Year FM Further Pure Mathematics Examination Revision : Health Check N° 1

Why did the mattress go to the doctor ?
It had spring fever

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 25

Question 1

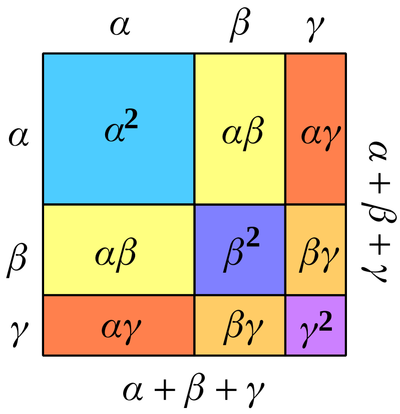
(i) A famous trigonometric identity is that, $\sin 3x = 3\sin x - 4\sin^3 x$
By use of Osborn's Rule, or otherwise, write down the similar identity for the hyperbolic function $\sinh 3x$

[1 mark]

(ii) Using the definition of $\sinh x$ in terms of exponentials prove that your part (i) identity is correct.

[3 marks]

Question 2



(i) With the aid of the diagram expand the brackets of $(\alpha + \beta + \gamma)^2$

[1 mark]

The roots of the equation $4x^3 - 12x^2 - x + 3 = 0$ are α, β and γ
Without solving the equation, write down the values of,

(ii) $\alpha + \beta + \gamma$

[1 mark]

(iii) $\alpha\beta + \beta\gamma + \gamma\alpha$

[1 mark]

(iv) $\alpha\beta\gamma$

[1 mark]

(v) $\alpha^2 + \beta^2 + \gamma^2$

[3 marks]

Question 3

Javier is revising Volumes of Revolution, and reads in his notes that;

Parametric Volumes of Revolution

The volume of revolution formed when the parametric curve with equations

$$x = f(\theta) \quad \text{and} \quad y = g(\theta)$$

is rotated through 2π radians about the x-axis between $x = a$ and $x = b$ is,

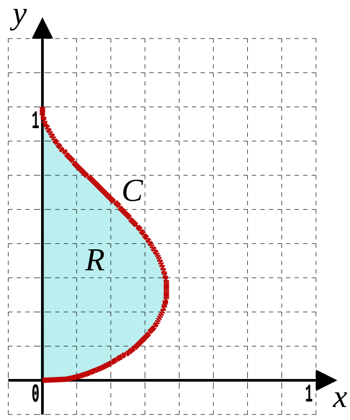
$$V = \pi \int_{x=a}^{x=b} y^2 \, dx = \pi \int_{\theta=q}^{\theta=p} y^2 \frac{dx}{d\theta} \, d\theta$$

The same curve rotated 2π radians about the y-axis between $y = a$ and $y = b$ is,

$$V = \pi \int_{y=a}^{y=b} x^2 \, dy = \pi \int_{\theta=q}^{\theta=p} x^2 \frac{dy}{d\theta} \, d\theta$$

The curve C, graphed below, has the parametric equations,

$$x = \sin^4\theta \sqrt{\cos\theta}, \qquad y = \cos\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$



The finite region R bounded by the curve and the y-axis is rotated through 2π radians about the y-axis. Find the volume of the solid of revolution formed.

[4 marks]

Question 4

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that,

$$\sum_{r=1}^{n} \left(r^2 - r - 1\right) = \frac{1}{3} n (n - 2) (n + 2)$$

for all positive integers n

[4 marks]

(b) Hence calculate $\sum_{r=10}^{40} \left(r^2 - r - 1\right)$

[3 marks]

Question 5

Further AS-Level Examination Question from June 2018, Paper 1, Q16 (AQA)

Two matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the equation $\mathbf{AB} = \mathbf{I} + 2\mathbf{A}$ where $\mathbf{I}$ is the identity matrix and $\mathbf{B} = \begin{pmatrix} 3 & -2 \\ -4 & 8 \end{pmatrix}$

Find $\mathbf{A}$

[3 marks]

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Health Check N° 1

Answer 1

(i) $sinh\, 3x \;=\; 4\, sinh^3 x \;+\; 3\, sinh\, x$

[1 mark]

(ii)
$$\begin{aligned}
\text{RHS} &= 4\, sinh^3 x + 3\, sinh\, x \\
&= 4\left(\frac{e^x - e^{-x}}{2} \right)^3 + 3\left(\frac{e^x - e^{-x}}{2} \right) \\
&= \frac{1}{2}\left(e^x - e^{-x} \right)^3 + \frac{3}{2}\left(e^x - e^{-x} \right) \\
&= \frac{1}{2}\left(e^{3x} - 3\, e^{2x} e^{-x} + 3\, e^x e^{-2x} - e^{-3x} \right) + \frac{3}{2}\left(e^x - e^{-x} \right) \\
&= \frac{1}{2}\left(e^{3x} - 3\, e^x + 3\, e^{-x} - e^{-3x} + 3\, e^x - 3\, e^{-x} \right) \\
&= \frac{e^{3x} - e^{-3x}}{2} \\
&= sinh\, 3x \\
&= \text{LHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 2

$$4\,x^3 - 12\,x^2 - x + 3 = 0$$

Divide through by 4

$$x^3 - 3\,x^2 - \frac{1}{4}x + \frac{3}{4} = 0$$

(i) $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

(ii) $\alpha + \beta + \gamma = 3$

(iii) $\alpha\beta + \beta\gamma + \gamma\alpha = -\dfrac{1}{4}$

(iv) $\alpha\beta\gamma = -\dfrac{3}{4}$

(v) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= (3)^2 - 2 \times \left(-\frac{1}{4}\right) = \frac{19}{2} \text{ or } 9.5$$

[1, 1, 1, 1, 3 marks]

Answer 3

$$x = \sin^4\theta\sqrt{\cos\theta}, \qquad y = \cos\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

$$\begin{aligned}
V &= \pi\int_{\theta=0}^{\theta=\frac{\pi}{2}} x^2\,\frac{dy}{d\theta}\,d\theta \\
&= \pi\int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^8\theta\cos\theta\,(-\sin\theta)\,d\theta \\
&= (-1)\pi\int_{\theta=0}^{\theta=\frac{\pi}{2}} \cos\theta\,(\sin\theta)^9\,d\theta
\end{aligned}$$

$f'(x)\ [f(x)]^n$ "Chain Rule Backwards"

$$\begin{aligned}
&= (-1)\pi\left[\frac{(\sin\theta)^{10}}{10}\right]_0^{\frac{\pi}{2}} \\
&= (-1)\pi\left[\frac{1}{10} - 0\right] \\
&= \frac{\pi}{10}
\end{aligned}$$

as volume cannot be negative

[4 marks]

Answer 4

(a)

$$\begin{aligned}
\sum_{r=1}^{n}\left(r^2 - r - 1\right) &= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\
&= \frac{1}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1) - n \\
&= \frac{1}{6}n\left\{(n+1)(2n+1) - 3(n+1) - 6\right\} \\
&= \frac{1}{6}n\left\{2n^2 + 3n + 1 - 3n - 3 - 6\right\} \\
&= \frac{1}{6}n\left\{2n^2 - 8\right\} \\
&= \frac{1}{3}n\left\{n^2 - 4\right\}
\end{aligned}$$

"Difference of two squares"

$$= \frac{1}{3}n(n-2)(n+2) \qquad \square$$

[4 marks]

$$\begin{aligned}
\sum_{r=10}^{40}\left(r^2 - r - 1\right) &= \sum_{r=1}^{40}\left(r^2 - r - 1\right) - \sum_{r=1}^{9}\left(r^2 - r - 1\right) \\
&= \frac{1}{3}\times 40\times 38\times 42 - \frac{1}{3}\times 9\times 7\times 11 \\
&= 21049
\end{aligned}$$

[3 marks]

Answer 5

$$\begin{aligned}
\mathbf{AB} &= \mathbf{I} + 2\mathbf{A} \\
\mathbf{AB} - 2\mathbf{A} &= \mathbf{I} \\
\mathbf{A}(\mathbf{B} - 2\mathbf{I}) &= \mathbf{I} \\
\mathbf{A} &= (\mathbf{B} - 2\mathbf{I})^{-1} \\
&= \left(\begin{pmatrix} 3 & -2 \\ -4 & 8 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right)^{-1} \\
&= \begin{pmatrix} 1 & -2 \\ -4 & 6 \end{pmatrix}^{-1} \\
&= \frac{1}{1 \times 6 - (-4)(-2)} \begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix} \\
&= \frac{1}{(-2)} \begin{pmatrix} 6 & 2 \\ 4 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -3 & -1 \\ -2 & -0.5 \end{pmatrix}
\end{aligned}$$

[3 marks]

Fortify Your Maths

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

The matrix $\mathbf{A} = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$, where k is a constant

(i) Find *det* **A** in terms of k

[2 marks]

Given that **A** is singular,

(ii) find the possible values of k

[2 marks]

Question 2

$$f\,(z) \;=\; z^3 \;+\; 3\,z^2 \;+\; kz \;+\; 48, \;\; k \;\in\; \mathbb{R}$$

Given that $f\,(4\mathrm{i}) \;=\; 0$

(a) find the value of k

[2 marks]

(b) find and list all the roots of the equation

[3 marks]

Question 3

Show that $\dfrac{\cos 2x \;+\; \mathrm{i}\,\sin 2x}{\cos 9x \;-\; \mathrm{i}\,\sin 9x}$ can be expressed in the form $\cos nx \;+\; \mathrm{i}\,\sin nx$, where n is an integer to be found

[4 marks]

Question 4

Given that $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n+1)(pn^2 + qn + r)$

(a) find the values of p, q and r

[4 marks]

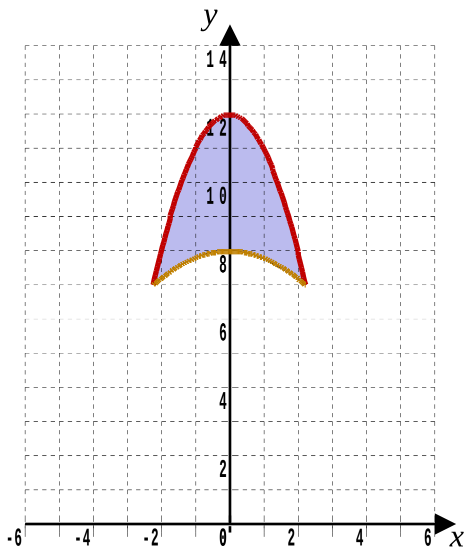
(b) Hence evaluate $\sum_{r=50}^{100} r^2(r-1)$

[2 marks]

Question 5

The diagram shows parts of the curves with equations,

$y = 12 - x^2$ (in red) and $y = 8 - 0.2x^2$ (in gold)



A jeweller models a gold ring as the volume of revolution formed when the area bounded by these two curves is rotated through 360° about the *x*-axis

(i) Given that the dimensions on the diagram are in millimetres, state the maximum outer diameter of the ring

[1 mark]

The density of gold is 19.3 g cm^{-3}

(ii) Find the mass of the ring according to this model, giving your answer in grams to 1 decimal place.

[9 marks]

(iii) Give one reason why the actual mass of the ring is likely to be different from your answer to part (ii).

[1 mark]

Year FM
Further Pure Mathematics Examination Revision
Health Check N° 2

Answer 1
(i)

$$\mathbf{A} = \begin{pmatrix} 3 & k & 0 \\ -2 & 1 & 2 \\ 5 & 0 & k+3 \end{pmatrix}$$

$$\begin{aligned} det\ \mathbf{A} &= 3\begin{vmatrix} 1 & 2 \\ 0 & k+3 \end{vmatrix} - k\begin{vmatrix} -2 & 2 \\ 5 & k+3 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 5 & 0 \end{vmatrix} \\ &= 3\left(k+3\right) - k(-2k-6-10) + 0 \\ &= 3k + 9 + 2k^2 + 16k \\ &= 2k^2 + 19k + 9 \end{aligned}$$

[2 marks]

(ii) If **A** is singular then its determinant will equal zero.

$$2k^2 + 19k + 9 = 0$$

$$(2k+1)(k+9) = 0$$

$$\text{Either } k = -\frac{1}{2} \text{ or } k = -9$$

[2 marks]

Answer 2

(a) As $f(z) = z^3 + 3z^2 + kz + 48$ and $f(4i) = 0$ it follows that,

$$(4i)^3 + 3(4i)^2 + k(4i) + 48 = 0$$

$$-64i - 48 + 4ki + 48 = 0$$

$$0 + 4i(k-16) = 0 + 0i$$

$$\therefore\ k = 16$$

[2 marks]

(b) As the coefficients of the polynomial are all real, the two complex roots occur as a conjugate pair.

$\therefore\ z = 4i$ is a root, along with $z = -4i$

The product of the two known factors $(z - 4i)$ with $(z + 4i)$ is $z^2 + 16$

and it's now obvious that the third factor is $(z + 3)$

Thus the three roots are, 4i, – 4i, – 3

[3 marks]

Answer 3

Method 1 : Core 1 Method

$$\frac{\cos 2x + \mathrm{i}\sin 2x}{\cos 9x - \mathrm{i}\sin 9x}$$

$$= \frac{\cos 2x + \mathrm{i}\sin 2x}{\cos 9x - \mathrm{i}\sin 9x} \times \frac{\cos 9x + \mathrm{i}\sin 9x}{\cos 9x + \mathrm{i}\sin 9x}$$

$$= \frac{\cos 2x\cos 9x - \sin 2x\sin 9x + \mathrm{i}(\sin 2x\cos 9x + \cos 2x\sin 9x)}{\cos^2 9x + \sin^2 9x}$$

$$= \cos(2x + 9x) + \mathrm{i}(\sin(2x + 9x))$$

$$= \cos 11x + \mathrm{i}\sin 11x$$

Method 2 : Core 2 Method

$$\frac{\cos 2x + \mathrm{i}\sin 2x}{\cos 9x - \mathrm{i}\sin 9x} = \frac{e^{\mathrm{i}2x}}{e^{-\mathrm{i}9x}} \text{ using Euler's Relation}$$

$$= e^{\mathrm{i}11x}$$

$$= \cos 11x + \mathrm{i}\sin 11x$$

[4 marks]

Answer 4

(a)

$$\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{1}{12}n(n+1)\big(3n(n+1) - 2(2n+1)\big)$$

$$= \frac{1}{12}n(n+1)(3n^2 + 3n - 4n - 2)$$

$$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$$

$$\therefore\ p = 3,\ q = -1 \text{ and } r = -2$$

[4 marks]

(b)

$$\sum_{r=50}^{100} r^2(r-1) = \sum_{r=1}^{100} r^2(r-1) - \sum_{r=1}^{49} r^2(r-1)$$

$$= \frac{1}{12} \times 100 \times 101 \times 29898 - \frac{1}{12} \times 49 \times 50 \times 7152$$

$$= 25164150 - 1460200$$

$$= 23703950$$

[2 marks]

Answer 5

(i) 24 mm

[1 mark]

(ii) Get the points of intersection of the two curves,

$$12 - x^2 = 8 - 0.2\,x^2$$
$$4 = 0.8\,x^2$$
$$x = \pm\sqrt{5} \qquad \text{(About 2.236 mm)}$$

$$\begin{aligned} Volume &= 2 \times \pi \int_0^{2.236} \left(12 - x^2\right)^2 - \left(8 - 0.2\,x^2\right)^2 \,dx \\ &= 2\pi \int_0^{2.236} 144 - 24\,x^2 + x^4 - 64 + 3.2\,x^2 - 0.04\,x^4 \,dx \\ &= 2\pi \int_0^{2.236} 80 - 20.8\,x^2 + 0.96\,x^4 \,dx \\ &= 2\pi \Big[\, 80x - 6.93\,x^3 + 0.192\,x^5 \,\Big]_0^{2.236} \\ &= 2\pi \left[\, 178.9 - 77.5 + 10.7 \,\right] \\ &= 704.3\ \text{mm}^3 \end{aligned}$$

$$\begin{aligned} mass &= density \times volume \\ &= 19.3 \times \left(\frac{704.3}{1000}\right) \\ &= 13.6\,g \end{aligned}$$

[9 marks]

(iii) Air bubble within the metal
Not “pure” gold
An engraving may have removed some gold

[1 mark]

Year FM Further Pure Mathematics Examination Revision : Health Check N° 3

The Doctor will see you now

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 30

Question 1

By considering $\left(z + \dfrac{1}{z}\right)^3$ where $z = \cos\theta + \mathrm{i}\sin\theta$, show that,

$$\cos^3\theta = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)$$

[4 marks]

Question 2

(i) Use the substitution $x = a\,tan\,\theta$ to show that,

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\,arctan\left(\frac{x}{a}\right) + c$$

(This "quotable without proof" result is given in the examination formula booklet, but, as here, you may be asked to prove it)

[3 marks]

(ii) Given that $f(x) = \dfrac{8x - 3}{4 + x^2}$ find $\int f(x)\,dx$, presenting your answer in the form $A\,ln\left(x^2 + 4\right) + B\,arctan\left(\dfrac{x}{2}\right) + c$ where c is an arbitrary constant and A and B are constants to be found.

