4.7 Answers

4.7.1 Solutions to 4.3 Example (Teaching Video)

Let w be the conjugate of h in which case,

$$\begin{aligned} w &= g\,h\,g^{-1} \\ &= (1\ 2)(3\ 4) \circ (1\ 2\ 3\ 4) \circ (1\ 2)(3\ 4) \\ &= (1\ 4\ 3\ 2) \ \text{ which is } r^3 \end{aligned}$$

So r and r^3 are in the same conjugacy class

[2 marks]

4.7.2 Solutions to 4.6 Exercise

Answer 1

(a) $h = (1\ 4\ 3\ 2) \qquad g = (1\ 2)(3\ 4)$

[2 marks]

(b) Let w be the conjugate of h in which case,

$$\begin{aligned} w &= g\,h\,g^{-1} \\ &= (1\ 2)(3\ 4) \circ (1\ 4\ 3\ 2) \circ (1\ 2)(3\ 4) \qquad \text{as } g = g^{-1} \\ &= (1\ 2\ 3\ 4) \ \text{ which has order 3} \end{aligned}$$

[2 marks]

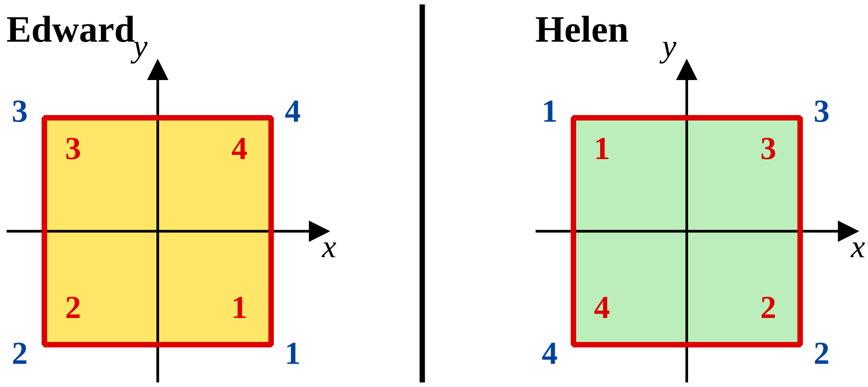
(c) The elements g and h are not conjugate.
Here are three reasons why;

- They are of different geometrical type.
  Conjugation preserves the property of reflection and the property of rotation.
  g is a reflection, h is a rotation.
- They are of different cycle type.
  Conjugation preserves cycle structure.
  g has the cycle structure (- -)(- -), h has the structure (- - - -)
- The order of each is different.
  Conjugation preserves the order of the elements.
  g is of order 2, h is of order 3

[2 marks]

Answer 2

(i)



The "renaming isomorphism" is $\phi = (1\;2\;4\;3)$ for Edward $\rightarrow$ Helen

For Edward's Square			For Helen's Square	
e	$(1)(2)(3)(4)$		E	$(1)(2)(3)(4)$
r	$(1\;4\;3\;2)$		R	$(1\;4\;2\;3)$
r^2	$(1\;3)(2\;4)$		R^2	$(1\;2)(3\;4)$
r^3	$(1\;2\;3\;4)$		R^3	$(1\;3\;2\;4)$
x	$(1\;4)(2\;3)$		X	$(1\;4)(2\;3)$
y	$(1\;2)(3\;4)$		Y	$(1\;3)(2\;4)$
p	$(1\;3)$		P	$(1\;2)$
n	$(2\;4)$		N	$(3\;4)$

[7 marks]

(ii) The renaming isomorphism does not match with any element in Edwards square. It will give rise to elements in Helen's square that are not in Edward's.
There is, in fact, no physical reflection or rotation that will provide a way to transform Edward's square onto Helen's.
An automorphism is a isomorphism from a group back onto itself but Helen's group has elements of S_4 in it that are not in Edwards.
So these two groups cannot be treated as if they were one and so no automorphism is possible.

