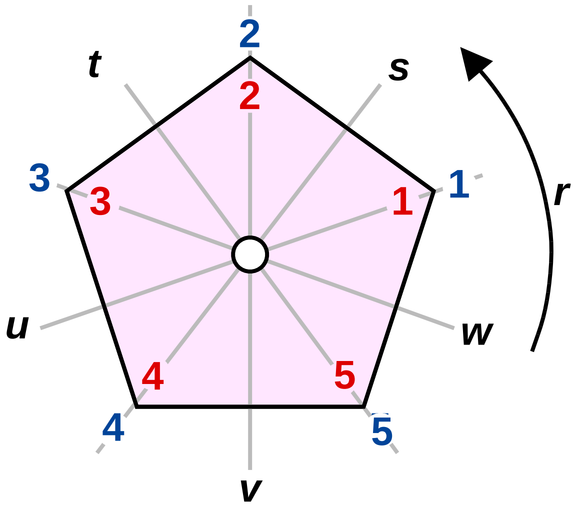
2.4 Answers

2.4.1 Solution to 2.2 Example (Teaching Video)

By continuing to work with the symmetry group of the pentagon and working in cycle notation, show that $s\,r^2 = w$



w will
- keep 3 fixed
- swap 2 and 4
- swap 1 and 5

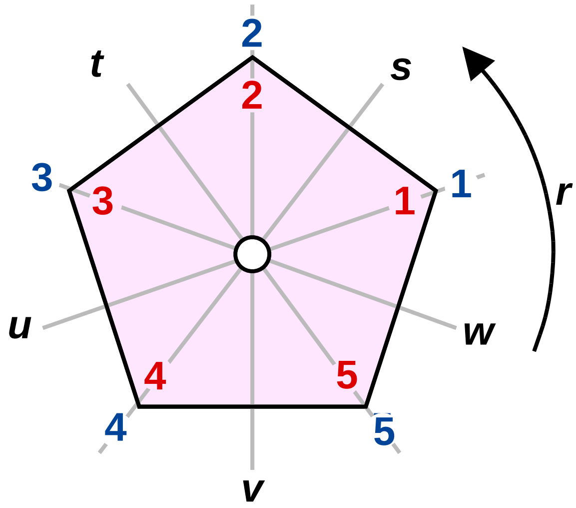
$\therefore\ w = (1\,5)(2\,4)$

$$
\begin{aligned}
\text{LHS} &= s\,r^2 \\
&= (1\,2)(3\,5) \circ (1\,3\,5\,2\,4) \\
&= (\,1\,5\,)(2\,4) \\
&= w \\
&= \text{RHS} \quad \square
\end{aligned}
$$

[4 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1



Symmetry	Cycle Notation
e	$(1)(2)(3)(4)(5)$
r	$(1\ 2\ 3\ 4\ 5)$
r^2	$(1\ 3\ 5\ 2\ 4)$
r^3	$(1\ 4\ 2\ 5\ 3)$
r^4	$(1\ 5\ 4\ 3\ 2)$
s	$(1\ 2)(3\ 5)$
t	$(1\ 4)(2\ 3)$
u	$(2\ 5)(3\ 4)$
v	$(1\ 3)(4\ 5)$
w	$(1\ 5)(2\ 4)$

[5 marks]

Answer 2

From the teaching video, it is shown that, $s\,r^2 \;=\; w$

Experimentation, perhaps in your mind palace, suggests $r^2\,s \;=\; t$

Proving that is easy using cycle notation.

First of all, note that the pieces in play are

$$s \;=\; (1\;2)(3\;5)\,,\quad r^2 \;=\; (1\;3\;5\;2\;4) \text{ and } t \;=\; (1\;4)(2\;3)$$

Proof:

$$\begin{aligned} \text{LHS} &= r^2\,s \\ &= (1\;3\;5\;2\;4) \circ (1\;2)(3\;5) \\ &= (1\;4)(2\;3) \\ &= t \\ &= \text{RHS} \end{aligned}$$

$$\therefore\; s\,r^2 \;\neq\; r^2\,s \qquad \square$$

[2 marks]