[4 marks]

Question 3

$$\mathbf{A} = \begin{pmatrix} 2p & p & 2 \\ 3 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} \text{ where } p \text{ is a real constant.}$$

Given that **A** is non-singular, find $\mathbf{A}^{-1}$ in terms of *p*.

[4 marks]

Question 4

Prove by induction that for all positive integers n, $2^{6n} + 3^{2n-2}$ is divisible by 5

[6 marks]

Question 5

The line l_1 has equation $\frac{x-2}{2} = \frac{y-4}{(-2)} = \frac{z+6}{1}$

The plane Π has equation $2x - 3y + z = 8$

The line l_2 is the reflection of line l_1 in the plane Π.

Find a vector equation of the line l_2

[9 marks]

Year FM
Further Pure Mathematics Examination Revision
Health Check N° 3

Answer 1
The key result needed, that can be quoted without proof, is,

- $$z^n + \frac{1}{z^n} = 2\cos(n\theta)$$

If forgotten it is easily proven...
Proof

$$\begin{aligned}
\text{LHS} &= z^n + \frac{1}{z^n} \\
&= z^n + z^{-n} \\
&= (\cos\theta + \mathrm{i}\sin\theta)^n + (\cos\theta + \mathrm{i}\sin\theta)^{-n} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(-n\theta) + \mathrm{i}\sin(-n\theta) && \text{de Moivre's theorem} \\
&= \cos(n\theta) + \mathrm{i}\sin(n\theta) + \cos(n\theta) - \mathrm{i}\sin(n\theta) && \text{even \& odd functions} \\
&= 2\cos(n\theta) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

And so to the question...

$$\begin{aligned}
\left(z + \frac{1}{z}\right)^3 &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \\
&= \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right) \\
(2\cos\theta)^3 &= 2\cos 3\theta + 3 \times 2\cos\theta \\
8\cos^3\theta &= 2\cos 3\theta + 6\cos\theta \\
\cos^3\theta &= \frac{1}{4}(\cos 3\theta + 3\cos\theta) \qquad \square
\end{aligned}$$

[4 marks]

Answer 2

(i)

$$\text{LHS} = \int \frac{1}{a^2 + x^2}\, dx$$

$$= \int \frac{1}{a^2 + (a \tan\theta)^2}\; a \sec^2\theta \; d\theta$$

$$= \int \frac{a \sec^2\theta}{a^2 \left(1 + \tan^2\theta \right)}\, d\theta$$

$$= \int \frac{a \sec^2\theta}{a^2 \sec^2\theta}\, d\theta$$

$$= \int \frac{1}{a}\, d\theta$$

$$= \frac{\theta}{a} + c$$

$$= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$= \text{RHS} \qquad \square$$

[3 marks]

$$\int f(x)\, dx = \int \frac{8x - 3}{4 + x^2}\, dx$$

$$= 4 \int \frac{2x}{4 + x^2}\, dx - 3 \int \frac{1}{2^2 + x^2}\, dx$$

Setting up a chain rule backwards, and preparing to use the part (i) result

$$= 4\, ln\left(4 + x^2 \right) - 3 \times \frac{1}{2} \arctan\left(\frac{x}{2} \right) + c$$

Which is of the form required with $A = 4$ and $B = -\frac{3}{2}$

[4 marks]

Answer 3

Step 1 : Find the determinant of of **A**, *det* **A**

There is a shortcut available by expanding along the middle row

$$\begin{vmatrix} 2p & p & 2 \\ 3 & 0 & 0 \\ -1 & 1 & -1 \end{vmatrix} = -3\begin{vmatrix} p & 2 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 2p & 2 \\ -1 & -1 \end{vmatrix} - 0\begin{vmatrix} 2p & p \\ -1 & 1 \end{vmatrix}$$

$$= -3\left(p \times (-1) - 1 \times 2\right)$$

$$= 3p + 6$$

Step 2 : Form, **M**, the matrix of minors of **A** by replacing each of the nine elements of the matrix **A** with that element's minor.

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} p & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2p & 2 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2p & p \\ -1 & 1 \end{vmatrix} \\ \begin{vmatrix} p & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2p & 2 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 2p & p \\ 3 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -3 & 3 \\ -p-2 & -2p+2 & 2p+p \\ 0 & -6 & -3p \end{pmatrix}$$

Step 3 : Form, **C**, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

$$\mathbf{C} = \begin{pmatrix} 0 & 3 & 3 \\ p+2 & -2p+2 & -3p \\ 0 & 6 & -3p \end{pmatrix}$$

Step 4 : Write down, $\mathbf{C}^T$, the transpose of the matrix of cofactors.

$$\mathbf{C}^T = \begin{pmatrix} 0 & p+2 & 0 \\ 3 & -2p+2 & 6 \\ 3 & -3p & -3p \end{pmatrix}$$

Step 5 : $\mathbf{A}^{-1} = \dfrac{1}{det\ \mathbf{A}}\mathbf{C}^T$

$$\mathbf{A}^{-1} = \frac{1}{3p+6}\begin{pmatrix} 0 & p+2 & 0 \\ 3 & -2p+2 & 6 \\ 3 & -3p & -3p \end{pmatrix}$$

[4 marks]

Answer 4

Statement

For all positive integers n, $f(n) = 2^{6n} + 3^{2n-2}$ is divisible by 5

Proof by Induction

When $n = 1$, $f(1) = 2^{6 \times 1} + 3^{2 \times 1 - 2}$

$= 65$ which is divisible by 5

Assume that when $n = k$, $f(k) = 2^{6k} + 3^{2k-2}$ is divisible by 5
in which case...

$$\begin{aligned} f(k+1) &= 2^{6k+6} + 3^{2k} \\ &= 2^{6k} \times 2^{6} + 3^{2k-2+2} \\ &= 64 \times 2^{6k} + 9 \times 3^{2k-2} \\ &= 9\left(2^{6k} + 3^{2k-2}\right) + 55 \times 2^{6k} \\ &= 9f(k) + 55 \times 2^{6k} \\ f(k+1) - 9f(k) &= 55 \times 2^{6k} \end{aligned}$$

$\therefore f(k+1)$ is divisible by 5

Therefore, if $f(n)$ is divisible by 5 when $n = k$,

then $f(n)$ is also divisible by 5 when $n = k + 1$

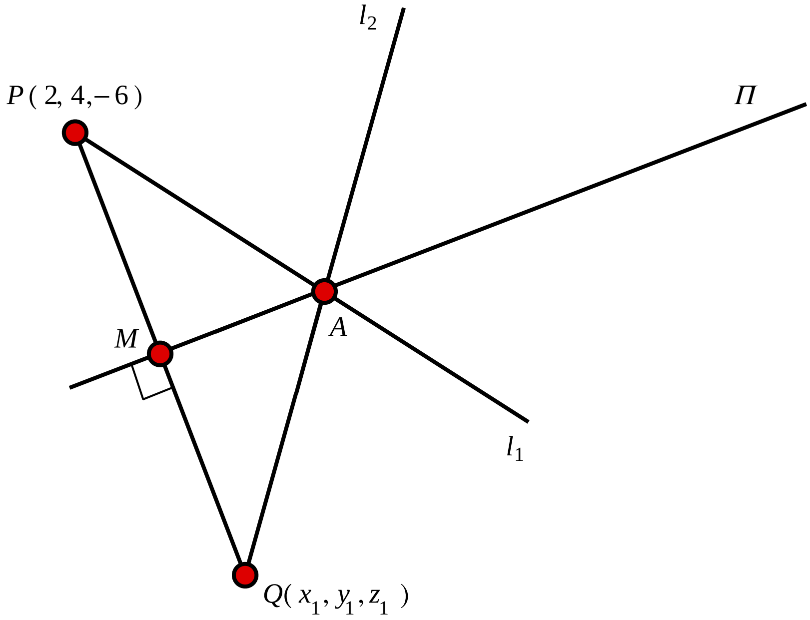
As $f(n)$ is divisible by 5 when $n = 1$, $f(n)$ is also divisible by 5 for all $n \in \mathbb{Z}^{+}$
by mathematical induction. □

[6 marks]

Answer 5

A vector equation of l_1 is $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ so $P(2, 4, -6)$ is a point on l_1

Let A be the point of intersection of l_1 and Π.



A lies on l_1 and on $2x - 3y + z = 8$

A has position vector $\begin{pmatrix} 2 + 2\lambda \\ 4 - 2\lambda \\ -6 + \lambda \end{pmatrix}$ and satisfies

$$2(2 + 2\lambda) - 3(4 - 2\lambda) - 6 + \lambda = 8$$
$$4 + 4\lambda - 12 + 6\lambda - 6 + \lambda = 8$$
$$11\lambda = 22$$
$$\lambda = 2$$

So A has coordinates $(6, 0, -4)$

A perpendicular direction vector to Π is $\boldsymbol{n} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

A vector equation of the line through P perpendicular to Π is,

$$r = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Let $Q(x_1, y_1, z_1)$ be the point of intersection of this line and l_2

Let M be the midpoint of PQ

M lies on line and on $2x - 3y + z = 8$ so satisfies

$$\begin{aligned} 2(2 + 2\mu) - 3(4 - 3\mu) - 6 + \mu &= 8 \\ 4 + 4\mu - 12 + 9\mu - 6 + \mu &= 8 \\ 14\mu &= 22 \\ \mu &= \frac{11}{7} \end{aligned}$$

So M has position vector $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \frac{11}{7}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

and Q has position vector $\begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + 2 \times \frac{11}{7}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{58}{7} \\ -\frac{38}{7} \\ -\frac{20}{7} \end{pmatrix}$

So Q has coordinates $\left(\frac{58}{7}, -\frac{38}{7}, -\frac{20}{7} \right)$

l_2 is the line through Q and A, so has direction,

$$\vec{AQ} = \begin{pmatrix} \frac{16}{7} \\ -\frac{38}{7} \\ \frac{8}{7} \end{pmatrix}$$

A vector equation of l_2 is $r = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix} + t\begin{pmatrix} 8 \\ -19 \\ 4 \end{pmatrix}$

[9 marks]

Year FM Further Pure Mathematics Examination Revision : Health Check N° 4

I told the Doctor I didn't want brain surgery.
He changed my mind !

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 32

Question 1

Find the value of x for which

$$2\tanh x - 1 = 0$$

giving your answer in terms of a natural logarithm.

[4 marks]

Question 2

$$y = \sin x \cosh x$$

(i) Show that $\dfrac{d^4y}{dx^4} = -4y$

[4 marks]

(ii) Hence find the first three non-zero terms of the Maclaurin series for y.
Give each coefficient in its simplest form.

[4 marks]

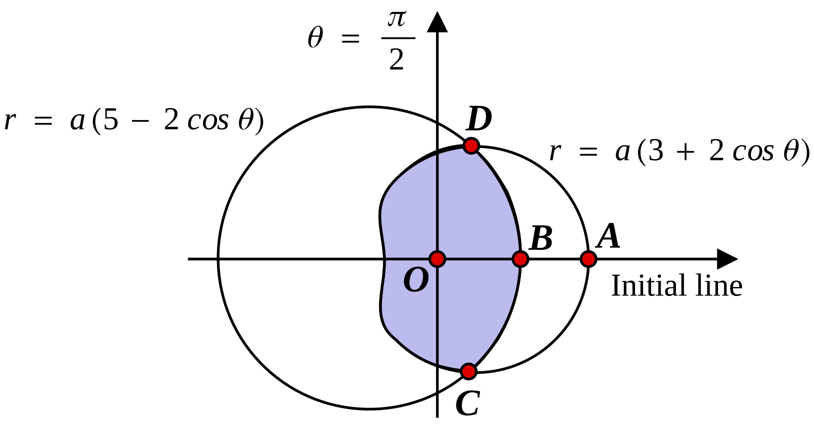
Question 3

A logo is designed which consists of two overlapping closed curves.
The polar equations of these curves are,

$$r = a(3 + 2\cos\theta), \quad 0 \leqslant \theta \leqslant 2\pi$$

$$r = a(5 - 2\cos\theta), \quad 0 \leqslant \theta \leqslant 2\pi$$

Given below is a sketch (not to scale) of these two curves.



(i) Write down the polar coordinates of the points A and B where the curves meet the initial line,

[2 marks]

(ii) Find the polar coordinates of the points C and D where the curves meet.

[4 marks]

(iii) Show that the area of the overlapping region, which is shaded in the diagram, is $\dfrac{a^2}{3}\left(49\pi - 48\sqrt{3}\right)$

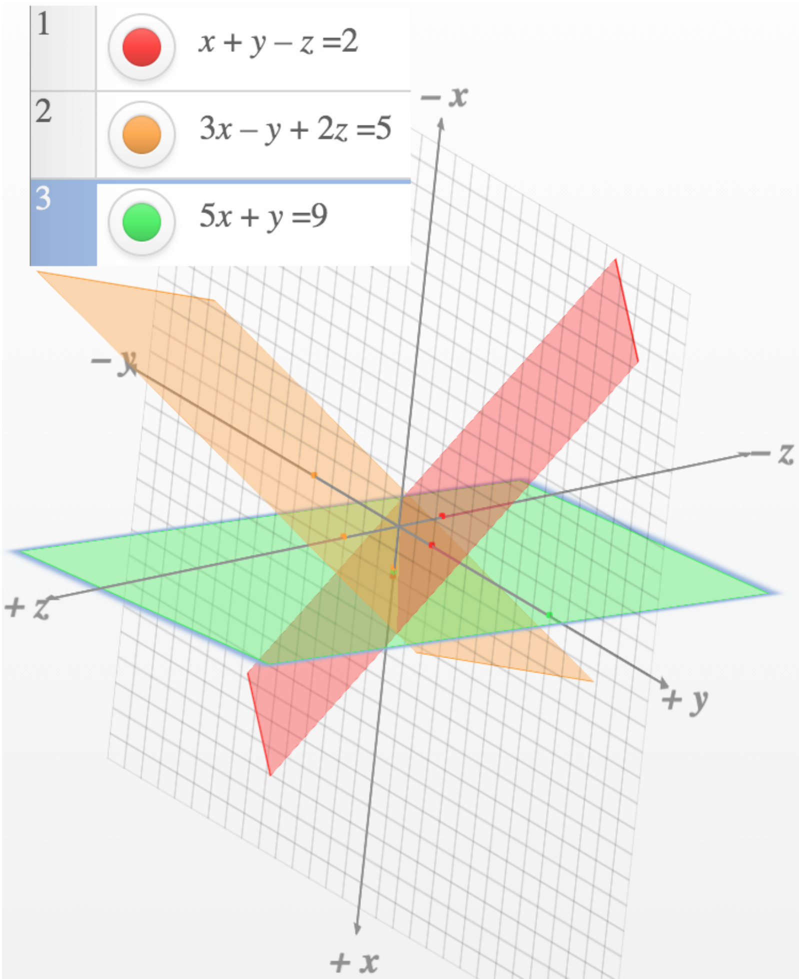
[6 marks]

Question 4

A three dimensional graph plotter is used to plot the planes with equations,

$$\begin{aligned} x + y - z &= 2 \qquad \text{Equation 1} \\ 3x - y + 2z &= 5 \qquad \text{Equation 2} \\ 5x + y \phantom{{}+2z} &= 9 \qquad \text{Equation 3} \end{aligned}$$

From the plot it looks as if, rather than intersecting at a common point, all three planes intersect along a line. This configuration of planes is called a sheaf.



(i) Consider the three planes written in matrix form,

$$\begin{pmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 5 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix}$$

Show that the 3×3 matrix in this equation has a determinant of zero.

[2 marks]

(ii) A determinant of zero means no inverse to the 3×3 matrix exists,
The matrix equation cannot be solved thus confirming that there is no unique point of intersection of all three planes.

The original equations were labelled Equation 1, Equation 2 and Equation 3. Show that, if z is eliminated by combining Equation 1 and Equation 2, then Equation 3 is obtained.

