3.6 Answers

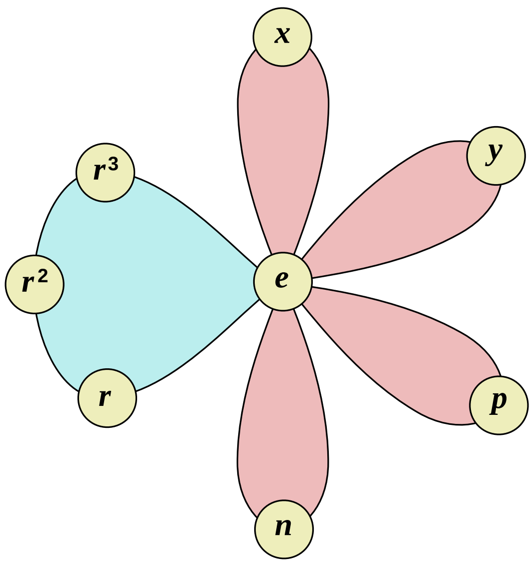
3.6.1 Solution to 3.4 Example (Teaching Video)

$*$	e	r	r^2	r^3	y	x	p	n
e	e	r	r^2	r^3	y	x	p	n
r	r	r^2	r^3	e	n	p	y	x
r^2	r^2	r^3	e	r	x	y	n	p
r^3	r^3	e	r	r^2	p	n	x	y
y	y	p	x	n	e	r^2	r^3	r
x	x	n	y	p	r^2	e	r	r^3
p	p	x	n	y	r	r^3	e	r^2
n	n	y	p	x	r^3	r	r^2	e

Cycle Graph Construction Strategy

Start with a graph containing only the identity element as a single node.

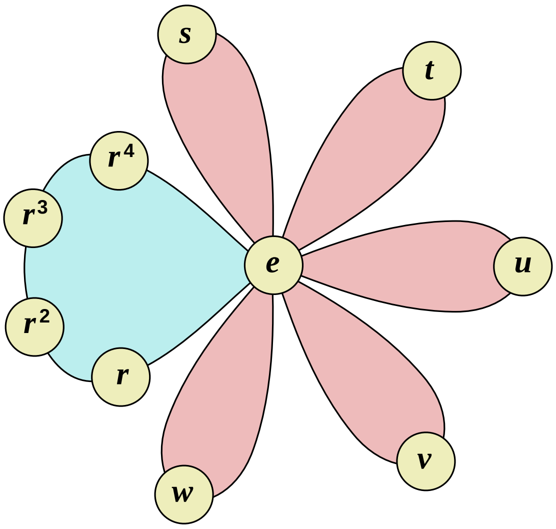
1. Pick an element not already in the graph.
2. Compute the cyclic subgroup generated by that element and add the cycle to the graph, connecting to already existing nodes as needed.
3. If the graph contains a sub-cycle of the new cycle, delete the sub-cycle.
4. Repeat steps 1 to 3 until all of the (finitely many) elements have a node.



[3 marks]

3.6.2 Solutions to 3.5 Exercise

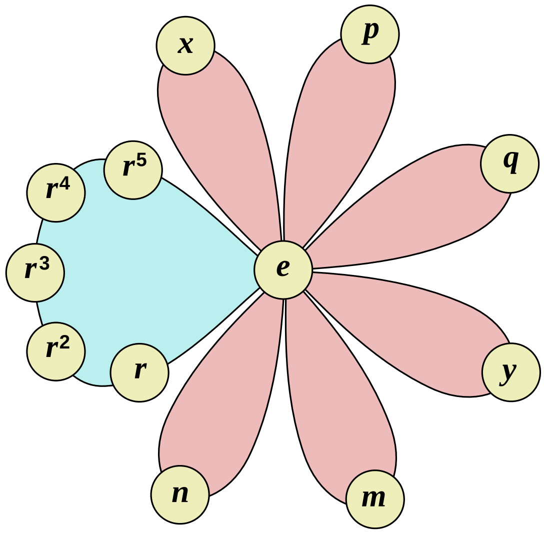
Answer 1



[3 marks]

