4.7 Answers

4.7.1 Solution to 4.2 Example (Teaching Video)

For the group D_4 find $\langle r^3 \rangle$, the subgroup generated by the element r^3

$*$	e	r	r^2	r^3	y	x	p	n
e	e	r	r^2	r^3	y	x	p	n
r	r	r^2	r^3	e	n	p	y	x
r^2	r^2	r^3	e	r	x	y	n	p
r^3	r^3	e	r	r^2	p	n	x	y
y	y	p	x	n	e	r^2	r^3	r
x	x	n	y	p	r^2	e	r	r^3
p	p	x	n	y	r	r^3	e	r^2
n	n	y	p	x	r^3	r	r^2	e

$$\begin{aligned}(r^3)^2 &= r^3 \circ r^3 \\ &= r^2 \qquad \text{from the Cayley table}\end{aligned}$$

$$\begin{aligned}(r^3)^3 &= r^3 \circ (r^3 \circ r^3) \\ &= r^3 \circ r^2 \\ &= r \qquad \text{from the Cayley table}\end{aligned}$$

$$\begin{aligned}(r^3)^4 &= (r^3 \circ r^3) \circ (r^3 \circ r^3) \\ &= r^2 \circ r^2 \\ &= e \qquad \text{from the Cayley table}\end{aligned}$$

In addition to r^2, r and e, for closure there is $r^3 \circ e = r^3$

The subgroup generated by r^3, $\langle r^3 \rangle$, is thus $H = \{ e,\ r,\ r^2,\ r^3 \}$

$|H| = 4$ The subgroup H has order 4

$|r^3| = 4$ The element r^3 has order 4

[3 marks]

4.7.2 Solutions to 4.6 Exercise

Answer 1

(i) The shape does not have any lines of mirror symmetry.

[1 mark]

(ii) The shape does have rotational symmetry, it is of order 4

[1 mark]

(iii) The symmetry group will be cyclic.

[1 mark]

(iv) a corresponds to is a rotation of 180°

[1 mark]

(v) $|a| = 2$, because $a^2 = e$ where e is the identity element.

[1 mark]

(vi) With e as identity element, and $*$ being composition of symmetries,

$*$	e	a
e	e	a
a	a	e

[2 marks]

Answer 2

(i)

Element	Subgroup Generated	Element's Order
0	$\langle 0\rangle = \{0\}$	1
1	$\langle 1\rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	12
2	$\langle 2\rangle = \{0, 2, 4, 6, 8, 10\}$	6
3	$\langle 3\rangle = \{0, 3, 6, 9\}$	4
4	$\langle 4\rangle = \{0, 4, 8\}$	3
5	$\langle 5\rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	12
6	$\langle 6\rangle = \{0, 6\}$	2
7	$\langle 7\rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	12
8	$\langle 8\rangle = \{0, 4, 8\}$	3
9	$\langle 9\rangle = \{0, 3, 6, 9\}$	4
10	$\langle 10\rangle = \{0, 2, 4, 6, 8, 10\}$	6
11	$\langle 11\rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$	12

[6 marks]

(ii) It's cyclic as the whole group can be generated from one element.

[2 marks]

Answer 3

(i) The group is of order 8

[1 mark]

(ii) $\langle a \rangle = \{e, a, a^2, a^3\} \qquad \therefore |a| = 4$

$\langle b \rangle = \{e, b, a^2, ba^2\} \qquad \therefore |b| = 4$

[2 marks]

(iii) $b * ba = a^3$

[1 mark]

(iv) $\left(ba^2\right)^{-1} = b$

[1 mark]

(v) The cyclic order 4 subgroup is $\langle a \rangle = \langle a^3 \rangle = H = \{e, a, a^2, a^3\}$

Less obviously, $I = \{e, a^2, b, ba^2\}$ and $J = \{e, a^2, ba, ba^3\}$

[1 mark]

(vi) Lagrange's theorem tells us that subgroups of order 3, 5, 6 and 7 can not exist for this group of order 8.