This shows that all three planes intersect in the line.

[2 marks]

(iii) Show that the points (0, 9, 7) and (1, 4, 3) are on all three planes and hence obtain the vector equation of the line of intersection in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

[4 marks]

Year FM
Further Pure Mathematics Examination Revision
Health Check N° 4

Answer 1

$$2\,\tanh x - 1 = 0$$

$$\frac{2\,e^{2x} - 2}{e^{2x} + 1} - 1 = 0$$

Multiply through by $e^{2x} + 1$

$$2\,e^{2x} - 2 - e^{2x} - 1 = 0$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \frac{1}{2}\ln 3$$

[4 marks]

Answer 2

(i)

$$y = \sin x \cosh x$$

$$\frac{dy}{dx} = \sin x \sinh x + \cos x \cosh x$$

$$\frac{d^2y}{dx^2} = \sin x \cosh x + \cos x \sinh x + \cos x \sinh x - \sin x \cosh x$$

$$= 2\cos x \sinh x$$

$$\frac{d^3y}{dx^3} = 2\cos x \cosh x - 2\sin x \sinh x$$

$$\frac{d^4y}{dx^4} = 2\cos x \sinh x - 2\sin x \cosh x - 2\sin x \cosh x - 2\cos x \sinh x$$

$$= -4\sin x \cosh x$$

$$= -4\,y \qquad \square$$

[4 marks]

(ii) Using $f(x) = f(0) + f'(0)\,x + \dfrac{f''(0)}{2!}x^2 + \ldots + \dfrac{f^{(r)}(0)}{r!}x^r + \ldots$

$$y = 0 + x + 0 + \frac{2}{3!}x^3 + 0 - \frac{4}{5!}x^5$$

$$= x + \frac{1}{3}x^3 - \frac{1}{30}x^5$$

[4 marks]

Answer 3

(i) To be on the initial line, $r = a(3 + 2\cos(0))$

$$= 5a \quad \Rightarrow \quad A(5a,\ 0)$$

$$r = a(5 - 2\cos(0))$$

$$= 3a \quad \Rightarrow \quad B(3a,\ 0)$$

[2 marks]

(ii) The curves intersect when $a(3 + 2\cos\theta) = a(5 - 2\cos\theta)$

$$4\cos\theta = 2$$

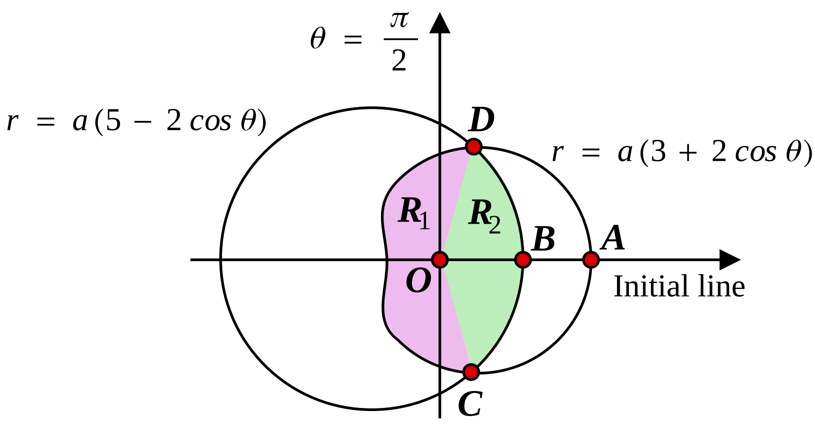
$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3},\ \frac{5\pi}{3}$$

$$\therefore\ C\left(4a,\ \frac{5\pi}{3}\right) \text{ and } D\left(4a,\ \frac{\pi}{3}\right)$$

[4 marks]

(iii)



$$R_1 = 2 \times \frac{1}{2}\int_{\frac{\pi}{3}}^{\pi} r^2\,d\theta = \int_{\frac{\pi}{3}}^{\pi} a^2\,(3 + 2\cos\theta)^2\,d\theta$$

$$= a^2\int_{\frac{\pi}{3}}^{\pi}\left(9 + 12\cos\theta + 4\cos^2\theta\right)d\theta$$

$$= a^2\int_{\frac{\pi}{3}}^{\pi}\left(9 + 12\cos\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\right)d\theta$$

$$= a^2\int_{\frac{\pi}{3}}^{\pi}\left(11 + 12\cos\theta + 2\cos 2\theta\right)d\theta$$

$$= a^2\left[\,11\theta + 12\sin\theta + \sin 2\theta\,\right]_{\frac{\pi}{3}}^{\pi}$$

$$= a^2\left(\frac{22\pi}{3} - \frac{13\sqrt{3}}{2}\right)$$

$$R_2 = 2 \times \frac{1}{2}\int_0^{\frac{\pi}{3}} r^2\, d\theta = \int_0^{\frac{\pi}{3}} a^2\,(5 - 2\cos\theta)^2\, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{3}} \left(25 - 20\cos\theta + 4\cos^2\theta \right) d\theta$$

$$= a^2 \int_0^{\frac{\pi}{3}} \left(25 - 20\cos\theta + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta \right) \right) d\theta$$

$$= a^2 \int_0^{\frac{\pi}{3}} \left(27 - 20\cos\theta + 2\cos 2\theta \right) d\theta$$

$$= a^2 \left[27\theta - 20\sin\theta + \sin 2\theta \right]_0^{\frac{\pi}{3}}$$

$$= a^2 \left(\frac{27\pi}{3} - \frac{19\sqrt{3}}{2} \right)$$

$$R_1 + R_2 = a^2 \left(\frac{22\pi}{3} - \frac{13\sqrt{3}}{2} \right) + a^2 \left(\frac{27\pi}{3} - \frac{19\sqrt{3}}{2} \right)$$

$$= \frac{a^2}{3}\left(49\pi - 48\sqrt{3} \right)$$

[6 marks]

Answer 4

(i) $$\begin{vmatrix} 1 & 1 & -1 \\ 3 & -1 & 2 \\ 5 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 5 & 0 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix}$$

$$= 1(-1 \times 0 - 1 \times 2) - (3 \times 0 - 5 \times 2) - 1(3 \times 1 - 5(-1))$$

$$= -2 + 10 - 8 = 0 \qquad \square$$

[2 marks]

(ii) $2x + 2y - 2z = 4$ (2 × Equation 1)

Add $3x - y + 2z = 5$ (Equation 2)

$5x + y = 9$ Which is Equation 3 ☐

[2 marks]

(iii) For the point (0, 9, 7),

LHS $= x + y - z = 0 + 9 - 7 = 2 =$ RHS $\therefore$ Eq 1 satisfied

LHS $= 3x - y + 2z = 0 - 9 + 14 = 5 =$ RHS $\therefore$ Eq 2 satisfied

LHS $= 5x + y = 0 + 9 = 9 =$ RHS $\therefore$ Eq 3 satisfied

Thus (0, 9, 7) is on all three planes ☐

For the point (1, 4, 3),

LHS $= x + y - z = 1 + 4 - 3 = 2 =$ RHS $\therefore$ Eq 1 satisfied

LHS $= 3x - y + 2z = 3 - 4 + 6 = 5 =$ RHS $\therefore$ Eq 2 satisfied

LHS $= 5x + y = 5 + 4 = 9 =$ RHS $\therefore$ Eq 3 satisfied

Thus (1, 4, 3) is on all three planes ☐

$$r = \begin{pmatrix} 0 \\ 9 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 0 - 1 \\ 9 - 4 \\ 7 - 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 9 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix}$$ (Or an equivalent version of this)

[4 marks]

Year FM Further Pure Mathematics Examination Revision : Health Check N° 5

A year ago the Doctor told me I was going deaf. I haven't heard from him since.

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

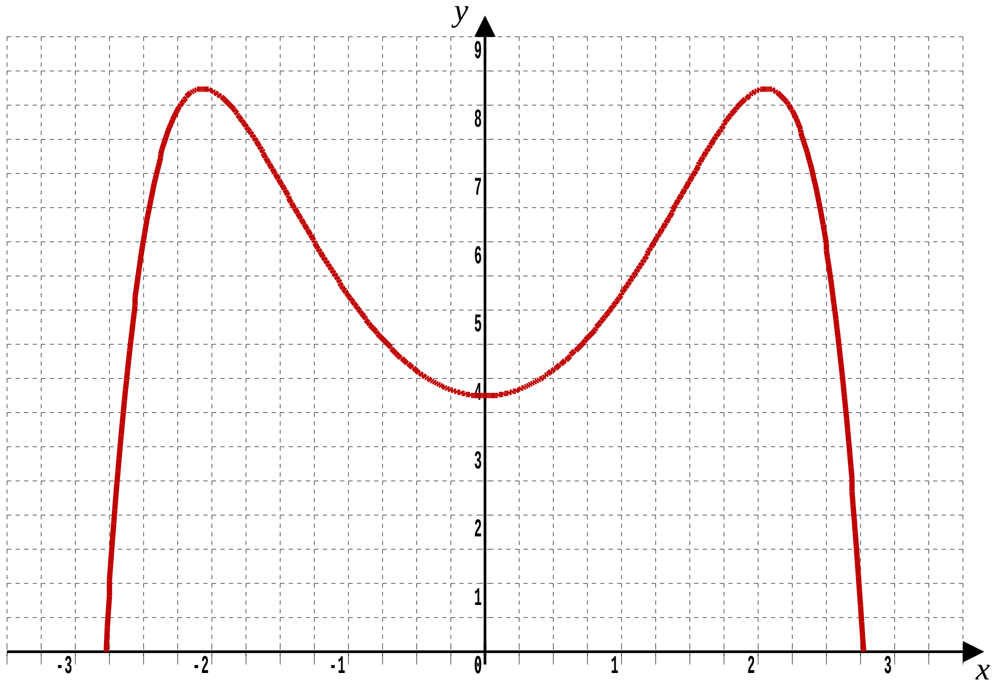
Find $\int \frac{1}{4 + 3x^2}\, dx$ giving your answer in the form $A \arctan(Bx) + c$ where c is an arbitrary constant and A and B are constants to be found.

[4 marks]

Question 2

The graph is of the function $f(x) = 4\cosh x - \dfrac{1}{4}\cosh(2x)$, $\quad x \in \mathbb{R}$

Determine the exact x coordinates of the stationary points.
Your answer should be in terms of natural logarithms.



[5 marks]

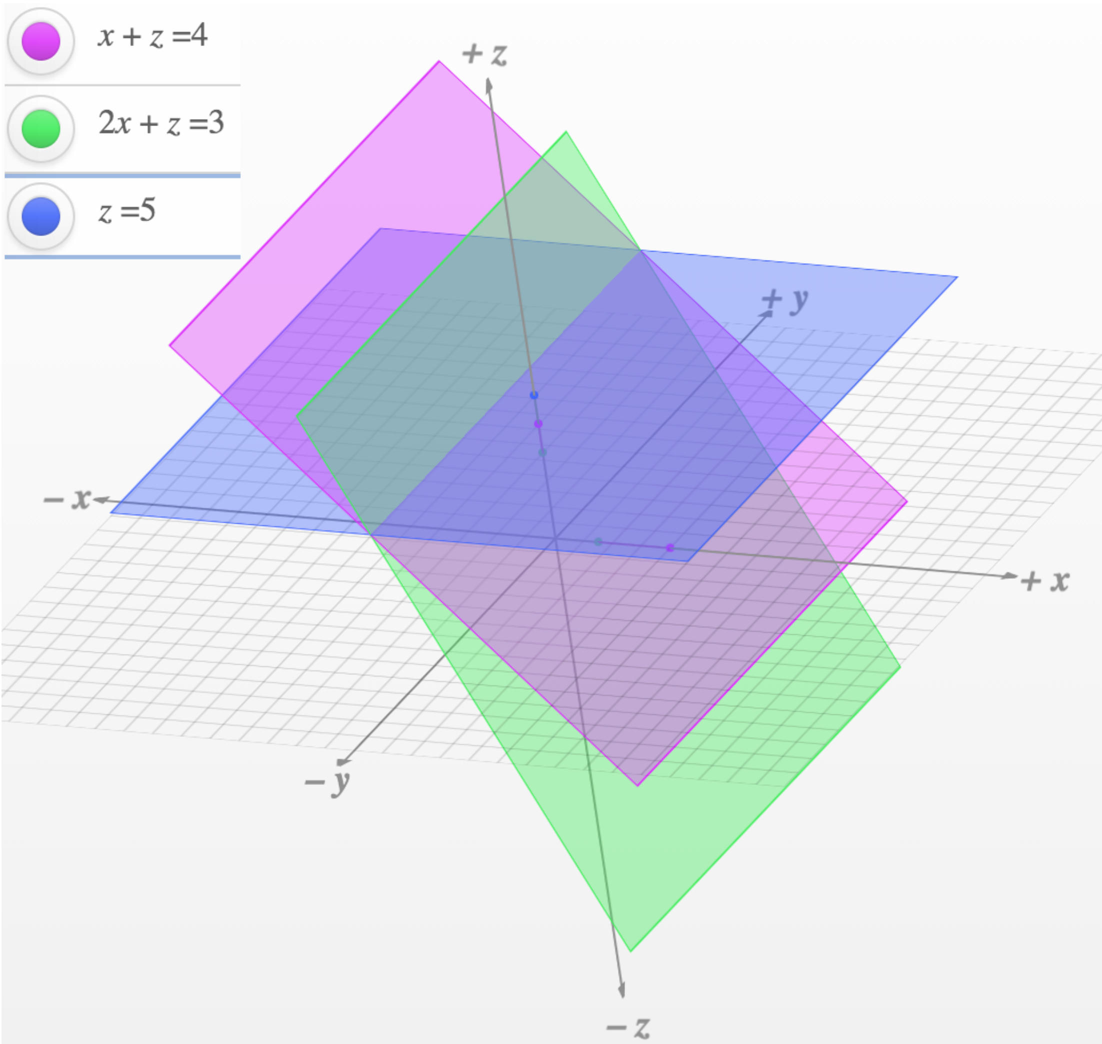
Question 3

The three dimensional graph below is of the three planes,

In purple : $x + z = 4$

In green : $2x + z = 3$

In blue : $z = 5$



The three planes have a common line of intersection.

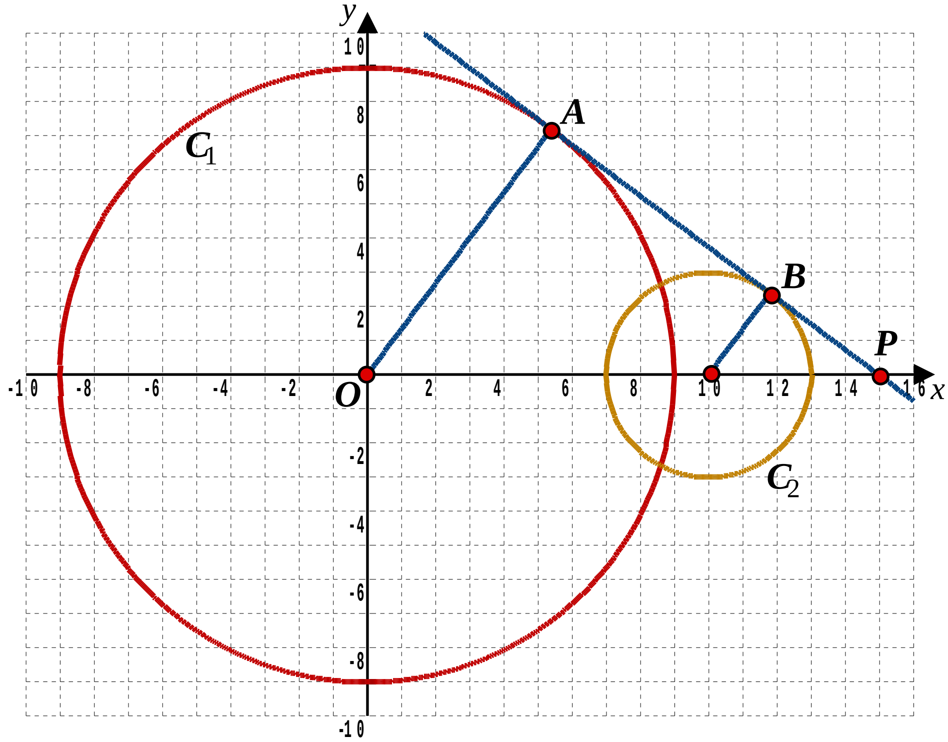
(i) What is this type of configuration of planes called ?

