1.8 Answers

1.8.1 Solutions to 1.4 Exercise

(i) $6 + 9 \equiv x \pmod{12}$

$x \equiv 15 \pmod{12}$

$x \equiv 3 \pmod{12}$

(ii) $10 + x \equiv 5 \pmod{12}$

$x \equiv 5 - 10 \pmod{12}$

$x \equiv -5 \pmod{12}$

$x \equiv 7 \pmod{12}$

(iii) $4 - 7 \equiv x \pmod{12}$

$x \equiv -3 \pmod{12}$

$x \equiv 9 \pmod{12}$

(iv) $42 - x \equiv 8 \pmod{12}$

$x \equiv 42 - 8 \pmod{12}$

$x \equiv 34 \pmod{12}$

$x \equiv 10 \pmod{12}$

[8 marks]

1.8.2 Solution to 1.6 Example (Teaching Video)

Note : This is a "beginners" method with a premium placed on "understanding"

$$3x + 14 \equiv 5 \pmod{6}$$
$$3x \equiv 5 - 14 \pmod{6}$$
$$3x \equiv -9 \pmod{6}$$
$$3x \equiv 3,\ 9,\ 15,\ 21,\ 27,\ \ldots \pmod{6}$$
$$x \equiv 1,\ 3,\ 5 \pmod{6}$$

This could be more succinctly written as $x \equiv 1 \pmod{2}$

[6 marks]

1.8.3 Solutions to 1.7 Exercise

Answer 1

(i) $29 \equiv 5 \pmod{12}$

(ii) $-7 \equiv 5 \pmod{12}$

(iii) $145 \equiv 1 \pmod{12}$

(iv) $47 + 21 \equiv 68 \equiv 8 \pmod{12}$

(iv) $3 - 19 \equiv -16 \equiv 8 \pmod{12}$

(vi) $5^3 \equiv 125 \equiv 5 \pmod{12}$

[4 marks]

Answer 2

(a) Least residues modulo 4 = { 0, 1, 2, 3 }

[1 mark]

(b) **(i)** $7^0 \equiv 1 \pmod 4$

(ii) $7^1 \equiv 7 \equiv 3 \pmod 4$

(iii) $7^2 \equiv 49 \equiv 1 \pmod 4$

(iv) $7^3 \equiv 343 \equiv 3 \pmod 4$

(v) $7^4 \equiv 2401 \equiv 1 \pmod 4$

(vi) $7^5 \equiv 16807 \equiv 3 \pmod 4$

[3 marks]

(c) From the pattern observed in part (b) it would seem that,

$$7^{EVEN} \equiv 1 \pmod 4$$

$$7^{ODD} \equiv 3 \pmod 4$$

(Proving this type of result is the focus of lesson 2)

(i) $7^{758} \equiv 1 \pmod{4}$

(ii) $7^{433} \equiv 3 \pmod 4$

(iii) $7^{501} + 7^{383} \equiv 3 + 3 \equiv 6 \equiv 2 \pmod 4$

[3 marks]

Answer 3

(i) $6 + 4 \equiv x \pmod 8$

$x \equiv 10 \pmod 8$

$x \equiv 2 \pmod{8}$

(ii) $3 + x \equiv 1 \pmod 8$

$x \equiv 1 - 3 \pmod 8$

$x \equiv -2 \pmod 8$

$x \equiv 6 \pmod 8$

(iii) $4 - 19 \equiv x \pmod 8$

$x \equiv -15 \pmod 8$

$x \equiv 1 \pmod{8}$

(iv) $37 - x \equiv 8 \pmod 8$

$x \equiv 37 - 8 \pmod 8$

$x \equiv 29 \pmod 8$

$x \equiv 5 \pmod 8$

[8 marks]

Answer 4

$$\begin{aligned} 2x + 11 &\equiv 1 && \pmod 6 \\ 2x &\equiv 1 - 11 && \pmod 6 \\ 2x &\equiv -10 && \pmod 6 \\ 2x &\equiv 2,\ 8,\ 14,\ \ldots && \pmod 6 \\ x &\equiv 1,\ 4,\ 7,\ \ldots && \pmod 6 \\ x &\equiv 1,\ 4 && \pmod 6 \end{aligned}$$

This could be more succinctly written as $x \equiv 1 \pmod 3$

[6 marks]