Answer 3

$*$	e	r	r^2	r^3	r^4	s	t	u	v	w
e	e	r	r^2	r^3	r^4	s	t	u	v	w
r	r	r^2	r^3	r^4	e	v	w	s	t	u
r^2	r^2	r^3	r^4	e	r	t	u	v	w	s
r^3	r^3	r^4	e	r	r^2	w	s	t	u	v
r^4	r^4	e	r	r^2	r^3	u	v	w	s	t
s	s	u	w	t	v	e	r^3	r	r^4	r^2
t	t	v	s	u	w	r^2	e	r^3	r	r^4
u	u	w	t	v	s	r^4	r^2	e	r^3	r
v	v	s	u	w	t	r	r^4	r^2	e	r^3
w	w	t	v	s	u	r^3	r	r^4	r^2	e

[6 marks]

Answer 4

(i) **Statement:** $(1\ 2) \circ (3\ 1) \circ (4\ 1) \circ (1\ 5) = r^4$

Proof: From Question 1, it is known that $r^4 = (1\ 5\ 4\ 3\ 2)$

$$\begin{aligned} \text{LHS} &= (1\ 2) \circ (3\ 1) \circ (4\ 1) \circ (1\ 5) \\ &= (1\ 5\ 4\ 3\ 2) \\ &= r^4 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[2 marks]

(ii) $(2\ 4) \circ (1\ 2) \circ (2\ 3) \circ (5\ 2) = (1\ 4\ 2\ 5\ 3)$

$= r^3$

[2 marks]

(iii) $(5\ 3) \circ (3\ 2) \circ (4\ 3) \circ (1\ 3) = (1\ 4\ 2\ 5\ 3)$

$= r^3$

[2 marks]

Answer 5

(i) $r^2 = (1\ 3\ 5\ 2\ 4)$

$= (1\ 4) \circ (1\ 2) \circ (1\ 5) \circ (1\ 3)$

[2 marks]

(ii) $r^2 = (5\ 2\ 4\ 1\ 3)$

$= (5\ 3) \circ (5\ 1) \circ (5\ 4) \circ (5\ 2)$

[2 marks]

(iii) Even

[1 mark]

Answer 6

(a) $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 3 & 6 & 5 \end{pmatrix} \Rightarrow p = (1\,4\,3)(5\,6)$

[1 mark]

$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 5 & 4 & 3 \end{pmatrix} \Rightarrow q = (1\,6\,3\,2)(4\,5)$

[1 mark]

$p^{-1} = (1\,3\,4)(5\,6)$

[1 mark]

$q^{-1} = (1\,2\,3\,6)(4\,5)$

[1 mark]

(b) $p \circ q = (1\,4\,3)(5\,6) \circ (1\,6\,3\,2)(4\,5)$

$= (\,1\,5\,3\,2\,4\,6\,)$

[1 mark]

$q \circ p = (1\,6\,3\,2)(4\,5) \circ (1\,4\,3)(5\,6)$

$= (1\,5\,3\,6\,4\,2)$

[1 mark]

(c) $p = (1\,4\,3)(5\,6)$

$= (1\,3) \circ (1\,4) \circ (5\,6)$ Parity of p is ODD

[2 marks]

$q = (1\,6\,3\,2)(4\,5)$

$= (1\,2) \circ (1\,3) \circ (1\,6) \circ (4\,5)$ Parity of q is EVEN

[2 marks]

(d) $r \circ s = q \circ r$

$r^{-1} \circ r \circ s = r^{-1} \circ q \circ r$ Pre-multiply both sides by r^{-1}

$s = r^{-1} \circ q \circ r$

$= (2\,3\,4\,5)^{-1} \circ (1\,6\,3\,2)(4\,5) \circ (2\,3\,4\,5)$

$= (2\,5\,4\,3) \circ (1\,6\,3\,2)(4\,5) \circ (2\,3\,4\,5)$

$= (1\,6\,2\,5)(3\,4)$

[2 marks]

Answer 7

(i) $s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix}$

$= (2\,4\,5\,3)$

[1 mark]

(ii) $s^2 = (2\,4\,5\,3) \circ (2\,4\,5\,3)$

$= (2\,5)(3\,4)$

$s^3 = (2\,4\,5\,3) \circ (2\,5)(3\,4)$

$= (2\,3\,5\,4)$

$s^4 = (2\,4\,5\,3) \circ (2\,3\,5\,4)$

$= (1)(2)(3)(4)(5)(6)$

$= e$ The identity permutation

[2 marks]

(iii) This question shows that after four perfect shuffles the deck of six cards is back in its original order.

[1 mark]

Here is a demonstration of the process;