Year FM

Further Pure Mathematics Examination Revision

Health Check N° 9

Answer 1

$$y = 6\,arcsin\,(2x)$$

$$\frac{dy}{dx} = \frac{12}{\sqrt{1 - (2x)^2}}$$

$$\text{When } x = \frac{1}{4},\ \frac{dy}{dx} = \frac{12}{\sqrt{1 - \left(\frac{1}{2}\right)^2}}$$

$$= \frac{12 \times 2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= 8\sqrt{3}$$

That is, $m = 8$ and $n = 3$

[4 marks]

Answer 2

METHOD 1

Use the fact that $R\,sin\,(\theta + \alpha) = R\,sin\,\theta\,cos\,\alpha + R\,cos\,\theta\,sin\,\alpha$

$$sin\,(\theta + \alpha) = sin\,\theta \times 2 + cos\,\theta \times 1$$

$$tan\,\alpha = \frac{R\,sin\,\alpha}{R\,cos\,\alpha}$$

$$= \frac{1}{2} \qquad R = \sqrt{2^2 + 1^2}$$

$$= \sqrt{5}$$

The maximum value of r is thus $\sqrt{5}\,a$

which occurs when $\theta + arctan\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$$\therefore \theta = \frac{\pi}{2} - arctan\left(\frac{1}{2}\right)$$

Requested polar coordinates are $(\,2.236a,\ 1.107^c\,)$

[7 marks]

METHOD 2

At a place on the curve where the distance from the origin stops increasing ahead of starting to decrease $\frac{dr}{d\theta} = 0$

$$r = a(\cos\theta + 2\sin\theta)$$

$$\frac{dr}{d\theta} = 0$$

$$a(-\sin\theta + 2\cos\theta) = 0$$

$$-\sin\theta + 2\cos\theta = 0 \quad \text{(As told } a \text{ is a positive constant)}$$

$$\sin\theta = 2\cos\theta \quad *$$

$$\frac{\sin\theta}{\cos\theta} = 2$$

$$\tan\theta = 2 \Rightarrow \theta = 1.107^c$$

Insert * into $\cos^2\theta + \sin^2\theta = 1$ to get...

$$\cos^2\theta + (2\cos\theta)^2 = 1$$

$$\cos^2\theta + 4\cos^2\theta = 1$$

$$5\cos^2\theta = 1$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{5} \quad \text{and so, from *,} \quad \sin\theta = \frac{2\sqrt{5}}{5}$$

$$r = a(\cos\theta + 2\sin\theta)$$

$$= a\left(\frac{\sqrt{5}}{5} + 2 \times \frac{2\sqrt{5}}{5}\right)$$

$$= \sqrt{5}\,a$$

Requested polar coordinates are $(2.236a, \ 1.107^c)$

[7 marks]

(b)

$$r = a\left(\cos\theta + 2\sin\theta \right)$$

multiply through by r

$$r^2 = a\,(r\cos\theta) + 2a\,(r\sin\theta)$$

$$x^2 + y^2 = ax + 2ay$$

$$x^2 - ax + y^2 - 2ay = 0$$

$$\left(x - \frac{a}{2} \right)^2 - \frac{a^2}{4} + (y - a)^2 - a^2 = 0$$

$$\left(x - \frac{a}{2} \right)^2 + (y - a)^2 = \frac{5a^2}{4}$$

which is recognizable as a circle with radius $\dfrac{\sqrt{5}\,a}{2}$

[6 marks]

(c) The question is set up so that you realise the part (b) radius is half of the part (a) answer, which must have been the diameter of the circle. So the direction to the circle's centre is the same as that found in (a).

Thus the polar coordinates of the circle's centre are,

$$\left(\frac{\sqrt{5}\,a}{2},\ 1.107^c \right)$$

[1 mark]

Answer 3

(i) An integral $\int_a^b f(x)\ dx$ is described as being improper if either one or both of the limits are infinite or if $f(x)$ is undefined at some point in the interval $a \leqslant x \leqslant b$

[1 mark]

(ii) Due to a division by zero, $f(x)$ is not defined when $x = 0$

The integral has an upper limit of ∞

[1 mark]

(iii) Using the hint given...

$$y = arctan\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{1 + (\sqrt{x})^2} \times \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}\ (1 + x)}$$

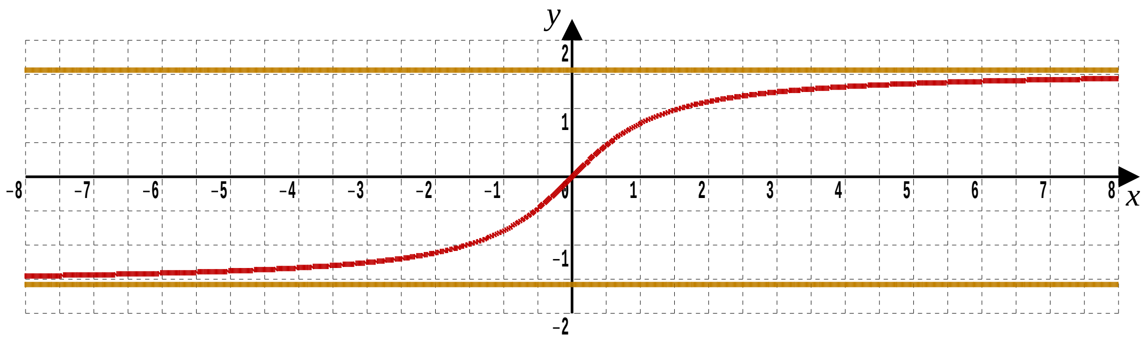
$$\int_0^\infty \frac{1}{(x + 1)\sqrt{x}}\ dx = 2 \underset{\substack{v \to \infty \\ u \to 0^+}}{limit} \int_u^v \frac{1}{2(x + 1)\sqrt{x}}\ dx$$

$$= 2 \underset{\substack{v \to \infty \\ u \to 0^+}}{limit} \left[arctan\sqrt{x} \right]_u^v$$

$$= 2 \underset{v \to \infty}{limit} \left[arctan\sqrt{v} \right] - 2 \underset{u \to 0^+}{limit} \left[arctan\sqrt{u} \right]$$

$$= 2 \times \frac{\pi}{2} - 2 \times 0$$

$$= \pi \qquad \text{(Convergent)}$$



Red: $y = arctan(x)$ Gold: $y = \pm\frac{\pi}{2}$

[5 marks]

