Year FM

**Further Pure Mathematics Examination Revision**

**Health Check N° 8**

**Answer 1**

$$\begin{aligned}
\text{LHS} &= \frac{z}{z^*} \\
&= \frac{a + b\mathrm{i}}{a - b\mathrm{i}} \times \frac{a + b\mathrm{i}}{a + b\mathrm{i}} \quad \Leftarrow \text{Key step must be seen} \\
&= \frac{a^2 + ab\mathrm{i} + ab\mathrm{i} + (b\mathrm{i})^2}{a^2 + ab\mathrm{i} - ab\mathrm{i} - (b\mathrm{i})^2} \\
&= \frac{a^2 + b^2(-1) + 2ab\mathrm{i}}{a^2 - b^2(-1)} \qquad \text{as } \mathrm{i}^2 = -1 \\
&= \frac{a^2 - b^2 + 2ab\mathrm{i}}{a^2 + b^2} \\
&= \left(\frac{a^2 - b^2}{a^2 + b^2}\right) + \left(\frac{2ab}{a^2 + b^2}\right)\mathrm{i} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

**[ 3 marks ]**

**Answer 2**

**( a )**

$$\begin{aligned}
(2z - z^*)^* &= z^2 \\
(2(a + b\mathrm{i}) - (a + b\mathrm{i})^*)^* &= (a + b\mathrm{i})^2 \\
(2a + 2b\mathrm{i} - (a - b\mathrm{i}))^* &= a^2 + 2ab\mathrm{i} + (b\mathrm{i})^2 \\
(2a + 2b\mathrm{i} - a + b\mathrm{i})^* &= a^2 + b^2(-1) + 2ab\mathrm{i} \\
(a + 3b\mathrm{i})^* &= a^2 - b^2 + 2ab\mathrm{i} \\
a - 3b\mathrm{i} &= a^2 - b^2 + 2ab\mathrm{i}
\end{aligned}$$

Thus. $a = a^2 - b^2$ and $-3b = 2ab$

$a^2 - a = b^2$ and $2ab + 3b = 0$

$a(a - 1) = b^2$ and $b(2a + 3) = 0$

Either $b = 0$ or $a = -\dfrac{3}{2}$

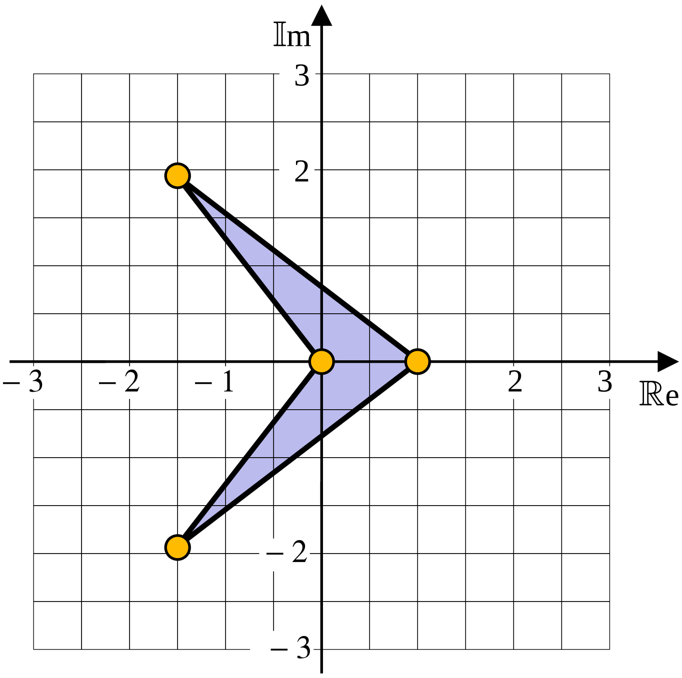
If $b = 0$ then $a(a - 1) = 0$ giving $a = 0$ or $a = 1$

If $a = -\dfrac{3}{2}$ then $b^2 = \left(-\dfrac{3}{2}\right)\left(-\dfrac{5}{2}\right)$

$= \dfrac{15}{4}$ and so $b = \pm\dfrac{\sqrt{15}}{2}$

The solution set is $z \in \left\{ 0 + 0\mathrm{i},\ 1 + 0\mathrm{i},\ -\dfrac{3}{2} \pm \dfrac{\sqrt{15}}{2} \right\}$

**( b )** **( i )** Plotting the solution set on the Argand diagram gives;




**[ 2 marks ]**

**( ii )** The area of the quadrilateral shaded is,

$$\begin{aligned} A &= 2 \times \ Area\ \triangle \\ &= 2 \times \frac{1}{2} \times 1 \times \frac{\sqrt{15}}{2} \\ &= \frac{\sqrt{15}}{2} \end{aligned}$$