[1 mark]

(ii) Find a vector equation of the line of intersection in the form **r** = **a** + λ**b**

[3 marks]

Question 4



The circle C_1 has equation $x^2 + y^2 = 81$

The circle C_2 has centre (10, 0) and radius 3

(a) Write down the equation of C_2

[1 mark]

The line ABP is a tangent to C_1 at A and is also a tangent to C_2 at B

It cuts the x-axis at the point P

(b) By considering similar triangles, show that the coordinates of P are (15, 0)

[3 marks]

(c) A line through P has gradient m.
Write down, in terms of m, the equation of this line.

[1 mark]

This line cuts C_1 in two points.

(d) Show that the x-coordinates of these two points satisfy the equation,

$$x^2\left(1 + m^2\right) - 30m^2x + \left(225m^2 - 81\right) = 0$$

[3 marks]

(e) **Hence** determine the coordinates of the point A

[5 marks]

Question 5

By solving a suitable matrix equation with the aid of your calculator, find the single point at which the following three planes intersect,

$$\begin{aligned} x - 3y - 4z &= 3 \\ 6x + 5y - 7z &= 30 \\ x + 4y + 6z &= -3 \end{aligned}$$

[4 marks]

Year FM
Further Pure Mathematics Examination Revision
Health Check N° 5

Answer 1
Using the quotable result that,

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\,arctan\left(\frac{x}{a}\right) + c,\ a > 0,\ |x| < a$$

$$\begin{aligned}
\int \frac{1}{4 + 3x^2}\,dx &= \frac{1}{3}\int \frac{1}{\left(\frac{4}{3}\right) + x^2}\,dx \\
&= \frac{1}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2 + x^2}\,dx \\
&= \frac{1}{3} \times \frac{\sqrt{3}}{2}\,arctan\left(\frac{\sqrt{3}\,x}{2}\right) + c \\
&= \frac{\sqrt{3}}{6}\,arctan\left(\frac{\sqrt{3}\,x}{2}\right) + c
\end{aligned}$$

[4 marks]

Answer 2

$$f(x) = 4\,cosh\,x - \frac{1}{4}\,cosh\,(2x)$$

$$f'(x) = 4\,sinh\,x - \frac{1}{2}\,sinh\,(2x)$$

For stationary points, $f'(x) = 0$

$$4\,sinh\,x - \frac{1}{2}\,sinh\,(2x) = 0$$

$$4\,sinh\,x - sinh\,x\,cosh\,x = 0$$

$$sinh\,x\,(4 - cosh\,x) = 0$$

Either $sinh\,x = 0$ in which case $x = 0$

Or $cosh\,x = 4$

$$\frac{e^x + e^{-x}}{2} = 4$$

$$e^x + e^{-x} = 8$$

Multiply through by e^x

$$(e^x)^2 - 8(e^x) + 1 = 0$$

$$e^x = \frac{8 \pm \sqrt{64 - 4 \times 1 \times 1}}{2}$$

$$= 4 \pm \sqrt{15} \qquad \Rightarrow\ x = ln\left(4 \pm \sqrt{15}\right)$$

[5 marks]

Answer 3

(i) A sheaf

[1 mark]

(ii) Putting the third equation, $z = 5$, into the first, $x + z = 4$, gives the same as putting it into the second, $2x + z = 3$. That is $x = -1$.
So, as they should be for a sheaf, the equations are consistent.
Thus, two point on the intersection line are $(-1,\ 0,\ 5)$ and $(-1,\ 1,\ 5)$

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} + \lambda\left(\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}\right)$$

$$= \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$ Other descriptions of this same line are possible

[3 marks]

Answer 4

(a) $(x - 10)^2 + y^2 = 3^2$

[1 mark]

(b) By considering the radius of each circle, $lsf = \frac{9}{3} = 3$

Let the length of OP be x so it is required to show that x is 15

$$\therefore \frac{x}{x - 10} = 3$$
$$x = 3(x - 10)$$
$$x = 3x - 30$$
$$2x = 30$$
$$x = 15 \quad \square$$

[3 marks]

(c) $y = mx + c \Rightarrow 0 = 15m + c \Rightarrow c = -15m$

$$\therefore y = mx - 15m$$

[1 mark]

(d)
$$x^2 + y^2 = 81$$
$$x^2 + (mx - 15m)^2 = 81$$
$$x^2 + m^2x^2 - 30m^2x + 225m^2 - 81 = 0$$
$$x^2\left(1 + m^2\right) - 30m^2x + \left(225m^2 - 81\right) = 0$$

[3 marks]

(e) The "two points" where the line cuts the circle C_1 will become "one point" at A as the line is a tangent to the circle at A.
So the discriminant of the part (d) equation will be zero.

$$\left(-30\,m^2\right)^2 - 4 \times \left(1 + m^2\right) \times \left(225\,m^2 - 81\right) = 0$$

$$900\,m^4 - 4\left(225\,m^2 - 81 + 225\,m^4 - 81\,m^2\right) = 0$$

Divide through by 4

$$225\,m^4 - 225\,m^2 + 81 - 225\,m^4 + 81\,m^2 = 0$$

$$-144\,m^2 + 81 = 0$$

$$144\,m^2 = 81$$

$$m^2 = \frac{9}{16}$$

$$m = -\frac{3}{4}$$

(As the gradient between A and P is negative)

Putting this back into the part (d) equation...

$$x^2\left(1 + m^2\right) - 30\,m^2\,x + \left(225\,m^2 - 81\right) = 0$$

$$x^2\left(1 + \frac{9}{16}\right) - 30 \times \frac{9}{16}\,x + \left(225 \times \frac{9}{16} - 81\right) = 0$$

$$\frac{25}{16}\,x^2 - \frac{270}{16}\,x + \frac{729}{16} = 0$$

$$25\,x^2 - 270x + 729 = 0$$

$$x = \frac{270 \pm 0}{50}$$

$$= \frac{27}{5}$$

$$y = -\frac{3}{4} \times \frac{27}{5} - 15 \times \left(-\frac{3}{4}\right)$$

$$= \frac{36}{5}$$

$\therefore$ The coordinates of A are $\left(\frac{27}{5}, \frac{36}{5}\right)$

[5 marks]

Answer 5

As a matrix equation,

$$\begin{pmatrix} 1 & -3 & -4 \\ 6 & 5 & -7 \\ 1 & 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 30 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{111}\begin{pmatrix} 58 & 2 & 41 \\ -43 & 10 & -17 \\ 19 & -7 & 23 \end{pmatrix}\begin{pmatrix} 3 \\ 30 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

$\therefore$ Point of intersection is $(1,\ 2,\ -2)$

[4 marks]

Year FM Further Pure Mathematics Examination Revision : Health Check N° 6

Doctor, doctor, I'm scared of Christmas.
Hmmm, I think you're suffering from Claus-trophobia.

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Given that $f(x) = \ln\cos x$

(i) show that $f'(x) = -\tan x$

[1 mark]

(ii) find the values of $f'(0)$, $f''(0)$, $f'''(0)$ and $f''''(0)$

[4 marks]

(iii) express $\ln\cos x$ as a series in ascending powers of x up to the term in x^4

[2 marks]

Question 2

Three planes A, B and C are defined by the following equations,

$$\begin{aligned} A &: x + ay + 2z = a \\ B &: x - y - z = a \\ C &: x + 4y + 4z = 0 \end{aligned}$$

Given that the planes do not meet at a single point,

(a) find the value of a

[4 marks]

(b) determine whether the three equations form a consistent system, and give a geometric interpretation of your answer.

[4 marks]

Question 3

$$f(n) = 2^n + 6^n$$

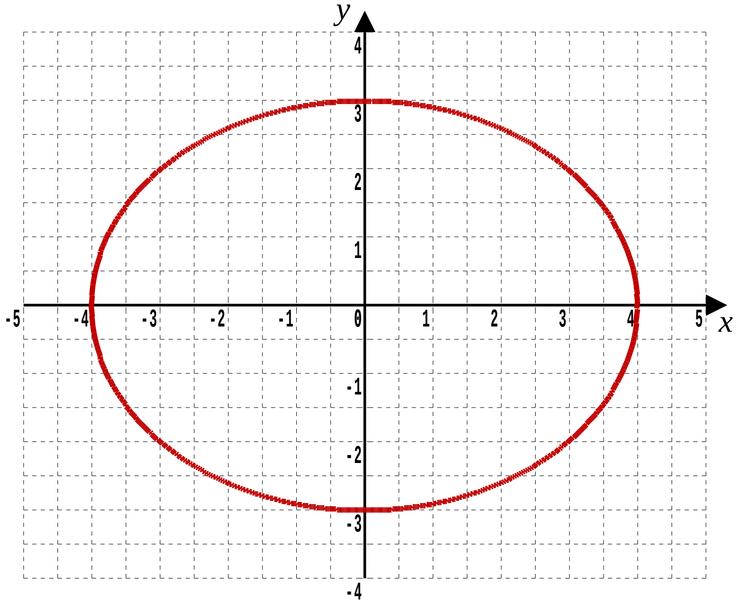
(a) Show that $f(k+1) = 6f(k) - 4(2^k)$

[3 marks]

(b) Prove by induction that that all $n \in \mathbb{Z}^+$ $f(n)$ is divisible by 8

[4 marks]

Question 4



An ellipse has parametric equations

$$x = 4\cos\theta, \qquad y = 3\sin\theta, \qquad 0 \leqslant \theta \leqslant 2\pi$$

(i) Find the area enclosed by the ellipse.

[5 marks]

(ii) Find the volume of the solid of revolution formed when this area is rotated through 2π radians about the x-axis.

[5 marks]

Question 5

The complex number w satisfies,

$$|w - 1 - i| = 3 \quad \text{and} \quad arg(w - 2) = \frac{\pi}{4}$$

Find, in simplest form, the exact value of $|w|^2$

[8 marks]

Year FM

Further Pure Mathematics Examination Revision
Health Check N° 6

Answer 1

(i)

$$f(x) = \ln \cos x$$

$$f'(x) = \frac{1}{\cos x} \times (-\sin x)$$

$$= -\tan x$$

[1 mark]

(ii) $f'(x) = -\tan x \Rightarrow f'(0) = 0$

$$f''(x) = -\sec^2 x \Rightarrow f''(0) = -\frac{1}{\cos^2(0)} = -1$$

$$f'''(x) = -2\sec^2 x \tan x \Rightarrow f'''(0) = 0$$

$$f'''(x) = -2\left(1 + \tan^2 x\right)\tan x$$

$$f''''(x) = -2\left(\tan x + \tan^3 x\right)$$

$$= -2\left(\sec^2 x + 3\tan^2 x \sec^2 x\right) \Rightarrow f''''(0) = -2(1 + 0)$$

$$= -2$$

[4 marks]

(iii) This is a Maclaurin series question without the words "Maclaurin series"

$$f(x) = f(0) + f'(0)\, x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f''''(0)}{4!} x^4 + \ldots$$

$$\therefore\ f(x) = 0 + 0\,x - \frac{1}{2!} x^2 + 0\,x^3 - \frac{2}{4!} x^4 + \ldots$$

$$= -\frac{1}{2} x^2 - \frac{1}{12} x^4 + \ldots$$

[2 marks]

Answer 2

(a) Writing the three planes as a matrix,

$$\begin{pmatrix} 1 & a & 2 \\ 1 & -1 & -1 \\ 1 & 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix}$$

Working out the determinant by expanding along the top row,

$$1\begin{vmatrix} -1 & -1 \\ 4 & 4 \end{vmatrix} - a\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= 1(-4 + 4) - a(4 + 1) + 2(4 + 1)$$

$$= 0 - 5a + 10$$

For the planes to not meet at a single point the determinant must be zero.

$$\therefore a = 2$$

[4 marks]

(b) With $a = 2$, the equations of the planes are,

$$A : x + 2y + 2z = 2$$

$$B : x - y - z = 2$$

$$C : x + 4y + 4z = 0$$

$$A - B \text{ gives } 3y + 3z = 0 \Rightarrow y = -z$$

Replacing y with $(-z)$ in all three equations gives,

$$A : x - 2z + 2z = 2 \Rightarrow x = 2$$

$$B : x + z - z = 2 \Rightarrow x = 2$$

$$C : x - 4z + 4z = 0 \Rightarrow x = 0$$

So the equations are not forming a consistent system: The planes form a prism.

[4 marks]

Answer 3

(a)

$$f(n) = 2^n + 6^n$$

$$\begin{aligned} \text{LHS} &= f(k + 1) \\ &= 2^{k+1} + 6^{k+1} \\ &= 2 \times 2^k + 6 \times 6^k \\ &= (6 - 4) \times 2^k + 6 \times 6^k \\ &= 6 \times 2^k + 6 \times 6^k - 4 \times 2^k \\ &= 6(2^k + 6^k) - 4(2^k) \\ &= 6f(k) - 4(2^k) \\ &= \text{RHS} \qquad \square \end{aligned}$$

[3 marks]

(b)
Proof by Induction

When $n = 1$, $f(1) = 2^1 + 6^1$

$= 8$ which is divisible by 8

Assume that when $n = k$, $f(k) = 2^k + 6^k$ is divisible by 8
in which case...

$$f(k+1) = 2^{k+1} + 6^{k+1}$$

$$= 6f(k) - 4\left(2^k\right) \quad \text{using part (a)}$$

$$f(k+1) - 6f(k) = 4\left(2^k\right)$$

$$= 8\left(2^{k-1}\right) \quad \text{Valid for } k \geqslant 1,$$

$$\therefore f(k+1) \text{ is divisible by } 8$$

Therefore, if $f(n)$ is divisible by 8 when $n = k$,

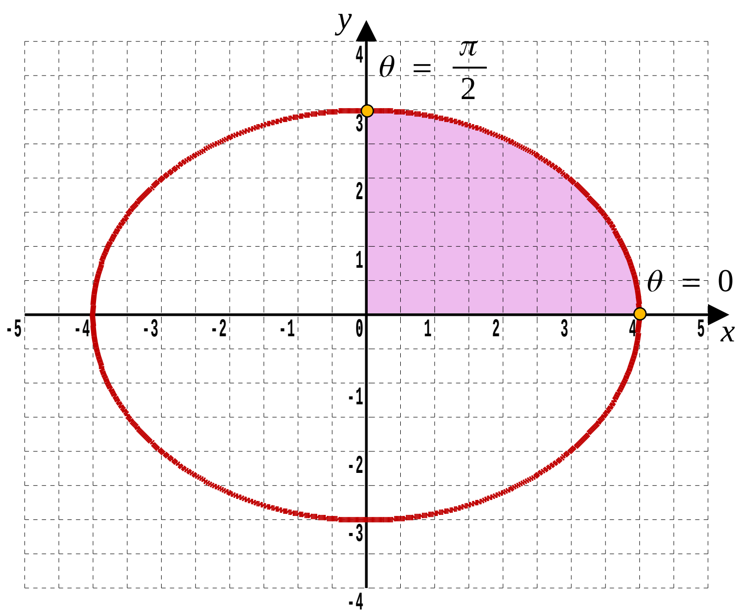
then $f(n)$ is also divisible by 8 when $n = k + 1$

As $f(n)$ is divisible by 8 when $n = 1$, $f(n)$ is also divisible by 8 for all $n \in \mathbb{Z}^+$
by mathematical induction. □

[4 marks]

Answer 4

$$x = 4\cos\theta, \qquad y = 3\sin\theta, \qquad 0 \leqslant \theta \leqslant 2\pi$$



(i) Of the various possible strategies here is one of the easiest which focuses on finding the area shaded which will be one quarter of the total area.

$$Area = 4 \int_{\theta=\frac{\pi}{2}}^{\theta=0} y \frac{dx}{d\theta}\, d\theta$$

$$= 4 \int_{\theta=\frac{\pi}{2}}^{\theta=0} 3\sin\theta\ (-4\sin\theta)\ d\theta$$

$$= -48 \int_{\theta=\frac{\pi}{2}}^{\theta=0} \sin^2\theta\, d\theta$$

$$= 48 \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= 24 \int_{\theta=0}^{\theta=\frac{\pi}{2}} 1 - \cos(2\theta)\ d\theta$$

$$= 24 \left[\theta - \frac{\sin(2\theta)}{2} \right]_0^{\frac{\pi}{2}}$$

$$= 24\left[\frac{\pi}{2} - \frac{\sin(\pi)}{2} - 0 - \frac{\sin(0)}{2} \right]$$

$$= 12\pi$$

[5 marks]

