

2.5 Answers

2.5.1 Solution to 2.3 Example (Teaching Video)

Prove $G = \{ 1, 2, 3, 4 \}$ under multiplication modulo 5 is a group.

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Identity: For this group, 1 is the identity element, the element that “does nothing” under the binary operation of multiplication modulo 5.

Closure : As the table shows all possible multiplications it can be seen that no number can occur as an answer that's not an element of $G = \{ 1, 2, 3, 4 \}$.

Inverse : Each element has a unique inverse. 2 and 3 are each the other's inverse and 4 is it's own inverse as is 1.

$$2 \times 3 \equiv 3 \times 2 \equiv 1 \pmod{5} \quad 4 \times 4 \equiv 1 \pmod{5}$$

Associativity : This is a property inherited from of multiplication of integers.

2.5.2 Solutions to 2.4 Exercise

Answer 1

(i)

$\times_{18}$	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

[2 marks]

(ii) The identity element is 1

[1 mark]

(iii) The inverse of 5 is 11 because $5 \times 11 = 1 \pmod{18}$

[1 mark]

(iv) The inverse of 17 is 17 because $17 \times 17 = 1 \pmod{18}$

[1 mark]

Answer 2

$$x * y = x^2 y$$

(i) $5 * 6 = 5^2 \times 6 = 150$

(ii) $6 * 5 = 6^2 \times 5 = 180$

[2 marks]

Answer 3

(i)

$\times_{12}$	1	3	7	9
1	1	3	7	9
3	3	9	9	3
7	7	9	1	3
9	9	3	3	9

[2 marks]

(ii) Yes, the closure property is satisfied.

[1 mark]

(iii) • The Latin square property does not hold.
• Two of the elements, the 3 and the 9, do not have an inverse.

[1 mark]

Answer 4

$$a * b = ab + 1$$

(a) (i) $4 * 7 = 4 \times 7 + 1$
 $= 29$

(ii) $(-3) * 5 = (-3) \times 5 + 1$
 $= -14$

[2 marks]

(b) $2 * (3 * 4) = 2 * (3 \times 4 + 1)$
 $= 2 * 13$
 $= 2 \times 13 + 1$
 $= 27$

$(2 * 3) * 4 = (2 \times 3 + 1) * 4$
 $= 7 * 4$
 $= 7 \times 4 + 1$
 $= 29$

Thus $2 * (3 * 4) \neq (2 * 3) * 4$

$\therefore *$ is not associative $\square$

[2 marks]

Answer 5**(i)**

$\circ$	0	1	2	4	5	6
0	0	1	2	4	5	6
1	1	4	0	6	2	5
2	2	0	5	1	6	4
4	4	6	1	5	0	2
5	5	2	6	0	4	1
6	6	5	4	2	1	0

[3 marks]**(ii)** The identity element is 0**[1 mark]****(iii)** The inverse of 5 is 4**[1 mark]****(iv)** The inverse of 1 is 2**[1 mark]****(v)** Other than 0, the self-inverse element is 6**[1 mark]****Answer 6****Method 1 :**

$$\begin{aligned}
 x \circ y &= x^2y \\
 odd \circ odd &= odd^2 \times odd \\
 &= odd \times odd \\
 &= odd
 \end{aligned}$$

So, when any two elements from G are combined the result is an element from G which is what is means to be closed $\square$

Method 2 :Let the two odd numbers be $2n + 1$ and $2m + 1$

$$\begin{aligned}
 (2n + 1) \circ (2m + 1) &= (2n + 1)^2(2m + 1) \\
 &= 2(4m n^2 + 4nm + m + 2 n^2 + 2n) + 1 \\
 &= 2k + 1, \text{ another odd number}
 \end{aligned}$$

 $\therefore G$ is closed under $\circ$ **[2 marks]**

Answer 7

(a) From the Cayley table provided,

$\otimes$	1	2	3	4	5
1	1	2	3	4	5
2	2	1	5	3	4
3	3	4	2	5	1
4	4	5	1	2	3
5	5	3	4	1	2

$$\begin{aligned} \text{(i)} \quad 2 \otimes (5 \otimes 3) &= 2 \otimes 4 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad (2 \otimes 5) \otimes 3 &= 4 \otimes 3 \\ &= 1 \end{aligned}$$

[2 marks]

(b) The table does not have associativity and so is not of a group.

[1 mark]

Answer 8

$$x \circ y = xy + x$$

Let's check to see if associativity holds.

In other words, for any three elements of G , say a , b and c , is it true that,

$$a \circ (b \circ c) = (a \circ b) \circ c ?$$

$$\begin{aligned} \text{LHS} &= a \circ (b \circ c) & \text{RHS} &= (a \circ b) \circ c \\ &= a \circ (bc + b) & &= (ab + a) \circ c \\ &= a(bc + b) + a & &= (ab + a)c + (ab + a) \\ &= abc + ab + a & &= abc + ac + ab + a \end{aligned}$$

As $\text{LHS} \neq \text{RHS}$ the proposed group lacks associativity and so is not a group under the binary operation $\circ$.

[4 marks]

Answer 9

Assume there exists a column in a Cayley table in which the same element occurs twice. Given two distinct elements, p and q , in column b the situation is;

$\circ$	...	b	...
...	...	...	...
p	...	$p \circ b$	...
...	...	...	...
q	...	$q \circ b$	...
...	...	...	...

Under the assumption that the same element occurs twice,

$$p \circ b = q \circ b$$

$$(p \circ b) \circ b^{-1} = (q \circ b) \circ b^{-1} \quad \text{Every element in a group has an inverse}$$

$$p \circ (b \circ b^{-1}) = q \circ (b \circ b^{-1}) \quad \text{A group has associativity}$$

$$p \circ e = q \circ e \quad e \text{ is the identity element}$$

$$p = q$$

But this contradicts the starting assumption that p and q are distinct elements.

$\therefore$ the same element cannot occur twice in any column in a group's Cayley table.

The two parts together prove that all group Cayley tables will be Latin squares.

[2 marks]

Answer 10**(a)**

$$\begin{aligned}
 ff(x) &= f\left(\frac{3x-1}{7x-2}\right) \\
 &= \frac{3\left(\frac{3x-1}{7x-2}\right) - 1}{7\left(\frac{3x-1}{7x-2}\right) - 2} \\
 &= \frac{9x - 3 - 7x + 2}{21x - 7 - 14x + 4} \\
 &= \frac{2x - 1}{7x - 3} \quad \square
 \end{aligned}$$

[3 marks]**(b)**

$$\begin{aligned}
 fff(x) &= ff\left(\frac{3x-1}{7x-2}\right) \\
 &= f\left(\frac{2x-1}{7x-3}\right) \\
 &= \frac{3\left(\frac{2x-1}{7x-3}\right) - 1}{7\left(\frac{2x-1}{7x-3}\right) - 2} \\
 &= \frac{6x - 3 - 7x + 3}{14x - 7 - 14x + 6} \\
 &= \frac{-x}{-1} \\
 &= x \quad \square
 \end{aligned}$$

[3 marks]**(c) (i)** The identity element is $fff(x)$ because $fff(x) = x$ **[1 mark]**

$$\begin{aligned}
 \text{(ii)} \quad fff(x) \circ ff(x) &= e \circ ff(x) \quad e \text{ is the identity} \\
 &= ff(x)
 \end{aligned}$$

[1 mark]**(iii)** The inverse of $ff(x)$ is $f(x)$ because,

$$ff(x) \circ f(x) = f(x) \circ ff(x) = e$$

[1 mark]