**[ 1 mark ]**

**Answer 3**

**( a )** By definition,

$$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

$$= \frac{1}{\frac{\pi}{2} - 0}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos 2x + 2}\, dx$$

$$= \frac{2}{\pi} \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2\sin x \cos x}{\cos 2x + 2}\, dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin 2x\,(\cos 2x + 2)^{-1}\, dx$$

$$= -\frac{1}{2\pi}\int_0^{\frac{\pi}{2}} (-2\sin 2x)(\cos 2x + 2)^{-1}\, dx$$

$$= -\frac{1}{2\pi}\Big[\ln|\cos 2x + 2|\Big]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2\pi}\Big[\ln|\cos \pi + 2| - \ln|\cos 0 + 2|\Big]$$

$$= -\frac{1}{2\pi}\Big[\ln(-1+2) - \ln(1+2)\Big]$$

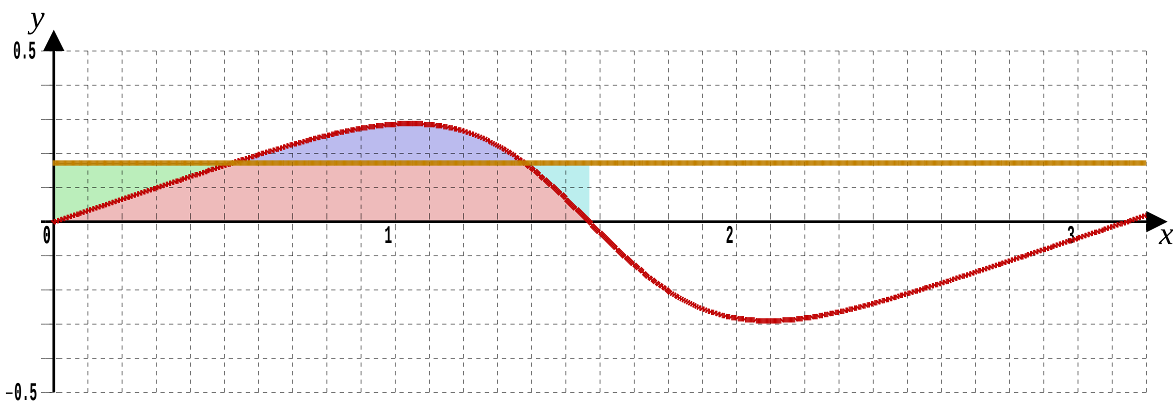
$$= -\frac{1}{2\pi}\Big[\ln 1 - \ln 3\Big]$$

$$= \frac{\ln 3}{2\pi}$$

(About 0.175 which is the gold line on the graph)

**[ 4 marks ]**

**( b )**




The areas shaded green and blue below the horizontal line must be equal to the area shaded purple above the horizontal line.
The rectangle shaded green, pink and blue has the same area as the area between the curve and the $x$-axis over the given interval.

**[ 2 marks ]**

**Answer 4**

**( a )** **( i )**

$$|z - 1 + 2\mathrm{i}| = 3$$
$$|x + y\mathrm{i} - 1 + 2\mathrm{i}| = 3$$
$$|(x - 1) + (y + 2)\mathrm{i}| = 3$$
$$(x - 1)^2 + (y + 2)^2 = 9$$

which is a circle centre $(1, -2)$ and radius 3

**[ 2 marks ]**

**( ii )**

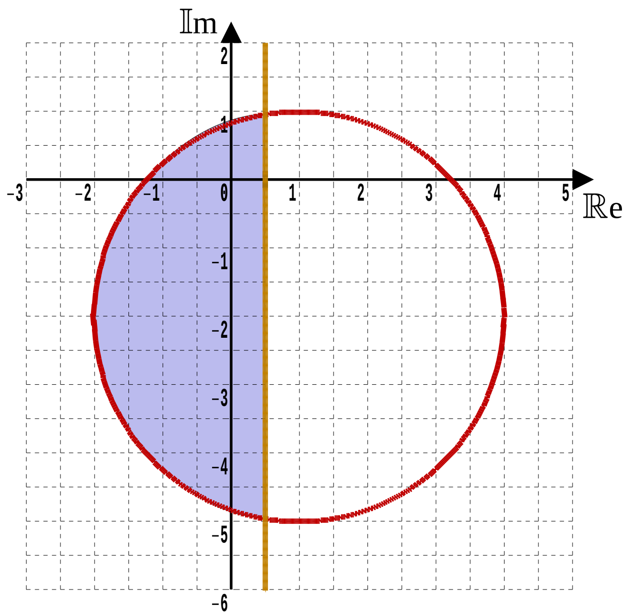
$$|z + 1| = |z - 2|$$
$$|x + y\mathrm{i} + 1| = |x + y\mathrm{i} - 2|$$
$$|x + 1 + y\mathrm{i}| = |x - 2 + y\mathrm{i}|$$
$$(x + 1)^2 + y^2 = (x - 2)^2 + y^2$$