(ii) The strategy used here is to rotate the shaded area about the x-axis and double that result to get the total volume.

$$V = 2\pi \int_{\theta=\frac{\pi}{2}}^{\theta=0} y^2 \frac{dx}{d\theta}\, d\theta$$

$$= 2\pi \int_{\theta=\frac{\pi}{2}}^{\theta=0} 9\sin^2\theta\ (-4\sin\theta)\ d\theta$$

$$= -72\pi \int_{\theta=\frac{\pi}{2}}^{\theta=0} \sin^2\theta \sin\theta\, d\theta$$

$$= 72\pi \int_{\theta=0}^{\theta=\frac{\pi}{2}} \left(1 - \cos^2\theta\right) \sin\theta\, d\theta$$

$$= 72\pi \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta\, d\theta - 72\pi \int_{\theta=0}^{\theta=\frac{\pi}{2}} \sin\theta \cos^2\theta\, d\theta$$

$$= 72\pi \left[-\cos\theta\right]_0^{\frac{\pi}{2}} - 72\pi \left[\frac{\cos^3\theta}{3} \right]_0^{\frac{\pi}{2}}$$

$$= 72\pi \left[-0 + \cos(0)\right] - 72\pi\left[0 - \frac{1}{3} \right]$$

$$= 72\pi - 24\pi$$

$$= 48\pi$$

[5 marks]

Answer 5

$$|w - 1 - i| = 3 \qquad\qquad arg(w - 2) = \frac{\pi}{4}$$

$$|x + yi - 1 - i| = 3 \qquad\qquad arg(x + yi - 2) = \frac{\pi}{4}$$

$$|(x - 1) + (y - 1)i| = 3 \qquad\qquad arg((x - 2) + yi) = \frac{\pi}{4}$$

$$(x - 1)^2 + (y - 1)^2 = 9 \qquad\qquad \frac{y}{x - 2} = tan\left(\frac{\pi}{4}\right)$$

$$y = x - 2$$

Substituting the straight line into the circle...

$$(x - 1)^2 + (x - 2 - 1)^2 = 9$$

$$x^2 - 2x + 1 + x^2 - 6x + 9 = 9$$

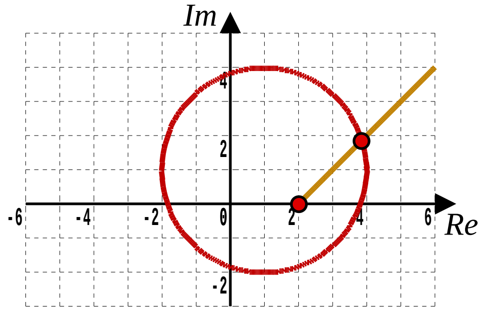
$$2x^2 - 8x + 1 = 0$$

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 2 \times 1}}{2 \times 2}$$

$$= \frac{8 \pm \sqrt{56}}{4}$$

$$= 2 \pm \frac{1}{2}\sqrt{14}$$

Only want positive value of x so the intersection is $\left(2 + \frac{1}{2}\sqrt{14}, \ \frac{1}{2}\sqrt{14}\right)$



$$|w|^2 = \left(\frac{4 + \sqrt{14}}{2}\right)^2 + \left(\frac{\sqrt{14}}{2}\right)^2$$

$$= \frac{16 + 8\sqrt{14} + 14 + 14}{4}$$

$$= 11 + 2\sqrt{14}$$

[8 marks]

Why did Count Dracula go to the doctor ?
He couldn't stop coffin !

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 42

Question 1

FM A-Level Examination Question from November 2021, Paper Core 2, Q1 (OCR)

Two matrices, **A** and **B** are given by,

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ a & 1 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -6 & 3 & -4 \\ -1 & 6 & -4 \\ 8 & -8 & -1 \end{pmatrix} \text{ where } a \text{ is a constant.}$$

Find the value of a for which $\mathbf{AB} = \mathbf{BA}$

[3 marks]

Question 2

FM A-Level Examination Question from June 2021, Paper 2, Q5 (AQA)

The equation $z^3 + 2z^2 - 5z - 3 = 0$ has roots α, β and γ

Find a cubic with roots $\frac{1}{2}\alpha - 1$, $\frac{1}{2}\beta - 1$ and $\frac{1}{2}\gamma - 1$

[5 marks]

Question 3

FM A-Level Examination Question from June 2017, Paper FP2, Q4 (Edexcel)

$$y = ln\left(\frac{1}{1 - 2x}\right), \qquad |x| < \frac{1}{2}$$

(a) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$

[4 marks]

(b) Hence, or otherwise, find the series expansion of $ln\left(\frac{1}{1 - 2x}\right)$ about $x = 0$, in ascending powers of x, up to and including the term in x^3. Give each coefficient in its simplest form.

[3 marks]

(c) Use your expansion to find an approximate value for $ln\left(\frac{3}{2}\right)$ giving your answer to 3 decimal places.

[3 marks]

Question 4

FM AS-Level Examination Question from May 2019, Paper Core, Q4 (OCR)

In this question you must show detailed reasoning.

You are given that $f(z) = 4z^4 - 12z^3 + 41z^2 - 128z + 185$ and that $2 + \mathrm{i}$ is a root of the equation $f(z) = 0$

(a) Express $f(z)$ as the product of two quadratic factors with integer coefficients.

[5 marks]

(b) Solve $f(z) = 0$

[3 marks]

Two loci on an Argand diagram are defined by

$$C_1 = \{ z : |z| = r_1 \} \text{ and } C_2 = \{z : |z| = r_2\} \text{ where } r_1 > r_2$$

You are given that two of the points representing the roots of $f(z) = 0$

(which you worked out in part (b)) are on C_1 and two are on C_2

Let R be the region on the Argand diagram between C_1 and C_2

(c) Find the exact area of R

[4 marks]

(d) ω is the sum of all the roots of $f(z) = 0$

Determine whether or not the point on the Argand diagram which represents ω lies in R.

[2 marks]

Question 5

Further Mathematics Examination Question from January 2012, Q7 (ii) (OCR)

It is given that x satisfies the equation $arsinh\, x - arcosh\, x = ln\, 2$

(i) Use the logarithmic forms for $arsinh\, x$ and $arcosh\, x$ to show that,

$$\sqrt{x^2 + 1} - 2\sqrt{x^2 - 1} = x$$

[2 marks]

(ii) Hence, by squaring this equation, find the exact value of x

[3 marks]

Question 6

FM AS-Level Examination Question from May 2019, Paper Core, Q6 (OCR)

A transformation T is represented by the matrix $\mathbf{T}$ where,

$$\mathbf{T} = \begin{pmatrix} x^2 + 1 & -4 \\ 3 - 2x^2 & x^2 + 5 \end{pmatrix}$$

A quadrilateral Q, whose area is 12 units, is transformed by T to Q'

Find the smallest possible value of the area of Q'

[5 marks]

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Answer 1

$$\mathbf{AB} = \begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ a & 1 & 1 \end{pmatrix}\begin{pmatrix} -6 & 3 & -4 \\ -1 & 6 & -4 \\ 8 & -8 & -1 \end{pmatrix} = \begin{pmatrix} -12 & -1 & 5 \\ -1 & -20 & 3 \\ -6a+7 & 3a-2 & -4a-5 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} -6 & 3 & -4 \\ -1 & 6 & -4 \\ 8 & -8 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ a & 1 & 1 \end{pmatrix} = \begin{pmatrix} -4a & -1 & 5 \\ 11-4a & -20 & 3 \\ -8-a & 7 & -17 \end{pmatrix}$$

$$a = 3 \text{ gives } \mathbf{AB} = \mathbf{BA} = \begin{pmatrix} -12 & -1 & 5 \\ -1 & -20 & 3 \\ -3 & 7 & -17 \end{pmatrix}$$

[3 marks]

Answer 2

Let the sought after cubic be in terms of the variable w in which case,

$$w = \frac{1}{2}z - 1$$

$$w + 1 = \frac{1}{2}z$$

$$z = 2w + 2$$

$$z^3 + 2z^2 - 5z - 3 = 0$$

$$\Downarrow$$

$$(2w+2)^3 + 2(2w+2)^2 - 5(2w+2) - 3 = 0$$

$$\begin{aligned} 8w^3 + 24w^2 + 24w &+ 8 \\ + 8w^2 + 16w &+ 8 \\ - 10w &- 10 \\ &- 3 = 0 \end{aligned}$$

$$8w^3 + 32w^2 + 30w + 3 = 0$$

[3 marks]

Answer 3

(a) $y = ln\left(\dfrac{1}{1 - 2x}\right), \qquad |x| < \dfrac{1}{2}$

$$= ln(1) - ln(1 - 2x)$$

$$\frac{dy}{dx} = 0 - \frac{1}{1 - 2x} \times (-2)$$

$$= 2(1 - 2x)^{-1}$$

$$\frac{d^2y}{dx^2} = 2(-1)(1 - 2x)^{-2}(-2)$$

$$= 4(1 - 2x)^{-2}$$

$$\frac{d^3y}{dx^3} = 4(-2)(1 - 2x)^{-3}(-2)$$

$$= 16(1 - 2x)^{-3}$$

[4 marks]

(b) $f(x) = f(0) + f'(0)\,x + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \dfrac{f''''(0)}{4!}x^4 + \ldots$

$$\therefore f(x) = 0 + 2x + \frac{4}{2!}x^2 + \frac{16}{3!}x^3 + \ldots$$

$$= 2x + 2x^2 + \frac{8}{3}x^3 + \ldots$$

[3 marks]

(c) We require,

$$\frac{3}{2} = \frac{1}{1 - 2x}$$

$$\frac{2}{3} = 1 - 2x$$

$$2x = 1 - \frac{2}{3}$$

$$x = \frac{1}{6}$$

$$ln\left(\frac{3}{2}\right) \approx 2 \times \frac{1}{6} + 2 \times \left(\frac{1}{6}\right)^2 + \frac{8}{3} \times \left(\frac{1}{6}\right)^3 + \ldots$$

$$\approx \frac{1}{3} + \frac{1}{18} + \frac{1}{81} + \ldots$$

$$\approx \frac{65}{162} \qquad \text{(About 0.401)}$$

[3 marks]

Answer 4

(a) $f(z) = 4z^4 - 12z^3 + 41z^2 - 128z + 185$

As the coefficients of the polynomial are all real, another root will be 2 – i

$$\begin{aligned}(z - 2 - i)(z - 2 + i) &= ((z - 2) - i)((z - 2) + i) \\ &= (z - 2)^2 - i^2 \\ &= z^2 - 4z + 5\end{aligned}$$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad\quad 4z^2 + 4z + 37 \\ \hline z^2 - 4z + 5 & 4z^4 - 12z^3 + 41z^2 - 128z + 185 \\ & \underline{4z^4 - 16z^3 + 20z^2} \\ & \quad\quad\quad 4z^3 + 21z^2 - 128z \\ & \quad\quad\quad \underline{4z^3 - 16z^2 + 20z} \\ & \quad\quad\quad\quad\quad 37z^2 - 148z + 185 \\ & \quad\quad\quad\quad\quad \underline{37z^2 - 148z + 185} \\ & \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0 \end{array}$$

[5 marks]

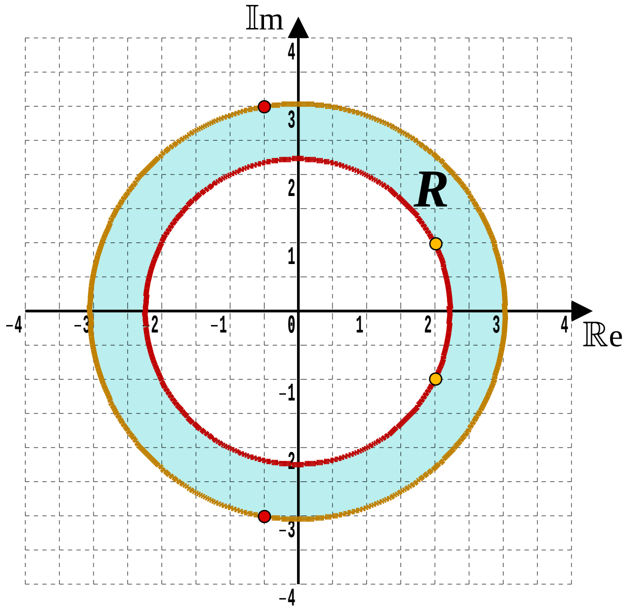
(b) Working out the roots of $4z^2 + 4z + 37 = 0$

$$\begin{aligned} z &= \frac{-4 \pm \sqrt{4^2 - 4 \times 4 \times 37}}{2 \times 4} \\ &= \frac{-4 \pm \sqrt{-576}}{8} \\ &= \frac{-4 \pm 24i}{8} \\ &= -\frac{1}{2} \pm 3i \end{aligned}$$

The roots of $f(x)$ are, $z \in \left\{ 2 \pm i,\ -\frac{1}{2} \pm 3i \right\}$

[3 marks]

(c) The following Argand Diagram will help with understanding the situation describes which is a large part of what this question is about,



The area of the region R is given by,

$$\frac{37\pi}{4} - 5\pi = \frac{17\pi}{4}$$

[4 marks]

(d) $\omega = 2 + i + 2 - i - \frac{1}{2} + 3i - \frac{1}{2} - 3i = 3 + 0i$

which is just inside the region R

[2 marks]

Answer 5

(i)

$$arsinh\, x - arcosh\, x = ln\, 2$$

$$ln\left(x + \sqrt{x^2 + 1}\right) - ln\left(x + \sqrt{x^2 - 1}\right) = ln\, 2$$

$$ln\left(\frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 - 1}}\right) = ln\, 2$$

$$\frac{x + \sqrt{x^2 + 1}}{x + \sqrt{x^2 - 1}} = 2$$

$$x + \sqrt{x^2 + 1} = 2x + 2\sqrt{x^2 - 1}$$

$$\sqrt{x^2 + 1} - 2\sqrt{x^2 - 1} = x \qquad \square$$

[2 marks]

(ii) Squaring the equation, as instructed,

$$x^2 + 1 - 4\sqrt{(x^2+1)(x^2-1)} + 4(x^2-1) = x^2$$

$$4x^2 - 4\sqrt{x^4-1} - 3 = 0$$

$$4\sqrt{x^4-1} = 4x^2 - 3$$

$$16(x^4-1) = 16x^4 - 24x^2 + 9$$

$$24x^2 - 25 = 0$$

$$x^2 = \frac{25}{24}$$

$$x = \pm\frac{5}{\sqrt{24}} \times \frac{\sqrt{24}}{\sqrt{24}}$$

$$= \pm\frac{10\sqrt{6}}{24}$$

$$= \pm\frac{5\sqrt{6}}{12}$$

The only solution that makes the original equation true is $x = \dfrac{5\sqrt{6}}{12}$

[3 marks]

Answer 6

$$|\mathbf{T}| = \begin{vmatrix} x^2+1 & -4 \\ 3-2x^2 & x^2+5 \end{vmatrix}$$

$$= (x^2+1)(x^2+5) - (3-2x^2)(-4)$$

$$= x^4 + 6x^2 + 5 + 12 - 8x^2$$

$$= x^4 - 2x^2 + 17$$

$$= (x^2-1)^2 + 16$$

$$\therefore\ |\mathbf{T}|_{MIN} = 16$$

Smallest possible area is $12 \times 16 = 192$ units

[5 marks]

“Doctor, doctor, I'm addicted to brake fluid”
“What nonsense, you can stop anytime”

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

Given a complex number $z = a + b\text{i}$, the conjugate of z, which is denoted z^*, is the complex number $z = a - b\text{i}$.