Answer 2

(i)



[3 marks]

(ii)

Subgroup	Order		Subgroup	Order		Subgroup	Order
$\{e\}$	1		$\{e,\ y\}$	2		$\{e,\ r^2,\ r^4\}$	3
$\{e,\ x\}$	2		$\{e,\ m\}$	2		$\{e,\ r,\ r^2,\ r^3,\ r^4,\ r^5\}$	6
$\{e,\ p\}$	2		$\{e,\ n\}$	2		$\{e,\ x,\ p,\ q,\ y,\ m,\ n$	
$\{e,\ q\}$	2		$\{e,\ r^3\}$	2		$r,\ r^2,\ r^3,\ r^4,\ r^5\}$	12

[3 marks]

(iii)

Symmetry	Order	Cycle Notation	Transpositions	Parity
e	1	$(1)(2)(3)(4)(5)(6)$	none	even
r	6	$(1\ 2\ 3\ 4\ 5\ 6)$	$(1\ 6)(1\ 5)(1\ 4)(1\ 3)(1\ 2)$	odd
r^2	3	$(1\ 3\ 5)(2\ 4\ 6)$	$(1\ 5)(1\ 3)(2\ 6)(2\ 4)$	even
r^3	2	$(1\ 4)(2\ 5)(3\ 6)$	$(1\ 4)(2\ 5)(3\ 6)$	odd
r^4	3	$(1\ 5\ 3)(2\ 6\ 4)$	$(1\ 3)(1\ 5)(2\ 4)(2\ 6)$	even
r^5	6	$(1\ 6\ 5\ 4\ 3\ 2)$	$(1\ 2)(1\ 3)(1\ 4)(1\ 5)(1\ 6)$	odd
x	2	$(2\ 6)(3\ 5)$	$(2\ 6)(3\ 5)$	even
p	2	$(1\ 2)(3\ 6)(4\ 5)$	$(1\ 2)(3\ 6)(4\ 5)$	odd
q	2	$(1\ 3)(4\ 6)$	$(1\ 3)(4\ 6)$	even
y	2	$(1\ 4)(2\ 3)(5\ 6)$	$(1\ 4)(2\ 3)(5\ 6)$	odd
m	2	$(1\ 5)(2\ 4)$	$(1\ 5)(2\ 4)$	even
n	2	$(1\ 6)(2\ 5)(3\ 4)$	$(1\ 6)(2\ 5)(3\ 4)$	odd

[7 marks]

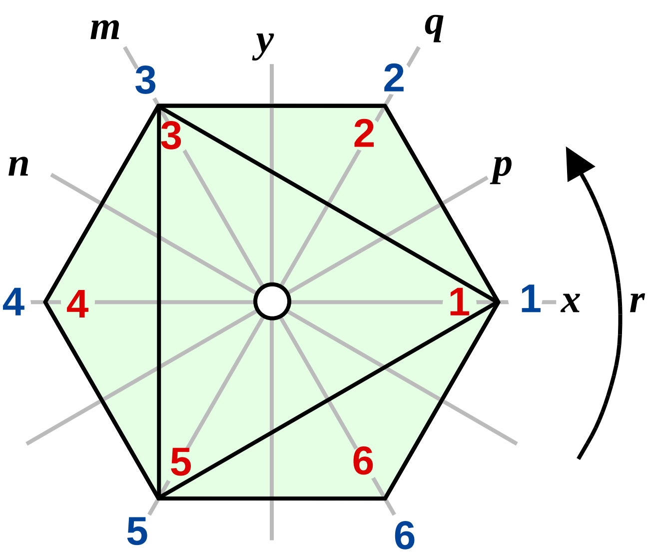
(iv) The elements with even parity are, $H = \{e,\ r^2,\ r^4,\ x,\ q,\ m\}$

$\circ$	e	r^2	r^4	x	q	m
e	e	r^2	r^4	x	q	m
r^2	r^2	r^4	e	q	m	x
r^4	r^4	e	r^2	m	x	q
x	x	m	q	e	r^4	r^2
q	q	x	m	r^2	e	r^4
m	m	q	x	r^4	r^2	e