[2 marks]

Answer 3

(a) The cycle $(1\,5\,3\,6\,4\,2)$ can be expressed in six variations that all describe the same cycle;

$$\begin{aligned}(1\,5\,3\,6\,4\,2) &= (5\,3\,6\,4\,2\,1) = (3\,6\,4\,2\,1\,5)\\ &= (6\,4\,2\,1\,5\,3) = (4\,2\,1\,5\,3\,6) = (2\,1\,5\,3\,6\,4)\end{aligned}$$

Thinking of the conjugating elements, g_k for $1 \leqslant k \leqslant 6$ as "renaming" functions means that the elements required are;

$$g_1 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 5 & 3 & 6 & 4 & 2 \end{pmatrix} = e \qquad g_2 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 3 & 6 & 4 & 2 & 1 \end{pmatrix} = (1\,5\,3\,6\,4\,2)$$

$$g_3 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 6 & 4 & 2 & 1 & 5 \end{pmatrix} = (1\,3\,4)(2\,5\,6) \qquad g_4 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} = (1\,6)(2\,3)(4\,5)$$

$$g_5 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 2 & 1 & 5 & 3 & 6 \end{pmatrix} = (1\,4\,3)(2\,6\,5) \qquad g_6 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 5 & 3 & 6 & 4 \end{pmatrix} = (1\,2\,4\,6\,3\,5)$$

[5 marks]

(b)

Element	Order	Parity
e	1	even
$(1\,5\,3\,6\,4\,2)$	6	odd
$(1\,3\,4)(2\,5\,6)$	3	even
$(1\,6)(2\,3)(4\,5)$	2	odd
$(1\,4\,3)(2\,6\,5)$	3	even
$(1\,2\,4\,6\,3\,5)$	6	odd

[5 marks]

(c)

$\circ$	e	(153642)	$(134)(256)$	$(16)(23)(45)$	$(143)(265)$	(124635)
e	e	(153642)	$(134)(256)$	$(16)(23)(45)$	$(143)(265)$	(124635)
(153642)	(153642)	$(134)(256)$	$(16)(23)(45)$	$(143)(265)$	(124635)	e
$(134)(256)$	$(134)(256)$	$(16)(23)(45)$	$(143)(265)$	(124635)	e	(153642)
$(16)(23)(45)$	$(16)(23)(45)$	$(143)(265)$	(124635)	e	(153642)	$(134)(256)$
$(143)(265)$	$(143)(265)$	(124635)	e	(153642)	$(134)(256)$	$(16)(23)(45)$
(124635)	(124635)	e	(153642)	$(134)(256)$	$(16)(23)(45)$	$(143)(265)$

Verifying the subgroup axioms;

Closure: No new elements were needed to complete the Cayley table.

If $H = \{e, (1\,5\,3\,6\,4\,2), (1\,3\,4)(2\,5\,6),$

$(1\,6)(2\,3)(4\,5), (1\,4\,3)(2\,6\,5), (1\,2\,4\,6\,3\,5)\}$

then, for all $h_1, h_2 \in H$, the composite $h_1 h_2 \in H$

Identity: The identity element of S_6 belongs to H. That is, $e \in H$

Inverses: $(1\,5\,3\,6\,4\,2)$ has inverse $(2\,4\,6\,3\,5\,1)$, i.e. $(1\,2\,4\,6\,3\,5)$

$(1\,3\,4)(2\,5\,6)$ has inverse $(4\,3\,1)(6\,5\,2)$ i.e. $(1\,4\,3)(2\,6\,5)$

$(1\,6)(2\,3)(4\,5)$ has inverse $(6\,1)(3\,2)(5\,4)$ i.e. $(1\,6)(2\,3)(4\,5)$

$(1\,4\,3)(2\,6\,5)$ has inverse $(3\,4\,1)(5\,6\,2)$ i.e. $(1\,3\,4)(2\,5\,6)$

$(1\,2\,4\,6\,3\,5)$ has inverse $(5\,3\,6\,4\,2\,1)$ i.e. $(1\,5\,3\,6\,4\,2)$

e has inverse e

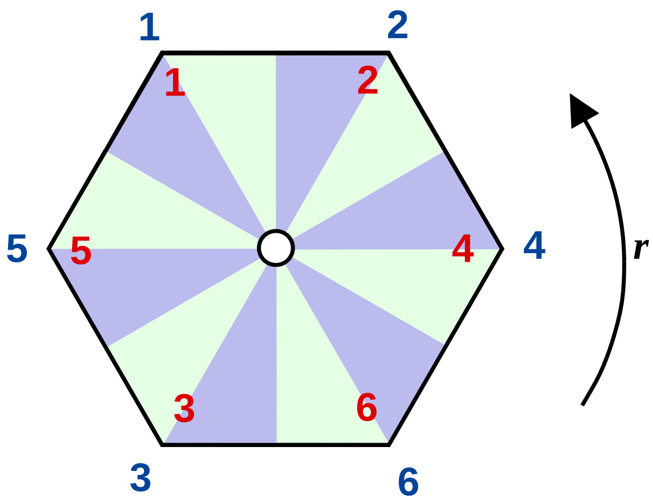
In other words, for each $h \in H$, the inverse $h^{-1} \in H$

$\therefore$ The set of elements do indeed form a subgroup of S_6

The subgroup of S_6 is isomorphic to the cyclic group $\mathbb{Z}_6$ of order 6

A geometric example of $\mathbb{Z}_6$ is the rotational symmetries of a regular hexagon.