Show that $\dfrac{z}{z^*} = \left(\dfrac{a^2 - b^2}{a^2 + b^2}\right) + \left(\dfrac{2ab}{a^2 + b^2}\right)\text{i}$

[3 marks]

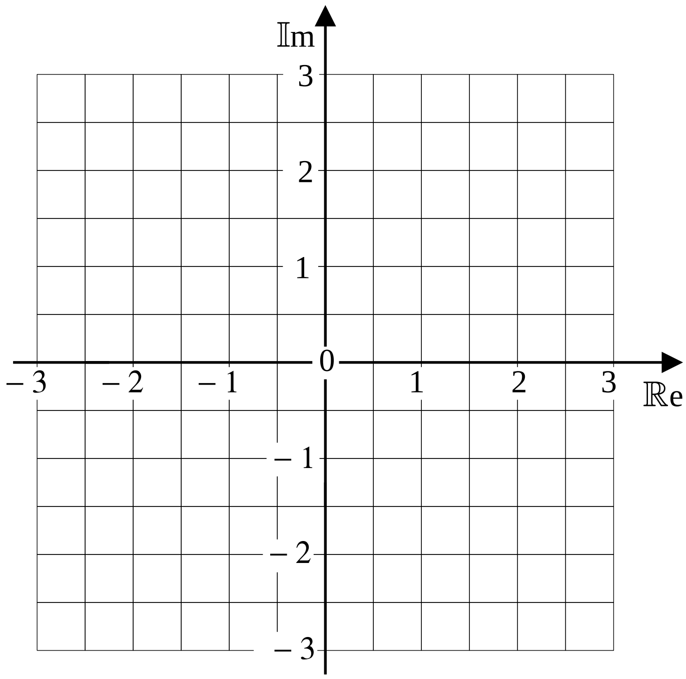
Question 2

FM A-Level Examination Question from June 2021, Paper 1, Q6 (AQA)

(a) Show that the equation $(2z - z^*)^* = z^2$ has exactly four solutions.
Find these solutions.

[7 marks]

(b) **(i)** Plot the four solutions to the equation in part (a) on the Argand diagram and join them together to form a quadrilateral with one line of symmetry.



[2 marks]

(ii) Show that the area of this quadrilateral is $\dfrac{\sqrt{15}}{2}$ square units.

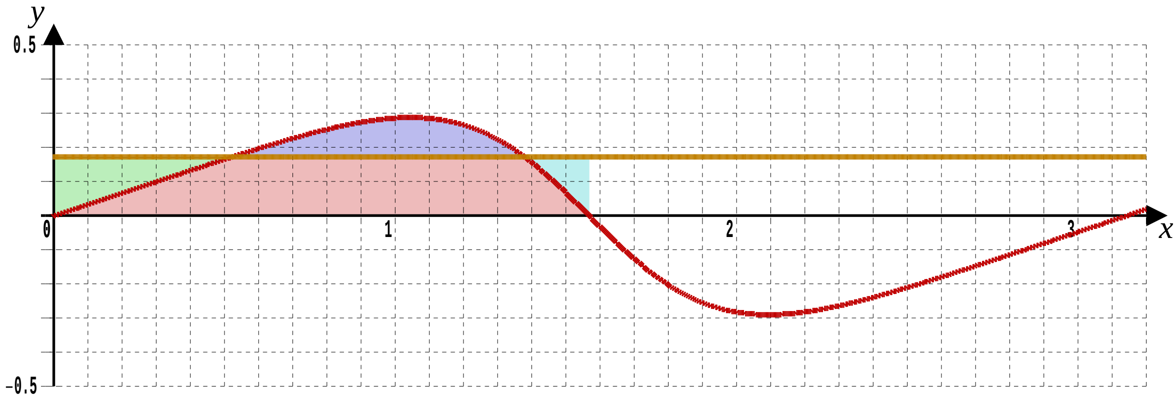
[1 mark]

Question 3

(a) Find the exact mean value of $f(x) = \dfrac{\sin x \cos x}{\cos 2x + 2}$ over the interval $\left[0, \dfrac{\pi}{2}\right]$

[4 marks]

(b) The graph is of the function $f(x) = \dfrac{\sin x \cos x}{\cos 2x + 2}$



Making use of the graph, explain the geometric significance of your part (a) answer.

[2 marks]

Question 4

FM A-Level Examination Question from October 2021, Paper Core 1, Q1 (OCR)

(a) Sketch on a single Argand diagram the loci given by,

(i) $|z - 1 + 2\mathrm{i}| = 3$

[2 marks]

(ii) $|z + 1| = |z - 2|$

[2 marks]

(b) Indicate, by shading, the region of the Argand diagram for which

$$|z - 1 + 2\mathrm{i}| \leqslant 3 \text{ and } |z + 1| \leqslant |z - 2|$$

[2 marks]

Question 5

(a) Use the substitution $x = \dfrac{a}{sinh\,\theta}$, where a is a constant, to show that,

for $x > 0, a > 0, \quad \displaystyle\int \frac{1}{x\sqrt{x^2 + a^2}}\,dx = -\frac{1}{a}\,arsinh\left(\frac{a}{x}\right) + constant$

[6 marks]

(b) Hence, or otherwise, find the exact value of $\displaystyle\int_1^2 \frac{1}{x\sqrt{x^2 + 4}}\,dx$

[4 marks]

Question 6

The Cartesian equation of a curve is $\left(x^2 + y^2 - 2x\right)^2 = 4\left(x^2 + y^2\right)$

Recast this equation in the polar form, $r = f(\theta)$

[5 marks]

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Answer 1

$$\begin{aligned}
\text{LHS} &= \frac{z}{z^*} \\
&= \frac{a + b\mathrm{i}}{a - b\mathrm{i}} \times \frac{a + b\mathrm{i}}{a + b\mathrm{i}} \qquad \Leftarrow \text{ Key step must be seen} \\
&= \frac{a^2 + ab\mathrm{i} + ab\mathrm{i} + (b\mathrm{i})^2}{a^2 + ab\mathrm{i} - ab\mathrm{i} - (b\mathrm{i})^2} \\
&= \frac{a^2 + b^2(-1) + 2ab\mathrm{i}}{a^2 - b^2(-1)} \qquad \text{as } \mathrm{i}^2 = -1 \\
&= \frac{a^2 - b^2 + 2ab\mathrm{i}}{a^2 + b^2} \\
&= \left(\frac{a^2 - b^2}{a^2 + b^2}\right) + \left(\frac{2ab}{a^2 + b^2}\right)\mathrm{i} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[3 marks]

Answer 2

(a)

$$\begin{aligned}
(2z - z^*)^* &= z^2 \\
(2(a + b\mathrm{i}) - (a + b\mathrm{i})^*)^* &= (a + b\mathrm{i})^2 \\
(2a + 2b\mathrm{i} - (a - b\mathrm{i}))^* &= a^2 + 2ab\mathrm{i} + (b\mathrm{i})^2 \\
(2a + 2b\mathrm{i} - a + b\mathrm{i})^* &= a^2 + b^2(-1) + 2ab\mathrm{i} \\
(a + 3b\mathrm{i})^* &= a^2 - b^2 + 2ab\mathrm{i} \\
a - 3b\mathrm{i} &= a^2 - b^2 + 2ab\mathrm{i}
\end{aligned}$$

$$\text{Thus. } a = a^2 - b^2 \text{ and } -3b = 2ab$$

$$a^2 - a = b^2 \text{ and } 2ab + 3b = 0$$

$$a(a - 1) = b^2 \text{ and } b(2a + 3) = 0$$

$$\text{Either } b = 0 \text{ or } a = -\frac{3}{2}$$

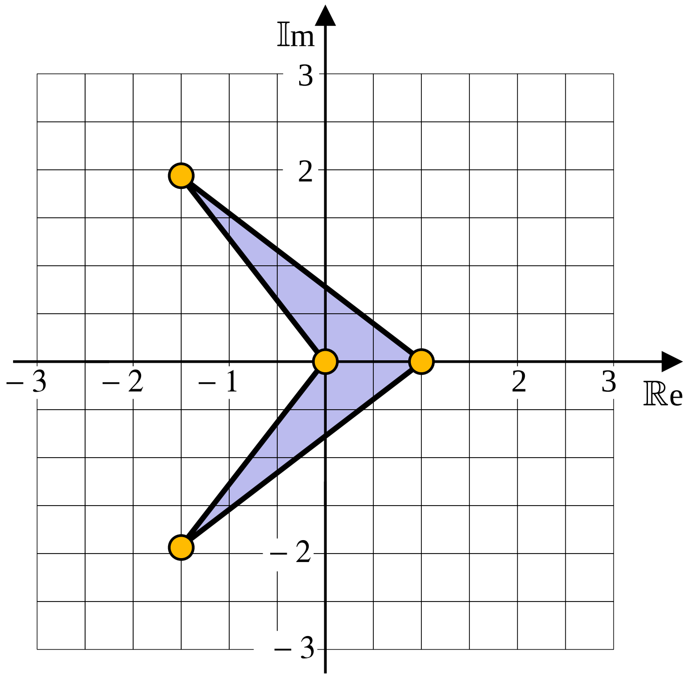
If $b = 0$ then $a(a - 1) = 0$ giving $a = 0$ or $a = 1$

$$\text{If } a = -\frac{3}{2} \text{ then } b^2 = \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$$

$$= \frac{15}{4} \text{ and so } b = \pm\frac{\sqrt{15}}{2}$$

The solution set is $z \in \left\{ 0 + 0\text{i},\ 1 + 0\text{i},\ -\dfrac{3}{2} \pm \dfrac{\sqrt{15}}{2} \right\}$

(b) **(i)** Plotting the solution set on the Argand diagram gives;



[2 marks]

(ii) The area of the quadrilateral shaded is,

$$A = 2 \times \ Area\ \Delta$$

$$= 2 \times \frac{1}{2} \times 1 \times \frac{\sqrt{15}}{2}$$

$$= \frac{\sqrt{15}}{2}$$

[1 mark]

Answer 3

(a) By definition,

$$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos 2x + 2}\,dx$$

$$= \frac{2}{\pi} \times \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2\sin x \cos x}{\cos 2x + 2}\,dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin 2x\,(\cos 2x + 2)^{-1}\,dx$$

$$= -\frac{1}{2\pi}\int_0^{\frac{\pi}{2}} (-2\sin 2x)(\cos 2x + 2)^{-1}\,dx$$

$$= -\frac{1}{2\pi}\Big[\ln|\cos 2x + 2|\Big]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2\pi}\Big[\ln|\cos \pi + 2| - \ln|\cos 0 + 2|\Big]$$

$$= -\frac{1}{2\pi}\Big[\ln(-1+2) - \ln(1+2)\Big]$$

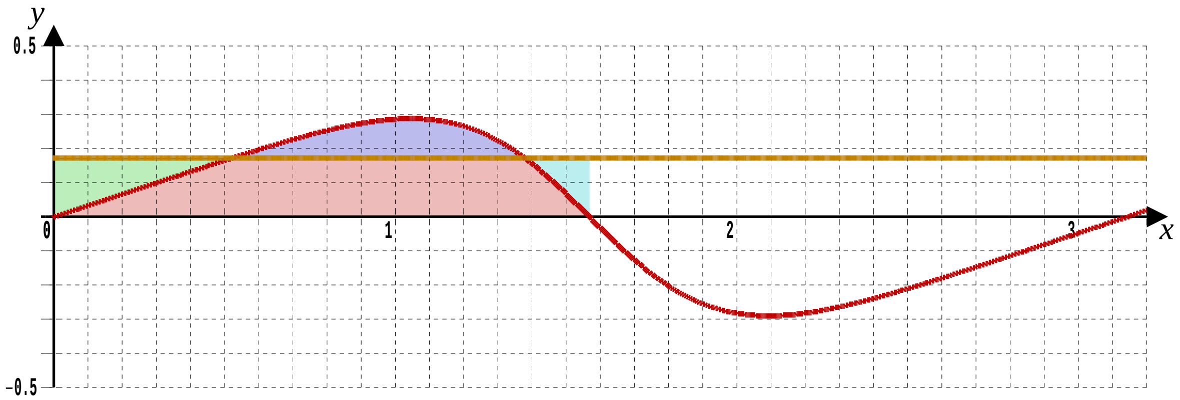
$$= -\frac{1}{2\pi}\Big[\ln 1 - \ln 3\Big]$$

$$= \frac{\ln 3}{2\pi}$$

(About 0.175 which is the gold line on the graph)

[4 marks]

(b)



The areas shaded green and blue below the horizontal line must be equal to the area shaded purple above the horizontal line.

The rectangle shaded green, pink and blue has the same area as the area between the curve and the x-axis over the given interval.

[2 marks]

Answer 4

(a) **(i)**

$$|z - 1 + 2\mathrm{i}| = 3$$
$$|x + y\mathrm{i} - 1 + 2\mathrm{i}| = 3$$
$$|(x - 1) + (y + 2)\mathrm{i}| = 3$$
$$(x - 1)^2 + (y + 2)^2 = 9$$

which is a circle centre $(1, -2)$ and radius 3

[2 marks]

(ii)

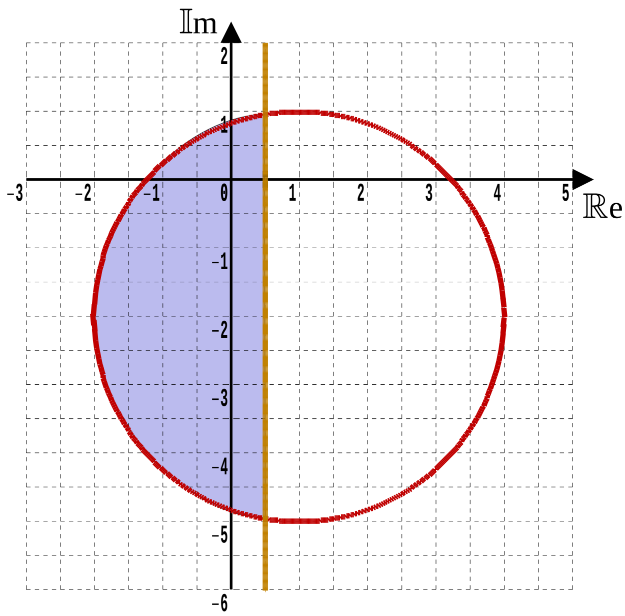
$$|z + 1| = |z - 2|$$
$$|x + y\mathrm{i} + 1| = |x + y\mathrm{i} - 2|$$
$$|x + 1 + y\mathrm{i}| = |x - 2 + y\mathrm{i}|$$
$$(x + 1)^2 + y^2 = (x - 2)^2 + y^2$$

$x + 1 = x - 2$ or $x + 1 = -x + 2$

(No solutions) $2x = 1$

$$x = \frac{1}{2}$$

[2 marks]



(b) Shown shaded is the region of the Argand diagram for which

$$|z - 1 + 2\mathrm{i}| \leqslant 3 \text{ and } |z + 1| \leqslant |z - 2|$$

[2 marks]

Answer 5

(i) $$x = \frac{a}{\sinh\theta} = a\,(\sinh\theta)^{-1} \Rightarrow \frac{dx}{d\theta} = -a\,(\sinh x)^{-2}\cosh x$$

$$= -\frac{a\cosh\theta}{\sinh^2\theta}$$

$$\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = \int \frac{\sinh\theta}{a\sqrt{\left(\frac{a}{\sinh x}\right)^2 + a^2}} \times (-1)\frac{a\cosh\theta}{\sinh^2\theta}\,d\theta$$

$$= (-1)\int \frac{a\sinh\theta\cosh\theta}{a^2\sqrt{\frac{1+\sinh^2\theta}{\sinh^2\theta}}} \times \frac{1}{\sinh^2\theta}\,d\theta$$

$$= (-1)\int \frac{\cosh\theta}{a\sqrt{\frac{\cosh^2\theta}{\sinh^2\theta}}} \times \frac{1}{\sinh\theta}\,d\theta$$

$$= -\frac{1}{a}\int 1\,d\theta$$

$$= -\frac{1}{a}\theta + constant$$

$$= -\frac{1}{a}\,arsinh\left(\frac{a}{x}\right) + constant$$

[6 marks]

(ii) $$\int_1^2 \frac{1}{x\sqrt{x^2+4}}\,dx = \int_1^2 \frac{1}{x\sqrt{x^2+2^2}}\,dx$$

$$= \left[-\frac{1}{2}\,arsinh\left(\frac{2}{x}\right)\right]_1^2$$

$$= -\frac{1}{2}\left[\ln\left\{\frac{2}{x} + \sqrt{\left(\frac{2}{x}\right)^2 + 1}\right\}\right]_1^2$$

$$= -\frac{1}{2}\left[\ln\left\{1+\sqrt{2}\right\} - \ln\left\{2+\sqrt{5}\right\}\right]$$

$$= -\frac{1}{2}\ln\left(\frac{1+\sqrt{2}}{2+\sqrt{5}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{1+\sqrt{2}}\right)$$ or equivalent

[4 marks]

Answers 6

If $x = r\cos\theta$ and $y = r\sin\theta$ then $r^2 = x^2 + y^2$

$$\left(x^2 + y^2 - 2x\right)^2 = 4\left(x^2 + y^2\right)$$

$$\left(r^2 - 2r\cos\theta\right)^2 = 4r^2$$

$$r^4 - 4r^3\cos\theta + 4r^2\cos^2\theta = 4r^2$$

As $r \neq 0$, divide through by r^2

$$r^2 - 4r\cos\theta + 4\cos^2\theta - 4 = 0$$

Viewing this as a quadratic in r...