[3 marks]

(v) $$H \cong S_3$$

There is a clever explanation as to why H is isomorphic to the permutation group S_3 which is most famously associated with the symmetry group for an equilateral triangle.
It's as if such a triangle has been drawn on the hexagon, like so;



[1 mark]

(vi) Prove that for any group G, the elements with even parity form a subgroup.
You may use the fact that,

- even and odd permutations combine according to the parity table;

$\circ$	even	odd
even	even	odd
odd	odd	even

Proof:
The three subgroup axioms need to be checked.

Closure: The subgroup is defined to be constructed from all elements of G which are even. Given any two elements of G that are even, the parity table shows that, when composed, the result is another even element. That is, $even \circ even = even$. $\therefore$ H is closed.

Identity: The identity of G is even, so this will be in H.

Inverses: Given any element $f \in H$, consider the parity of

$$f \circ f^{-1} = e$$

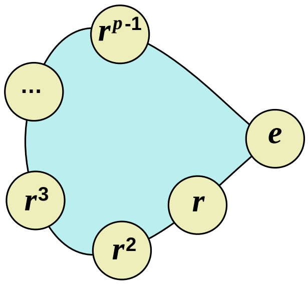
The parity table that the inverse is even and so in H

As H satisfies all three subgroup axioms, H is a subgroup of G □

[6 marks]

Answer 3

(i)



[2 marks]

(ii) The cycle graph for a cyclic group consists of a single loop. There can be no sub-cycles within this loop because a prime number only divides by itself and the number one.

The only subgroups are therefore $\{e\}$ and $\{e, r, r^2, r^3, \ldots, r^{p-1}\}$

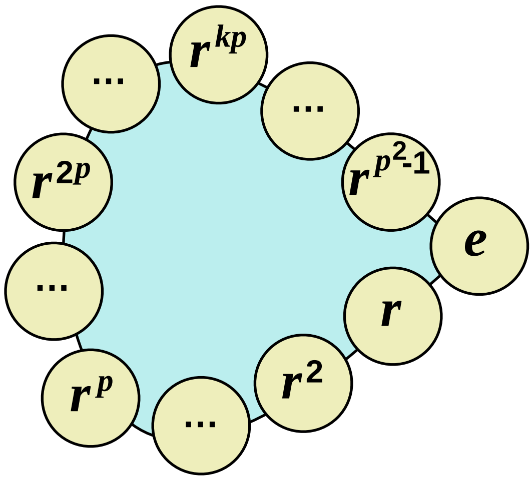
Neither of these is a proper sub-group.

$\therefore$ A cyclic group has no proper subgroups. $\square$

[2 marks]

Answer 4

In the cycle diagram k is a positive integer with $k = p - 1$



(i) From the above cycle diagram, there will be one proper subgroup.

[2 marks]

(ii) $\langle r^p \rangle = \{e, r^p, r^{2p}, r^{3p}, \ldots, r^{kp}\}$ where $k = p - 1$

[2 marks]

Answer 5
Statement: A cyclic group of even order has exactly one element of order 2

Proof : If G is a finite cyclic group of order n, then G has a unique subgroup of order m for each m dividing n.
Now. if n is even, then G has a unique subgroup of order 2, hence one element of order 2 (as if G had two distinct elements of order 2, then there would be two distinct subgroups of order 2)

This question could be generalised to say that,
"A cyclic group of order n has exactly one subgroup of order m for each value of m that divides n"

[3 marks]

Answer 6
Statement:
Let G be a group, with identity element e, containing finite subgroups H and K.
If $|H|$ and $|K|$ are coprime, then $H \cap K = \{e\}$

Proof:
By Lagrange's theorem $|H \cap K|$ must divide $|H|$ and must also divide $|K|$
Since the orders of H and K are relatively prime, necessarily $|H \cap K| = 1$
Therefore, $H \cap K = \{e\}$ □

[3 marks]