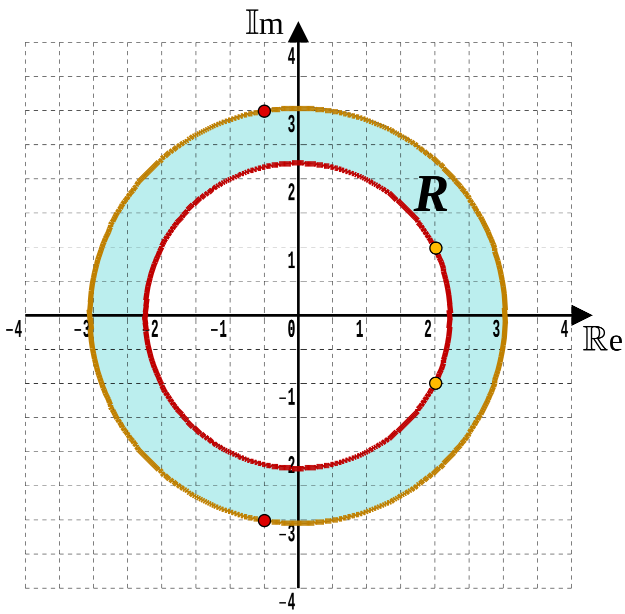
(b) Working out the roots of $4z^2 + 4z + 37 = 0$

$$\begin{aligned}z &= \frac{-4 \pm \sqrt{4^2 - 4 \times 4 \times 37}}{2 \times 4} \\ &= \frac{-4 \pm \sqrt{-576}}{8} \\ &= \frac{-4 \pm 24\mathrm{i}}{8} \\ &= -\frac{1}{2} \pm 3\mathrm{i}\end{aligned}$$

The roots of $f(x)$ are, $z \in \left\{ 2 \pm \mathrm{i},\ -\frac{1}{2} \pm 3\mathrm{i} \right\}$

[3 marks]

(c) The following Argand Diagram will help with understanding the situation describes which is a large part of what this question is about,



The area of the region R is given by,

$$\frac{37\pi}{4} - 5\pi = \frac{17\pi}{4}$$

[4 marks]

(d) $\omega = 2 + i + 2 - i - \dfrac{1}{2} + 3i - \dfrac{1}{2} - 3i = 3 + 0i$

which is just inside the region R

[2 marks]

Answer 5

(i)

$$arsinh\, x - arcosh\, x = ln\, 2$$

$$ln\left(x + \sqrt{x^2 + 1}\right) - ln\left(x + \sqrt{x^2 - 1}\right) = ln\, 2$$

$$ln\left(\frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 - 1}}\right) = ln\, 2$$

$$\frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 - 1}} = 2$$

$$x + \sqrt{x^2 + 1} = 2x + 2\sqrt{x^2 - 1}$$

$$\sqrt{x^2 + 1} - 2\sqrt{x^2 - 1} = x \qquad \square$$

[2 marks]

(ii) Squaring the equation, as instructed,

$$x^2 + 1 - 4\sqrt{(x^2+1)(x^2-1)} + 4(x^2 - 1) = x^2$$

$$4x^2 - 4\sqrt{x^4 - 1} - 3 = 0$$

$$4\sqrt{x^4-1} = 4x^2 - 3$$

$$16(x^4 - 1) = 16x^4 - 24x^2 + 9$$

$$24x^2 - 25 = 0$$

$$x^2 = \frac{25}{24}$$

$$x = \pm\frac{5}{\sqrt{24}} \times \frac{\sqrt{24}}{\sqrt{24}}$$

$$= \pm\frac{10\sqrt{6}}{24}$$

$$= \pm\frac{5\sqrt{6}}{12}$$

The only solution that makes the original equation true is $x = \dfrac{5\sqrt{6}}{12}$

[3 marks]

Answer 6

$$|\mathbf{T}| = \begin{vmatrix} x^2+1 & -4 \\ 3-2x^2 & x^2+5 \end{vmatrix}$$

$$= (x^2+1)(x^2+5) - (3-2x^2)(-4)$$

$$= x^4 + 6x^2 + 5 + 12 - 8x^2$$

$$= x^4 - 2x^2 + 17$$

$$= (x^2-1)^2 + 16$$

$$\therefore |\mathbf{T}|_{MIN} = 16$$

Smallest possible area is $12 \times 16 = 192$ units

[5 marks]