3.8 Answers

3.8.1 Solution to 3.5 An Example of Combining Symmetries (Teaching Video)

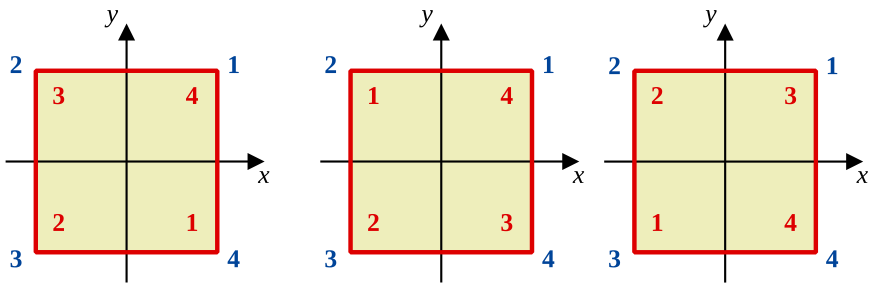
Question :

Show for a square that, using permutations, a rotation of 90° followed by a reflection in the x-axis is equivalent to a reflection in the line $y = -x$

Solution :

The three symmetries involved and their associated permutations are;

- x is a reflection in the x-axis
- r is a rotation of 90°
- n is a reflection in the line $y = -x$



$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ $\quad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ $\quad n = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

So, the question is asking that it be shown that, $x * r = n$

(Take care over getting this the correct way round !)

$$\begin{aligned} \text{LHS} &= x * r \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \\ &= n \\ &= \text{RHS} \qquad \square \end{aligned}$$

Thus, a rotation of 90° followed by a reflection in the x-axis is indeed equivalent to a reflection in the line $y = -x$

[4 marks]

3.8.2 Solutions to 3.7 Exercise

Answer 1

(i)

$$v \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix}$$

[2 marks]

(ii)

$$w \circ v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 1 & 2 \end{pmatrix}$$

[2 marks]

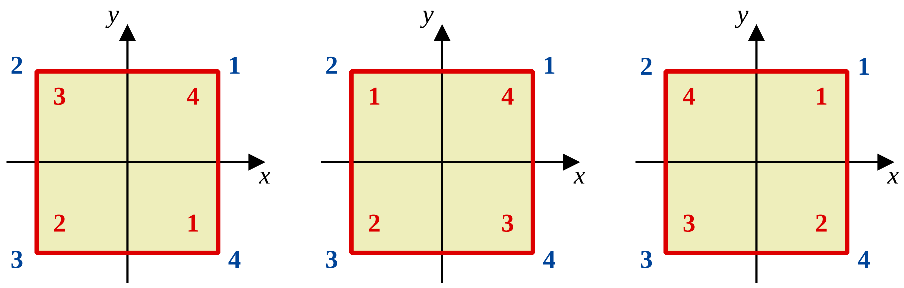
(iii)

$$v^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

[2 marks]

Answer 2



$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \qquad p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Note that reflection in the x-axis, x, followed by a rotation of 90°, r is $r \circ x$
So, the question is requesting a proof that $r \circ x = p$

$$\text{LHS} = r \circ x$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

$$= \text{RHS} \qquad \square$$

Thus, reflection in the x-axis followed by rotation of 90° is indeed equivalent to a reflection in the line $y = x$

[4 marks]

Answer 3

(i)

$*$	e	y
e	e	y
y	y	e

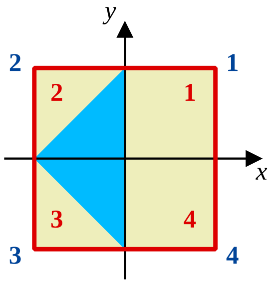
[1 mark]

(ii) This subgroup is of order 2

[1 mark]

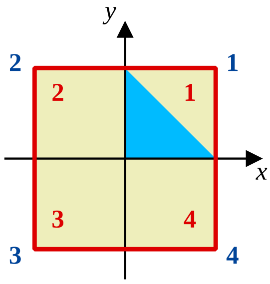
Answer 4

(i)



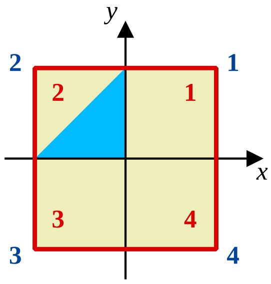
$*$	e	x
e	e	x
x	x	e

(ii)



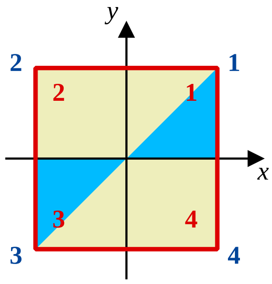
$*$	e	p
e	e	p
p	p	e

(iii)



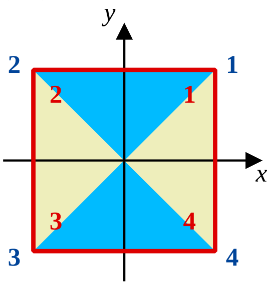
$*$	e	n
e	e	n
n	n	e

(iv)



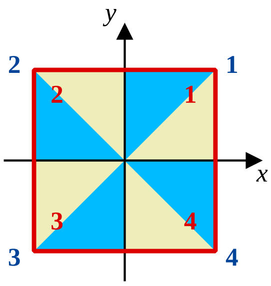
$*$	e	r^2
e	e	r^2
r^2	r^2	e

(v)



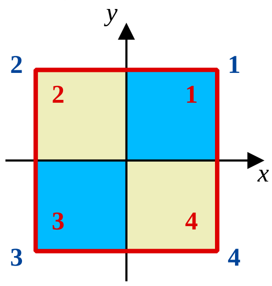
$*$	e	r^2	x	y
e	e	r^2	x	y
r^2	r^2	e	y	x
x	x	y	e	r^2
y	y	x	r^2	e

(vi)



$*$	e	r	r^2	r^3
e	e	r	r^2	r^3
r	r	r^2	r^3	e
r^2	r^2	r^3	e	r
r^3	r^3	e	r	r^2

(vii)



$*$	e	r^2	p	n
e	e	r^2	p	n
r^2	r^2	e	n	p
p	p	n	e	r^2
n	n	p	r^2	e

[14 marks]

Answer 5

Lagrange's Theorem
If H is a subgroup of a finite group G, the order of H divides the order of G.

A proof of this is a highlight of a first course in Group Theory at University. Historically it was first proven by Cauchy in 1844, about 30 years before Lagrange's major contributions to the subject.

[2 marks]

Answer 6

$*$	e	r	r^2	x	v	w
e	e	r	r^2	x	v	w
r	r	r^2	e	w	x	v
r^2	r^2	e	r	v	w	x
x	x	v	w	e	r	r^2
v	v	w	x	r^2	e	r
w	w	x	v	r	r^2	e

[6 marks]

Answer 7

Subgroup	Order	Number Found
$\{e\}$	1	1
$\{e, x\}, \{e, v\}, \{e, w\}$	2	3
$\{e, r, r^2\}$	3	1
$\{e, r, r^2, x, v, w\}$	6	1

[6 marks]