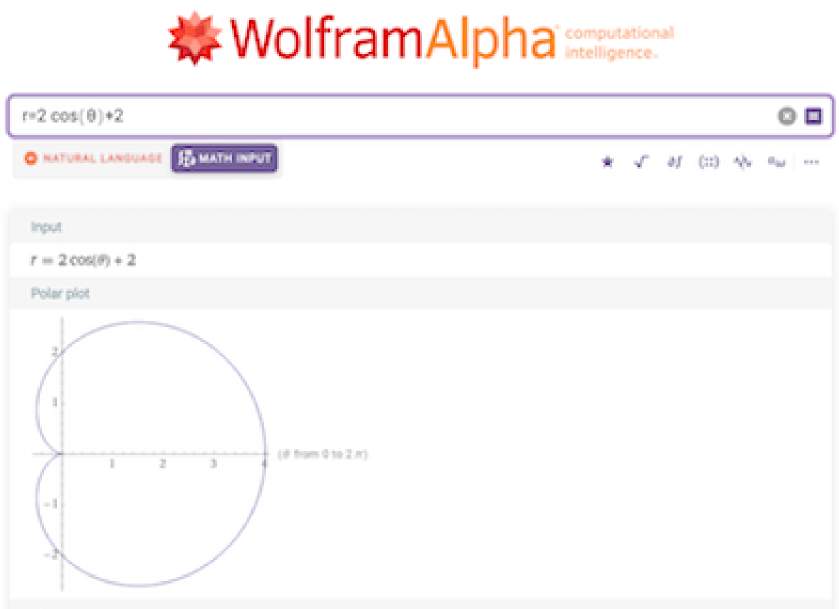
$$r = \frac{4\cos\theta \pm \sqrt{16\cos^2\theta - 4 \times 1 \times \left(4\cos^2\theta - 4\right)}}{2 \times 1}$$

$$= \frac{4\cos\theta \pm \sqrt{16\cos^2\theta - 16\cos^2\theta + 16}}{2}$$

$$= \frac{4\cos\theta \pm \sqrt{16}}{2}$$

$$= 2\cos\theta \pm 2$$

Both $r = 2\cos\theta + 2$ and $r = 2\cos\theta - 2$ give the same curve.



[5 marks]

Year FM Further Pure Mathematics Examination Revision : Health Check N° 9

“Doctor, doctor, Will this ointment clear my spots ?”
“I never make rash promises”

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

FM A-Level Examination Question from October 2021, Paper Core Pure, Q2 (MEI)

Find the gradient of the curve $y = 6\,arcsin\,(2x)$ at the point with x-coordinate $\frac{1}{4}$

Express the result in the form $m\sqrt{n}$, where m and n are integers.

[4 marks]

Question 2

FM A-Level Examination Question from October 2021, Paper Core Pure, Q14 (MEI)

A curve has polar equation $r = a(\cos\theta + 2\sin\theta)$, where a is a positive constant and $0 \leqslant \theta \leqslant \pi$

(a) Determine the polar coordinates of the point on the curve which is furthest from the pole.

[7 marks]

(b) **(i)** Show that the curve is a circle whose radius should be specified.

[6 marks]

(ii) Write down the polar coordinates of the centre of the circle.

[1 mark]

Question 3

(i) Explain what it means for an integral to be improper.

[1 mark]

(ii) Identify two features of $\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}\, dx$ which make it improper.

[1 mark]

(iii) By differentiating $arctan\sqrt{x}$, or otherwise, show that

$\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}\, dx$ is convergent and find its exact value.

[5 marks]

Question 4

FM A-Level Question from October 2020, Paper Core Pure 1, Q9 (OCR)

You are given that the cubic equation $2x^3 + px^2 + qx - 3 = 0$, where p and q are real numbers, has a complex root $\alpha = 1 + i\sqrt{2}$

(a) Write down a second complex root, β

[1 mark]

(b) Determine the third root, γ

[2 marks]

(c) Find the value of p and the value of q

[2 marks]

(d) Show that if n is an integer then

$$\alpha^n + \beta^n + \gamma^n = 2 \times 3^{\frac{n}{2}} \times \cos n\theta + \frac{1}{2^n} \quad \text{where } \tan\theta = \sqrt{2}$$

[4 marks]

Question 5

FM AS-Level Examination Question from October 2020, Paper Pure Core, Q1 (OCR)

In this question you must show detailed reasoning.

Use an algebraic method to find the square roots of $(-77 - 36\mathrm{i})$

[6 marks]

Year FM

Further Pure Mathematics Examination Revision

Health Check N° 9

Answer 1

$$y = 6\,arcsin\,(2x)$$

$$\frac{dy}{dx} = \frac{12}{\sqrt{1 - (2x)^2}}$$

$$\text{When } x = \frac{1}{4},\ \frac{dy}{dx} = \frac{12}{\sqrt{1 - \left(\frac{1}{2}\right)^2}}$$

$$= \frac{12 \times 2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= 8\sqrt{3}$$

That is, $m = 8$ and $n = 3$

[4 marks]

Answer 2

METHOD 1

Use the fact that $R\,sin\,(\theta + \alpha) = R\,sin\,\theta\,cos\,\alpha + R\,cos\,\theta\,sin\,\alpha$

$$sin\,(\theta + \alpha) = sin\,\theta \times 2 + cos\,\theta \times 1$$

$$tan\,\alpha = \frac{R\,sin\,\alpha}{R\,cos\,\alpha}$$

$$= \frac{1}{2} \qquad\qquad R = \sqrt{2^2 + 1^2}$$

$$= \sqrt{5}$$

The maximum value of r is thus $\sqrt{5}\,a$

which occurs when $\theta + arctan\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$$\therefore \theta = \frac{\pi}{2} - arctan\left(\frac{1}{2}\right)$$

Requested polar coordinates are $(\,2.236a,\ 1.107^c\,)$

[7 marks]

METHOD 2

At a place on the curve where the distance from the origin stops increasing ahead of starting to decrease $\frac{dr}{d\theta} = 0$

$$r = a(\cos\theta + 2\sin\theta)$$

$$\frac{dr}{d\theta} = 0$$

$$a(-\sin\theta + 2\cos\theta) = 0$$

$$-\sin\theta + 2\cos\theta = 0 \quad \text{(As told } a \text{ is a positive constant)}$$

$$\sin\theta = 2\cos\theta \quad *$$

$$\frac{\sin\theta}{\cos\theta} = 2$$

$$\tan\theta = 2 \Rightarrow \theta = 1.107^c$$

Insert * into $\cos^2\theta + \sin^2\theta = 1$ to get...

$$\cos^2\theta + (2\cos\theta)^2 = 1$$

$$\cos^2\theta + 4\cos^2\theta = 1$$

$$5\cos^2\theta = 1$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

$$= \frac{\sqrt{5}}{5} \quad \text{and so, from *,} \quad \sin\theta = \frac{2\sqrt{5}}{5}$$

$$r = a(\cos\theta + 2\sin\theta)$$

$$= a\left(\frac{\sqrt{5}}{5} + 2 \times \frac{2\sqrt{5}}{5}\right)$$

$$= \sqrt{5}\,a$$

Requested polar coordinates are $(2.236a,\ 1.107^c)$

[7 marks]

(b)

$$r = a\left(\cos\theta + 2\sin\theta \right)$$

multiply through by r

$$r^2 = a\,(r\cos\theta) + 2a\,(r\sin\theta)$$

$$x^2 + y^2 = ax + 2ay$$

$$x^2 - ax + y^2 - 2ay = 0$$

$$\left(x - \frac{a}{2} \right)^2 - \frac{a^2}{4} + (y - a)^2 - a^2 = 0$$

$$\left(x - \frac{a}{2} \right)^2 + (y - a)^2 = \frac{5a^2}{4}$$

which is recognizable as a circle with radius $\dfrac{\sqrt{5}\,a}{2}$

[6 marks]

(c) The question is set up so that you realise the part (b) radius is half of the part (a) answer, which must have been the diameter of the circle. So the direction to the circle's centre is the same as that found in (a).

Thus the polar coordinates of the circle's centre are,

$$\left(\frac{\sqrt{5}\,a}{2},\ 1.107^c \right)$$

[1 mark]

Answer 3

(i) An integral $\int_a^b f(x)\ dx$ is described as being improper if either one or both of the limits are infinite or if $f(x)$ is undefined at some point in the interval $a \leqslant x \leqslant b$

[1 mark]

(ii) Due to a division by zero, $f(x)$ is not defined when $x = 0$

The integral has an upper limit of ∞

[1 mark]

(iii) Using the hint given...

$$y = arctan\sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{1 + (\sqrt{x})^2} \times \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}\ (1 + x)}$$

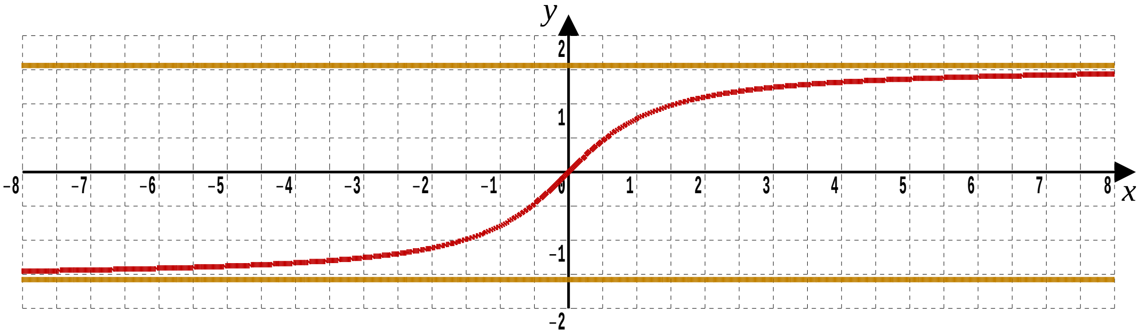
$$\int_0^\infty \frac{1}{(x + 1)\sqrt{x}}\ dx = 2 \underset{\substack{v \to \infty \\ u \to 0^+}}{limit} \int_u^v \frac{1}{2(x + 1)\sqrt{x}}\ dx$$

$$= 2 \underset{\substack{v \to \infty \\ u \to 0^+}}{limit} \left[arctan\sqrt{x} \right]_u^v$$

$$= 2 \underset{v \to \infty}{limit} \left[arctan\sqrt{v} \right] - 2 \underset{u \to 0^+}{limit} \left[arctan\sqrt{u} \right]$$

$$= 2 \times \frac{\pi}{2} - 2 \times 0$$

$$= \pi \qquad \text{(Convergent)}$$



Red: $y = arctan(x)$ Gold: $y = \pm \frac{\pi}{2}$

[5 marks]

Answer 4

$$2x^3 + px^2 + qx - 3 = 0$$

$$x^3 + \frac{p}{2}x^2 + \frac{q}{2}x - \frac{3}{2} = 0$$

(i) $\beta = 1 - i\sqrt{2}$

[1 mark]

(ii) "Product of the roots"

$$\alpha\beta\gamma = \frac{3}{2}$$

$$\left(1 + i\sqrt{2}\right)\left(1 - i\sqrt{2}\right)\gamma = \frac{3}{2}$$

$$\left(1 + 2\right)\gamma = \frac{3}{2}$$

$$\gamma = \frac{1}{2}$$

[2 marks]

(iii) $\left(x - 1 - i\sqrt{2}\right)\left(x - 1 + i\sqrt{2}\right)\left(x - \frac{1}{2}\right) = 0$

$$\left((x - 1)^2 + 2\right)\left(x - \frac{1}{2}\right) = 0$$

multiply both sides by 2

$$\left(x^2 - 2x + 3\right)\left(2x - 1\right) = 0$$

$$2x^3 - 4x^2 + 6x - x^2 + 2x - 3 = 0$$

$$2x^3 - 5x^2 + 8x - 3 = 0$$

Thus, $p = -5$ and $q = 8$

[2 marks]

(iv) $\alpha = 1 + \mathrm{i}\sqrt{2} \Rightarrow |\alpha| = \sqrt{3}$

$$\alpha = \sqrt{3}\left(\frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\,\mathrm{i} \right)$$

$$= 3^{\frac{1}{2}}\,(\cos\theta + \mathrm{i}\sin\theta) \quad \text{where } \cos\theta = \frac{1}{\sqrt{3}} \text{ and } \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \alpha^n = 3^{\frac{n}{2}}\,(\cos\theta + \mathrm{i}\sin\theta)^n \qquad \text{Note } \cos^2\theta + \sin^2\theta = 1 \text{ as it should}$$

$$= 3^{\frac{n}{2}}\,(\cos n\theta + \mathrm{i}\sin n\theta) \quad \text{by De Moivre's Theorem}$$

Similarly,

$\beta = 1 - \mathrm{i}\sqrt{2} \Rightarrow |\beta| = \sqrt{3}$

$$\beta = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}\,\mathrm{i} \right)$$

$$= 3^{\frac{1}{2}}\,(\cos\theta - \mathrm{i}\sin\theta) \text{ where } \cos\theta = \frac{1}{\sqrt{3}} \text{ and } \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \beta^n = 3^{\frac{n}{2}}\,(\cos\theta - \mathrm{i}\sin\theta)^n \qquad \text{Note } \cos^2\theta + \sin^2\theta = 1 \text{ as it should}$$

$$= 3^{\frac{n}{2}}\,(\cos n\theta - \mathrm{i}\sin n\theta) \quad \text{by De Moivre's Theorem}$$

$$\text{LHS} = \alpha^n + \beta^n + \gamma^n$$

$$= 3^{\frac{n}{2}}\,(\cos n\theta + \mathrm{i}\sin n\theta) + 3^{\frac{n}{2}}\,(\cos n\theta - \mathrm{i}\sin n\theta) + \left(\frac{1}{2}\right)^n$$

$$= 3^{\frac{n}{2}} \cos n\theta + 3^{\frac{n}{2}}\sin n\theta + 3^{\frac{n}{2}} \cos n\theta - 3^{\frac{n}{2}} \sin n\theta + \frac{1}{2^n}$$

$$= 2 \times 3^{\frac{n}{2}} \times \cos n\theta + \frac{1}{2^n}$$

$$= \text{RHS} \qquad \square$$

$$\text{Note that } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \div \frac{1}{\sqrt{3}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{1} \qquad \text{KOF}$$

$$= \sqrt{2} \qquad \text{as required}$$

[4 marks]

Answer 5

Let a root of $(-77 - 36\text{i})$ be $a + b\text{i}$ with $a, b \in \mathbb{R}$, in which case,

$$(a + b\text{i})^2 = -77 - 36\text{i}$$

$$a^2 + 2ab\text{i} + b^2\text{i}^2 = -77 - 36\text{i}$$

$$\therefore a^2 - b^2 = -77 \text{ and } 2ab = -36$$

$$b = -\frac{18}{a}$$

$$a^2 - \left(-\frac{18}{a}\right)^2 = -77$$

$$a^2 - \frac{324}{a^2} = -77$$

multiply through by a^2

$$a^4 + 77a^2 - 324 = 0$$

$$\left(a^2 - 4\right)\left(a^2 + 81\right) = 0$$

$$\text{Either } a^2 = 4 \text{ or } a^2 = -81$$

$$\therefore a = \pm 2 \text{ as } a \in \mathbb{R}$$

$$\therefore b = \mp 9$$

$$\text{Thus } \sqrt{-77 - 36\text{i}} = \pm(2 - 9\text{i})$$

[6 marks]

Year FM Further Pure Mathematics Examination Revision : Health Check N° 10

“Doctor, doctor, I keep seeing into the future”
“When did this start ?”
“Next Tuesday”

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

The line L has equation $r = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$

Identify which **one** of the following lines is perpendicular to L and prove your claim.

$r_1 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}$ $\quad$ $r_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ $\quad$ $r_3 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \mu_3 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

[3 marks]

Question 2

Complex Numbers Division Rule

For any two complex numbers z_1 and z_2

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$

This result may be quoted in examinations; worth knowing !