Answer 5

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

[4 marks]

Answer 6

(a) **(i)** $1! \equiv 1 \pmod 7$

(ii) $2! \equiv 2 \pmod 7$

(iii) $3! \equiv 6 \pmod 7$

(iv) $4! \equiv 24 \equiv 3 \pmod 7$

(v) $5! \equiv 120 \equiv 1 \pmod 7$

(vi) $6! \equiv 720 \equiv 6 \pmod 7$

(vii) $7! \equiv 5040 \equiv 0 \pmod 7$

(viii) $8! \equiv 40320 \equiv 0 \pmod 7$

(ix) $9! \equiv 362880 \equiv 0 \pmod 7$

[3 marks]

(b) For $n \geqslant 7$, $n = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times \ldots \times n$

As it contains a 7, it will divide by 7 and so be congruent to 0 modulo 7

[2 marks]

Answer 7

(a) **(i)** $13 \equiv 4 \pmod 9$ $\quad 1 + 3 = 4$

(ii) $572 \equiv 5 \pmod 9$ $\quad 5 + 7 + 2 = 14 \Rightarrow 1 + 4 = 5$

(iii) $334 \equiv 1 \pmod 9$ $\quad 3 + 3 + 4 = 10 \Rightarrow 1 + 0 = 1$

(iv) $3102 \equiv 6 \pmod 9$ $\quad 3 + 1 + 0 + 2 = 6$

(v) $1002001 \equiv 4 \pmod 9$ $\quad 1 + 0 + 0 + 2 + 0 + 0 + 1 = 4$

(vi) $9994999 \equiv 4 \pmod 9$ $\quad 9 + 9 + 9 + 4 + 9 + 9 + 9 = 58$

$$\Rightarrow 5 + 8 = 13$$

$$\Rightarrow 1 + 3 = 4$$

[3 marks]

(b) The answers can be obtained by summing the digits modulo 9, as shown above for the part (a) questions.

$$12345678987654321$$

$$\Rightarrow 2(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) + 9$$

$$= 2(\ (1 + 8) + (2 + 7) + (3 + 6) + (4 + 5)) + 9$$

$$= 2 \times 4 \times 9 + 9$$

$$\equiv 0 \pmod 9$$

[2 marks]

Answer 8

When $n \geqslant 5$, $n! \equiv 0 \pmod{15}$ because $n!$ then contains factors of both 3 and 5 and so contains a factor of 15.

$$1! + 2! + 3! + 4! + ... + 100! \pmod{15}$$

$$\equiv 1! + 2! + 3! + 4! + 0 + 0 + 0 + ... + 0 \pmod{15}$$

$$\equiv 1 + 2 + 6 + 24 \pmod{15}$$

$$\equiv 3 \pmod{15}$$

So the remainder is 3

[5 marks]

Answer 9

(i) For 51177875501

$$5 + 1 + 1 + 7 + 7 + 8 + 7 + 5 + 5 + 0 = 46$$

$$\Rightarrow 4 + 6 = 10$$

$$\Rightarrow 1 + 0 = 1$$

As 1 is the check digit, this serial number passes the security check.

(ii) For 88100245327

$$8 + 8 + 1 + 0 + 0 + 2 + 4 + 5 + 3 + 2 + 7 = 40$$

$$\Rightarrow 4 + 0 = 4$$

As 4 is not the check digit, this serial number does not pass.

[2 marks]

Answer 10

$1^2 \equiv 1 \pmod 4$

$2^2 \equiv 4 \equiv 0 \pmod 4$

$3^2 \equiv 9 \equiv 1 \pmod 4$

$4^2 \equiv 16 \equiv 0 \pmod 4$

$5^2 \equiv 25 \equiv 1 \pmod 4$

$6^2 \equiv 36 \equiv 0 \pmod 4$

...

To prove the pattern suggested by the above, show that any odd number squared is congruent to 1 modulo 4 and that any even number squared is congruent to 0 modulo 4, which is not difficult.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \ldots + 100^2 \pmod 4$$

$$\equiv 1 + 0 + 1 + 0 + 1 + 0 + \ldots + 0 \pmod 4$$

$$\equiv 50 \pmod 4$$

$$\equiv 2 \pmod 4$$

[4 marks]