Answer 4

$$2x^3 + px^2 + qx - 3 = 0$$

$$x^3 + \frac{p}{2}x^2 + \frac{q}{2}x - \frac{3}{2} = 0$$

(i) $\beta = 1 - \mathrm{i}\sqrt{2}$

[1 mark]

(ii) "Product of the roots"

$$\alpha\beta\gamma = \frac{3}{2}$$

$$\left(1 + \mathrm{i}\sqrt{2}\right)\left(1 - \mathrm{i}\sqrt{2}\right)\gamma = \frac{3}{2}$$

$$\left(1 + 2\right)\gamma = \frac{3}{2}$$

$$\gamma = \frac{1}{2}$$

[2 marks]

(iii) $\left(x - 1 - \mathrm{i}\sqrt{2}\right)\left(x - 1 + \mathrm{i}\sqrt{2}\right)\left(x - \frac{1}{2}\right) = 0$

$$\left((x - 1)^2 + 2\right)\left(x - \frac{1}{2}\right) = 0$$

multiply both sides by 2

$$\left(x^2 - 2x + 3\right)\left(2x - 1\right) = 0$$

$$2x^3 - 4x^2 + 6x - x^2 + 2x - 3 = 0$$

$$2x^3 - 5x^2 + 8x - 3 = 0$$

Thus, $p = -5$ and $q = 8$

[2 marks]

(iv) $\alpha = 1 + \mathrm{i}\sqrt{2} \Rightarrow |\alpha| = \sqrt{3}$

$$\alpha = \sqrt{3}\left(\frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\,\mathrm{i}\right)$$

$$= 3^{\frac{1}{2}}(\cos\theta + \mathrm{i}\sin\theta) \quad \text{where } \cos\theta = \frac{1}{\sqrt{3}} \text{ and } \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \alpha^n = 3^{\frac{n}{2}}(\cos\theta + \mathrm{i}\sin\theta)^n \qquad \text{Note } \cos^2\theta + \sin^2\theta = 1 \text{ as it should}$$

$$= 3^{\frac{n}{2}}(\cos n\theta + \mathrm{i}\sin n\theta) \quad \text{by De Moivre's Theorem}$$

Similarly,

$\beta = 1 - \mathrm{i}\sqrt{2} \Rightarrow |\beta| = \sqrt{3}$

$$\beta = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}\,\mathrm{i}\right)$$

$$= 3^{\frac{1}{2}}(\cos\theta - \mathrm{i}\sin\theta) \text{ where } \cos\theta = \frac{1}{\sqrt{3}} \text{ and } \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \beta^n = 3^{\frac{n}{2}}(\cos\theta - \mathrm{i}\sin\theta)^n \qquad \text{Note } \cos^2\theta + \sin^2\theta = 1 \text{ as it should}$$

$$= 3^{\frac{n}{2}}(\cos n\theta - \mathrm{i}\sin n\theta) \quad \text{by De Moivre's Theorem}$$

$$\text{LHS} = \alpha^n + \beta^n + \gamma^n$$

$$= 3^{\frac{n}{2}}(\cos n\theta + \mathrm{i}\sin n\theta) + 3^{\frac{n}{2}}(\cos n\theta - \mathrm{i}\sin n\theta) + \left(\frac{1}{2}\right)^n$$

$$= 3^{\frac{n}{2}}\cos n\theta + 3^{\frac{n}{2}}\sin n\theta + 3^{\frac{n}{2}}\cos n\theta - 3^{\frac{n}{2}}\sin n\theta + \frac{1}{2^n}$$

$$= 2 \times 3^{\frac{n}{2}} \times \cos n\theta + \frac{1}{2^n}$$

$$= \text{RHS} \qquad \square$$

$$\text{Note that } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \div \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{1} \qquad \text{KOF}$$

$$= \sqrt{2} \qquad \text{as required}$$

[4 marks]

Answer 5

Let a root of $(-77 - 36\mathrm{i})$ be $a + b\mathrm{i}$ with $a, b \in \mathbb{R}$, in which case,

$$(a + b\mathrm{i})^2 = -77 - 36\mathrm{i}$$

$$a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2 = -77 - 36\mathrm{i}$$

$$\therefore a^2 - b^2 = -77 \quad \text{and} \quad 2ab = -36$$

$$b = -\frac{18}{a}$$

$$a^2 - \left(-\frac{18}{a}\right)^2 = -77$$

$$a^2 - \frac{324}{a^2} = -77$$

multiply through by a^2

$$a^4 + 77a^2 - 324 = 0$$

$$\left(a^2 - 4\right)\left(a^2 + 81\right) = 0$$

$$\text{Either } a^2 = 4 \quad \text{or} \quad a^2 = -81$$

$$\therefore \quad a = \pm 2 \quad \text{as } a \in \mathbb{R}$$

$$\therefore \quad b = \mp 9$$

$$\text{Thus } \sqrt{-77 - 36\mathrm{i}} = \pm(2 - 9\mathrm{i})$$

[6 marks]