(i) Prove this result by by letting

- $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$
- $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

and working out $\frac{z_1}{z_2}$

[5 marks]

(ii) If $z_1 = 1 - \sqrt{3}\, i$ and $z_2 = \sqrt{3} + i$ determine,

(a) $arg(z_1)$

[1 mark]

(b) $arg(z_2)$

[1 mark]

(c) $arg\left(\frac{z_1}{z_2}\right)$

[1 mark]

Question 3

(i) Use the facts that $cosh\, x = \dfrac{e^{x} + e^{-x}}{2}$ and $sinh\, x = \dfrac{e^{x} - e^{-x}}{2}$

to prove that $tanh\, x = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

[3 marks]

(ii) As x becomes large and positive, what value does $tanh\, x$ tend towards ?
Justify your answer.

[2 marks]

(iii) As x becomes large and negative, what value does $tanh\, x$ tend towards ?
Justify your answer.

[2 marks]

(iv) Hence, sketch the graph of $y = tanh\, x$

[2 marks]

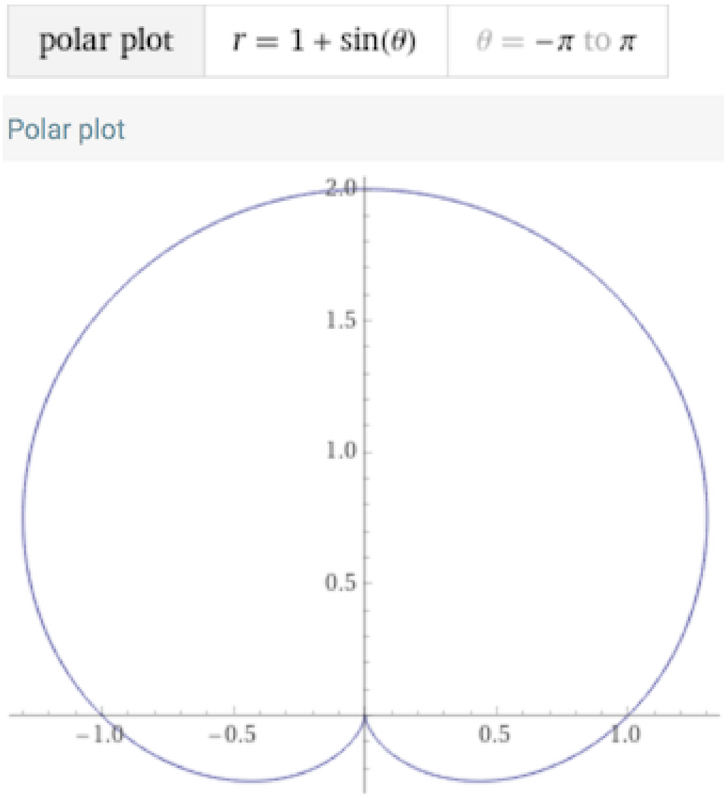
(v) Given that $3\, sinh\, x = p\, cosh\, x$ has real solutions, determine the range of possible values for p

[2 marks]

Question 4

The graph is a Wolfram Alpha™ plot of the polar equation

$$r = 1 + sin\,\theta \text{ over the interval } -\pi \leqslant \theta \leqslant \pi$$



Use differentiation as part of a proof that shows that there could be a vertical tangent to the curve that touches the curve at the point with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{6}\right)$

[6 marks]

Question 5

(i) Given that $y = arccos\, x$ over the interval $-1 \leqslant x \leqslant 1$, show that,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

If your solution involves the taking of square roots, explain carefully why you select the negative option and discard the positive, if appropriate.

[4 marks]

(ii) $f(x) = arccos\left(e^x\right) \qquad x \leqslant 0$

(a) Explain why domain restriction $x \leqslant 0$ is necessary.

[2 marks]

(b) Prove that $f(x)$ has no stationary points.

[2 marks]

Question 6

(i) Express as partial fractions; $\dfrac{4x^2 + 3x + 14}{(x + 2)(x^2 + 4)}$

[3 marks]

(ii) Hence find the exact value of $\displaystyle\int_0^1 \frac{4x^2 + 3x + 14}{(x + 2)(x^2 + 4)}\,dx$

Simplify your answer.

[5 marks]

Question 7

After you have done this question you may like to compare the method this old examination question guides you towards with that used in question 2.

$$z_1 = 4 + 2\mathrm{i} \qquad z_2 = -3 + \mathrm{i}$$

(i) Find the exact value of $| z_1 - z_2 |$

[2 marks]

Given that $w = \dfrac{z_1}{z_2}$

(ii) express w in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$

[2 marks]

(iii) find *arg w*, giving your answer in radians.

[2 marks]

Year FM

Further Pure Mathematics Examination Revision
Health Check N° 10

Answer 1

$r \ = \ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$ is perpendicular to $r_3 \ = \ \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \mu_3 \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ and this is proven by showing that the scalar product of the two line directions is zero.

So the claim is that, $\begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \ = \ 0$

$$\begin{aligned} \text{LHS} \ &= \ \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \\ &= \ (-1)(4) \ + \ (-2)(3) \ + \ (5)(2) \\ &= \ -4 \ - \ 6 \ + \ 10 \\ &= \ 0 \\ &= \ \text{RHS} \qquad \qquad \square \end{aligned}$$

[3 marks]

Answer 2

(i)

From $z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ we get that $|z_1| = r_1$ and $arg(z_1) = \theta_1$

From $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$ we get that $|z_2| = r_2$ and $arg(z_2) = \theta_2$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{r_2(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1 r_2}{r_2 r_2} \times \frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i\cos\theta_2\sin\theta_2 + i\cos\theta_2\sin\theta_2 - i^2\sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}$$

$$= \frac{r_1}{r_2} \times (\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

Now, referring back to the first two lines of our proof we can write,

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} \quad \text{and} \quad arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

$$= \frac{|z_1|}{|z_2|} \quad \text{and} \quad = arg(z_1) - arg(z_2)$$

This answer tidies up the ragged ending to the similar proof in Core Book 1, page 25

[5 marks]

(ii) **(a)**

$$arg(z_1) = arg(1 - \sqrt{3}\, i) = arctan\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

(b)

$$arg(z_2) = arg(\sqrt{3} + i) = arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

(c)

$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) = -\frac{\pi}{3} - \frac{\pi}{6} = -\frac{\pi}{2}$$

[1, 1, 1 marks]

Answer 3

(i) LHS $= \tanh x$

$$= \frac{\sinh x}{\cosh x}$$

$$= \sinh x \div \cosh x$$

$$= \frac{e^{x} - e^{-x}}{2} \times \frac{2}{e^{x} + e^{-x}}$$

$$= \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \times \frac{e^{x}}{e^{x}}$$

$$= \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$= \text{RHS} \qquad \square$$

[3 marks]

(ii) As $x \to \infty$, $e^{2x} - 1 \to e^{2x}$ and $e^{2x} + 1 \to e^{2x}$

and so $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1} \to \dfrac{e^{2x}}{e^{2x}} = 1$

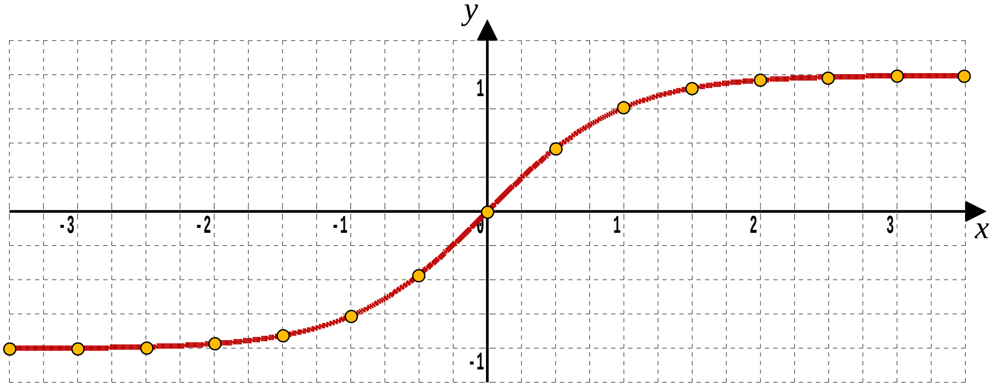
[2 marks]

(iii) As $x \to -\infty$, $e^{2x} - 1 \to -1$ and $e^{2x} + 1 \to 1$

and so $\tanh x = \dfrac{e^{2x} - 1}{e^{2x} + 1} \to \dfrac{-1}{1} = -1$

[2 marks]

(iv)



[2 marks]

(v) From $3 \sinh x = p \cosh x$ we get that, $\tanh x = \dfrac{p}{3}$

From the graph, $-1 < \tanh x < 1$

$$-1 < \frac{p}{3} < 1$$

$$-3 < p < 3$$

[2 marks]

Answer 4

$$\begin{aligned}
\text{Let } x &= r\cos\theta \\
&= (1 + \sin\theta)\cos\theta \\
&= \cos\theta + \sin\theta\cos\theta \\
&= \frac{1}{2}\sin(2\theta) + \cos\theta \\
\frac{dx}{d\theta} &= \cos(2\theta) - \sin\theta
\end{aligned}$$

$$\begin{aligned}
\text{For a vertical tangent, } \frac{dx}{d\theta} &= 0 \\
\cos(2\theta) - \sin\theta &= 0 \\
\cos^2\theta - \sin^2\theta - \sin\theta &= 0 \\
1 - 2\sin^2\theta - \sin\theta &= 0 \\
2\sin^2\theta + \sin\theta - 1 &= 0 \\
(2\sin\theta - 1)(\sin\theta + 1) &= 0
\end{aligned}$$

Either $\sin\theta = -1$ in which case $\theta = -\frac{\pi}{2}$ (in the specified interval)

Or $\sin\theta = \frac{1}{2}$ in which case $\theta = \frac{\pi}{6}$ (which is 30°)

When $\theta = \frac{\pi}{6}$, $r = 1 + \sin\left(\frac{\pi}{6}\right)$ giving $r = \frac{3}{2}$

Thus there could indeed be a vertical tangent to the curve that touches it at the point with polar coordinates $\left(\frac{3}{2}, \frac{\pi}{6}\right)$ □

[6 marks]

Answer 5

(i) $y = arccos\,x \;\Rightarrow\; x = cos\,y$

$$\frac{dx}{dy} = -\,sin\,y$$

Now use the trigonometry identity $cos^2 y + sin^2 y = 1$ to get...

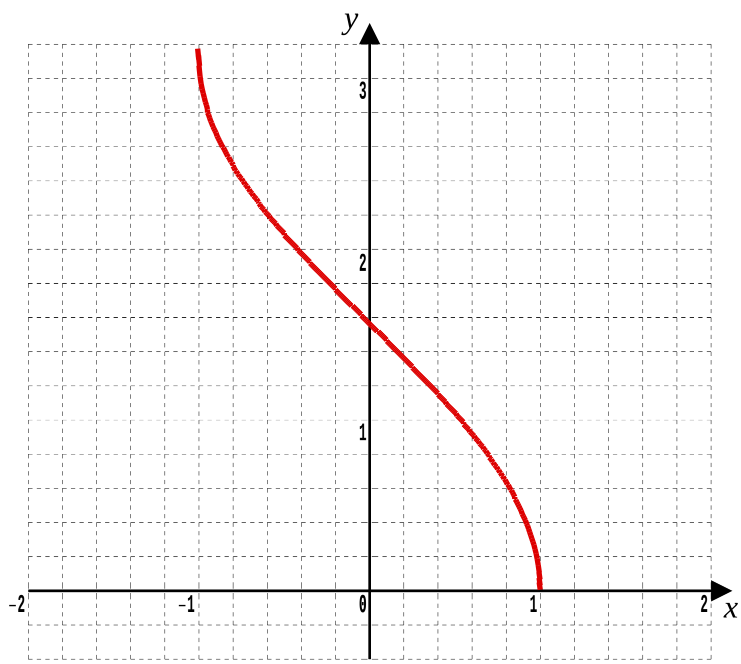
$$\frac{dx}{dy} = \pm\sqrt{1 - cos^2 y}$$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1$$

(Need to remove ± 1 to avoid a division by zero)

From the graph of $y = arccos\,x$ it can be seen that the gradient is always negative

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad -1 < x < 1 \qquad \Box$$



The graph of $y = arccos\,x$

It is the inverse of a one-to-one piece of the $cos\,x$ function

[4 marks]

(ii) **(a)** The function $g(x) = arccos\,x$ is only defined on domain $-1 \leqslant x \leqslant 1$ so the related function $f(x) = arccos\left(e^x\right)$ requires that e^x be constrained such that $-1 \leqslant e^x \leqslant 1$. As e^x is always positive (and an increasing function) there is no issue at the left end of this interval. To prevent e^x exceeding the maximum value of +1 requires the constraint $x \leqslant 0$.

[2 marks]

(b)

$$f(x) = arccos\left(e^x\right) \qquad x \leqslant 0$$

$$f'(x) = -\frac{1}{\sqrt{1 - (e^x)^2}} \times e^x \text{ by the chain rule, } x < 0$$

At a stationary point $f'(x) = 0$.

However, we have $x < 0$ or, more usefully, $0 < e^x < 1$

from which it is clear $f'(x)$ cannot equal zero.

$\therefore$ There can be no stationary points. $\square$

[2 marks]

Answer 6

(i)

$$\frac{4x^2 + 3x + 14}{\left[\,(x+2)(x^2+4)\,\right]} = \frac{A}{(x+2)} + \frac{Bx + C}{(x^2+4)}$$

Multiply through by the red bracket...

$$4x^2 + 3x + 14 = A\left(x^2 + 4\right) + \left(Bx + C\right)\left(x + 2\right)$$

Let $x = -2$ in which case, $24 = 8A \Rightarrow A = 3$

Let $x = 0$ in which case, $14 = 12 + 2C \Rightarrow C = 1$

Let $x = 1$ in which case, $21 = 15 + (B + 1)(3)$

$$6 = 3(B + 1) \Rightarrow B = 1$$

Conclusion:

$$\frac{4x^2 + 3x + 14}{(x+2)(x^2+4)} = \frac{3}{(x+2)} + \frac{x + 1}{(x^2+4)}$$

[3 marks]

(ii) $$\int_0^1 \frac{4x^2 + 3x + 14}{(x+2)(x^2+4)}\,dx$$

$$= 3\int_0^1 \frac{1}{x+2}\,dx + \frac{1}{2}\int_0^1 \frac{2x}{(x^2+4)}\,dx + \int_0^1 \frac{1}{(x^2+2^2)}\,dx$$

$$= \left[\,3\,ln\left|x + 2\right| + \frac{1}{2}\,ln\left|x^2 + 4\right| + \frac{1}{2}\,arctan\left(\frac{x}{2}\right)\right]_0^1$$

$$= 3\,ln\,3 + \frac{1}{2}\,ln\,5 + \frac{1}{2}arctan\left(\frac{1}{2}\right) - 3\,ln\,2 - \frac{1}{2}ln\,4$$

$$= ln\,27 + ln\sqrt{5} - ln\,8 - ln\,2 + \frac{1}{2}arctan\left(\frac{1}{2}\right)$$

$$= ln\left(\frac{27\sqrt{5}}{16}\right) + \frac{1}{2}arctan\left(\frac{1}{2}\right)$$

[5 marks]

Answer 7

(i)

$$
\begin{aligned}
|z_1 - z_2| &= |(4 + 2\mathrm{i}) - (-3 + \mathrm{i})| \\
&= |4 + 2\mathrm{i} + 3 - \mathrm{i}| \\
&= |7 + \mathrm{i}| \\
&= \sqrt{7^2 + 1^2} \\
&= \sqrt{50} \\
&= 5\sqrt{2}
\end{aligned}
$$

[2 marks]

(ii)

$$
\begin{aligned}
w &= \frac{z_1}{z_2} \\
&= \frac{4 + 2\mathrm{i}}{-3 + \mathrm{i}} \times \frac{-3 - \mathrm{i}}{-3 - \mathrm{i}} \\
&= \frac{-12 - 4\mathrm{i} - 6\mathrm{i} - 2\,\mathrm{i}^2}{9 - \mathrm{i}^2} \\
&= \frac{-10 - 10\mathrm{i}}{10} \\
&= -1 - \mathrm{i}
\end{aligned}
$$

[2 marks]

(iii)

$$
\begin{aligned}
arg\, w &= arg(-1 - \mathrm{i}) \\
&= arctan\left(\frac{-1}{-1}\right) \\
&= arctan\,(1) \\
&= -\pi + \frac{\pi}{4} \text{ or } \frac{\pi}{4} \text{ (Reject)} \\
&= -\frac{3\pi}{4}
\end{aligned}
$$

[2 marks]