$x + 1 = x - 2$ or $x + 1 = -x + 2$

(No solutions) $\quad 2x = 1$

$$x = \frac{1}{2}$$

**[ 2 marks ]**




**( b )** Shown shaded is the region of the Argand diagram for which

$$|z - 1 + 2\mathrm{i}| \leqslant 3 \text{ and } |z + 1| \leqslant |z - 2|$$

**[ 2 marks ]**

**Answer 5**

**( i )** $x = \dfrac{a}{\sinh\theta} = a\,(\sinh\theta)^{-1} \Rightarrow \dfrac{dx}{d\theta} = -a\,(\sinh x)^{-2}\cosh x$

$$= -\frac{a\cosh\theta}{\sinh^2\theta}$$

$$\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = \int \frac{\sinh\theta}{a\sqrt{\left(\frac{a}{\sinh x}\right)^2 + a^2}} \times (-1)\frac{a\cosh\theta}{\sinh^2\theta}\,d\theta$$

$$= (-1)\int \frac{a\sinh\theta\cosh\theta}{a^2\sqrt{\frac{1+\sinh^2\theta}{\sinh^2\theta}}} \times \frac{1}{\sinh^2\theta}\,d\theta$$

$$= (-1)\int \frac{\cosh\theta}{a\sqrt{\frac{\cosh^2\theta}{\sinh^2\theta}}} \times \frac{1}{\sinh\theta}\,d\theta$$

$$= -\frac{1}{a}\int 1\,d\theta$$

$$= -\frac{1}{a}\theta + constant$$

$$= -\frac{1}{a}\,arsinh\left(\frac{a}{x}\right) + constant$$

**[ 6 marks ]**

**( ii )** $\displaystyle\int_1^2 \frac{1}{x\sqrt{x^2+4}}\,dx = \int_1^2 \frac{1}{x\sqrt{x^2+2^2}}\,dx$

$$= \left[-\frac{1}{2}\,arsinh\left(\frac{2}{x}\right)\right]_1^2$$

$$= -\frac{1}{2}\left[\ln\left\{\frac{2}{x} + \sqrt{\left(\frac{2}{x}\right)^2 + 1}\right\}\right]_1^2$$

$$= -\frac{1}{2}\left[\ln\{1+\sqrt{2}\} - \ln\{2+\sqrt{5}\}\right]$$

$$= -\frac{1}{2}\ln\left(\frac{1+\sqrt{2}}{2+\sqrt{5}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right) \text{ or equivalent}$$

**[ 4 marks ]**

**Answers 6**

If $x = r\cos\theta$ and $y = r\sin\theta$ then $r^2 = x^2 + y^2$

$$\left(x^2 + y^2 - 2x\right)^2 = 4\left(x^2 + y^2\right)$$

$$\left(r^2 - 2r\cos\theta\right)^2 = 4r^2$$

$$r^4 - 4r^3\cos\theta + 4r^2\cos^2\theta = 4r^2$$

As $r \neq 0$, divide through by $r^2$

$$r^2 - 4r\cos\theta + 4\cos^2\theta - 4 = 0$$

Viewing this as a quadratic in $r$...

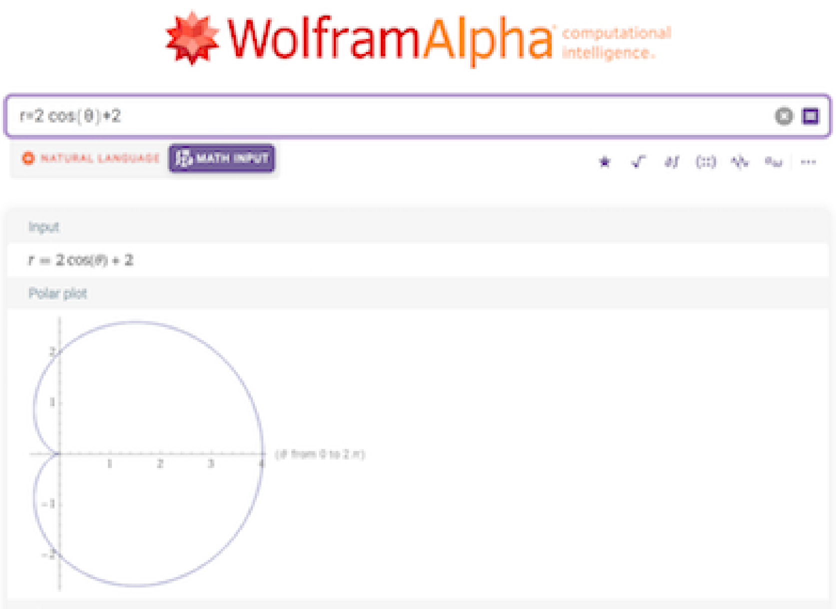
$$r = \frac{4\cos\theta \pm \sqrt{16\cos^2\theta - 4 \times 1 \times \left(4\cos^2\theta - 4\right)}}{2 \times 1}$$

$$= \frac{4\cos\theta \pm \sqrt{16\cos^2\theta - 16\cos^2\theta + 16}}{2}$$

$$= \frac{4\cos\theta \pm \sqrt{16}}{2}$$

$$= 2\cos\theta \pm 2$$

Both $r = 2\cos\theta + 2$ and $r = 2\cos\theta - 2$ give the same curve.




**[ 5 marks ]**