[1 mark]

Answer 4

(i) The identity element is e

[1 mark]

(ii) $\langle a \rangle = \{e, a, c\} \qquad \therefore |a| = 3$

$\langle b \rangle = \{e, a, b, c, d, f\} \qquad \therefore |b| = 6$

$\langle c \rangle = \{e, a, c\} \qquad \therefore |c| = 3$

$\langle d \rangle = \{e, a, b, c, d, f\} \qquad \therefore |d| = 6$

$\langle e \rangle = \{e\} \qquad \therefore |e| = 1$

$\langle f \rangle = \{e, f\} \qquad \therefore |f| = 2$

[3 marks]

(iii) $a^{-1} = c \qquad b^{-1} = d \qquad c^{-1} = a$

$d^{-1} = b \qquad e^{-1} = e \qquad f^{-1} = f$

[3 marks]

(iv) G is cyclic as it can be generated by a single element, b or d

[2 marks]

(v) $\langle f \rangle = \{e, f\}$ of order 2, $\langle a \rangle = \langle c \rangle = \{e, a, c\}$ of order 3

(The subgroups of a cyclic group have to be cyclic)

Lagrange states that the possible subgroups of a group of order 6 must be divisors of 6, that is, 1, 2, 3 or 6, which is consistent with those listed.

[3 marks]

Answer 5

(i) The identity element of G is 6
Various reasons can be given.
For example, it's the only element (in this case) for which $e \times e = e$

[2 marks]

(ii) $8^{-1} = 2$

[1 mark]

(iii)
$\langle 2 \rangle = \{2,\ 4,\ 6,\ 8\}$
$\langle 4 \rangle = \{4,\ 6\}$
$\langle 6 \rangle = \{6\}$
$\langle 8 \rangle = \{2,\ 4,\ 6,\ 8\}$

Thus, G has 3 subgroups, $\{6\}$ generated by 6
$\{4,\ 6\}$ generated by 4
$\{2,\ 4,\ 6,\ 8\}$ generated by 2 or by 8

[3 marks]

(iv) Yes, G is a cyclic group because it can be generated by a single element, either 2 or 8

[2 marks]

Answer 6

It may help to be shown the Cayley table for the group;

$\times_{20}$	1	3	7	9	11	13	17	19
1	1	3	7	9	11	13	17	19
3	3	9	1	7	13	19	11	17
7	7	1	9	3	17	11	19	13
9	9	7	3	1	19	17	13	11
11	11	13	17	19	1	3	7	9
13	13	19	11	17	3	9	1	7
17	17	11	19	13	7	1	9	3
19	19	17	13	11	9	7	3	1

(i) The group S is of order 8
Lagrange's theorem states that the order of any subgroup must be a divisor of the order of the group.
The divisors of 8 are 1, 2, 4, 8.
$\therefore$ As 3 is not a divisor of 8, S can not have a subgroup of order 3 ☐

[1 mark]

(ii)

$\langle 1 \rangle = \{1\} \quad \therefore \quad |1| = 1$

$\langle 3 \rangle = \{1, 3, 7, 9\} \quad \therefore \quad |3| = 4$

$\langle 7 \rangle = \{1, 3, 7, 9\} \quad \therefore \quad |7| = 4$

$\langle 9 \rangle = \{1, 9\} \quad \therefore \quad |9| = 2$

$\langle 11 \rangle = \{1,11\} \quad \therefore \quad |11| = 2$

$\langle 13 \rangle = \{1, 9, 13, 17\} \quad \therefore \quad |13| = 4$

$\langle 17 \rangle = \{1, 9, 13, 17\} \quad \therefore \quad |17| = 4$

$\langle 19 \rangle = \{1, 19\} \quad \therefore \quad |19| = 2$

[3 marks]

(iii) Two cyclic subgroups of order 4 were found above.
They are $\{1, 3, 7, 9\}$ along with $\{1, 9, 13, 17\}$.
The question wants us to find another subgroup of order 4.
It can't be cyclic, or it would have been detected in the part (ii) answer.
After some playing around I found $\{1, 9, 11, 19\}$

[4 marks]

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