If r is a rotation of 60° about the hexagon's centre, then;



$*$	e	r	r^2	r^3	r^4	r^5
e	e	r	r^2	r^3	r^4	r^5
r	r	r^2	r^3	r^4	r^5	e
r^2	r^2	r^3	r^4	r^5	e	r
r^3	r^3	r^4	r^5	e	r	r^2
r^4	r^4	r^5	e	r	r^2	r^3
r^5	r^5	e	r	r^2	r^3	r^4

$$\phi \;:\; H \to \mathbb{Z}_6 \quad \text{where} \;\; r^6 = e$$

$$\begin{aligned} e &\to e \\ (1\,5\,3\,6\,4\,2) &\to r \\ (1\,3\,4)(2\,5\,6) &\to r^2 \\ (1\,6)(2\,3)(4\,5) &\to r^3 \\ (1\,4\,3)(2\,6\,5) &\to r^4 \\ (1\,2\,4\,6\,3\,5) &\to r^5 \end{aligned}$$

[6 marks]

(d) The cycle $(1\,2\,3\,4\,5\,6)$ can be expressed in six variations that all describe the same cycle;

$$\begin{aligned} & (1\,2\,3\,4\,5\,6) = (2\,3\,4\,5\,6\,1) = (3\,4\,5\,6\,1\,2) \\ = & (4\,5\,6\,1\,2\,3) = (5\,6\,1\,2\,3\,4) = (6\,1\,2\,3\,4\,5) \end{aligned}$$

Thinking of the conjugating elements, g_k for $7 \leqslant k \leqslant 12$ as "renaming" functions means that the elements required are;

$$g_7 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = (2\,6\,4\,5) \qquad g_8 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} = (1\,2)(3\,4\,6\,5)$$

$$g_9 = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix} = (1\,3\,5\,4) \qquad g_{10} = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix} = (1\,4\,2\,3\,6)$$

$$g_{11} = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} = (1\,5\,6\,2\,4\,3) \qquad g_{12} = \begin{pmatrix} 1 & 5 & 3 & 6 & 4 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} = (1\,6\,3\,2\,5)$$

[5 marks]

(e) I'll choose $g_{11} = (1\,5\,6\,2\,4\,3)$ for my p and now take each x in turn;

- $p \circ g_1 = p \circ e = p = (1\,5\,6\,2\,4\,3) = g_{11}$
- $p \circ g_2 = (1\,5\,6\,2\,4\,3) \circ (1\,5\,3\,6\,4\,2) = (1\,6\,3\,2\,5) = g_{12}$
- $p \circ g_3 = (1\,5\,6\,2\,4\,3) \circ (1\,3\,4)(2\,5\,6) = (2\,6\,4\,5) = g_7$
- $p \circ g_4 = (1\,5\,6\,2\,4\,3) \circ (1\,6)(2\,3)(4\,5) = (1\,2)(3\,4\,6\,5) = g_8$
- $p \circ g_5 = (1\,5\,6\,2\,4\,3) \circ (1\,4\,3)(2\,6\,5) = (1\,3\,5\,4) = g_9$
- $p \circ g_6 = (1\,5\,6\,2\,4\,3) \circ (1\,2\,4\,6\,3\,5) = (1\,4\,2\,3\,6) = g_{10}$

Thus, the elements of $p \circ x$ are precisely the elements of part (d) $\square$

[4 marks]

4.7.3 Question 3 : Seeing the bigger picture

In part (a) the question looked at the six possible ways of labelling, for example, a hexagon without the anticlockwise order of the labels being altered.

In part (c) the question showed that the conjugating elements that performed the various possible relabelling actions did, in themselves, form a group, and that group was isomorphic to the group of symmetries of the hexagon. So, effectively, possible rotations of the labels were being looked art.

In part (d) the question looked at the six possible ways of relabelling the unconventionally labelled shape with the more conventional (1 2 3 4 5 6)

In part (e) the focus was on one particular relabelling.

It was shown, for example, $g_{11} \circ g_2 = g_{12}$