5.8 Answers to 5.7 Exercise

Answer 1

(i)

Equivalence Class	Elements of G in class
$[1]$	$\{1\}$
$[2]$	$\{2,\ 3,\ 5,\ 7,\ 11,\ 13\}$
$[3]$	$\{4,\ 9\}$
$[4]$	$\{6,\ 8,\ 10,\ 14,\ 15\}$
$[5]$	$\{16\}$
$[6]$	$\{12\}$

[3 marks]

(ii) The prime numbers all have exactly two factors.

[1 mark]

(iii) The square numbers cannot have an even number of factors.

$\therefore$ They can't be in the equivalence classes $[2]$, $[4]$, $[6]$, ... , $[2k]$, ...

for k being a positive integer. Also excluded is $[1]$.

[2 marks]

Answer 2

(i)

$$\begin{aligned} \text{LHS} &= (x - y)^2 (xy - 1)^2 \\ &= (2 - 1)^2 (2 \times 1 - 1)^2 \qquad \text{For } x = 2,\ y = 1 \\ &= 1 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[1 mark]

(ii)

$$\begin{aligned} \text{LHS} &= (x - y)^2 (xy - 1)^2 \\ &= (1 - 0)^2 (1 \times 0 - 1)^2 \qquad \text{For } x = 1,\ y = 0 \\ &= 1 \\ &= \text{RHS} \qquad \square \end{aligned}$$

[1 mark]

(iii)

$$\begin{aligned} \text{LHS} &= (x - z)^2 (xz - 1)^2 \\ &= (2 - 0)^2 (2 \times 0 - 1)^2 \qquad \text{For } x = 2,\ z = 0 \\ &= 4 \\ &\neq \text{RHS} \qquad \therefore \text{Not transitive} \end{aligned}$$

[1 mark]

(iv) The relation is symmetric.

Proof :

$$x \sim y \iff (x - y)^2 (xy - 1)^2 = 1$$

is algebraically the same as

$$y \sim x \iff (y - x)^2 \ (yx - 1)^2 = 1$$

$\therefore$ For all $x, y \in \mathbb{R}$, if $x \sim y$ then $y \sim x$ and so the relation is symmetric. $\square$

[2 marks]

(v) The relation is not reflexive.

Proof :

$$\begin{aligned} x \sim x \iff (x - x)^2 (xx - 1)^2 &= 0^2 \left(x^2 - 1 \right)^2 \\ &= 0 \\ &\neq 1 \end{aligned}$$

$\therefore$ x is not related to itself and so the relation is not reflexive. $\square$

[2 marks]

(vi) The relation is not an equivalence relation.

Proof :

Of the three axioms that would have to be satisfied only the symmetric property holds. The relation fails to be reflexive or transitive.

[1 mark]

Answer 3

The union of all the individual subsets must be all of X for the subsets to be a partition of X. For option A this is not the case but for option B it is.

$$\begin{aligned} A_1 \cup A_2 \cup A_3 &= \{1,\ 3,\ 5\} \cup \{2\} \cup \{4,\ 7\} \\ &= \{1,\ 2,\ 3,\ 4,\ 5,\ 7\} \qquad \text{Notice: the 6 is missing} \end{aligned}$$

$\therefore$ $\left\{ A_1, A_2, A_3 \right\}$ is not a partition of X and so will not give rise to an equivalence relation.

$$\begin{aligned} B_1 \cup B_2 \cup B_3 &= \{1,\ 2,\ 5,\ 7\} \cup \{3\} \cup \{4,\ 6\} \\ &= \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\} \\ &= X \end{aligned}$$

$\therefore$ $\left\{ B_1, B_2, B_3 \right\}$ is a partition of X and so will will give rise to an equivalence relation.

[3 marks]

Answer 4

$$x \sim y \quad \text{if} \quad x + 3y \text{ is a multiple of 4,} \qquad x, y \in \mathbb{Z}$$

(i) The relation is reflexive.

Proof :

$$x \sim x \iff x + 3x = 4 \text{ which is a multiple of 4}$$

Hence x is related to itself which is what it means to be reflexive. □

The relation is symmetric.

Proof :

If it is assumed that $x \sim y$ is a multiple of 4 then, in consequence,

$$x + 3y = 4k \qquad \text{for some integer } k$$

$$3x + 9y = 12k \qquad \text{multiplying through by 3}$$

$$3x + y + 8y = 12k$$

$$y + 3x = 12k - 8y$$

$$y + 3x = 4(3k - 2y)$$

and so $y + 3x$ is a multiple of 4

$\therefore$ For any $x \sim y$, a consequence is that $y \sim x$ which is what it means to be symmetric. □

The relation is transitive.

Proof :

Assuming that $x \sim y$ and $y \sim z$ are multiples of 4 gives that,

$$x + 3y = 4m \quad \text{and} \quad y + 3z = 4n \qquad \text{for} \quad m, n \in \mathbb{Z}$$

$$\therefore \quad x + 3(4n - 3z) = 4m$$

$$x + 12n - 9z = 4m$$

$$x + 12n + 3z - 12z = 4m$$

$$x + 3z = 4m - 12n + 12z$$

$$x + 3z = 4(m - 3n + 3z)$$

and so $x + 3z$ is a multiple of 4

$\therefore$ For any $x \sim y$ and $y \sim x$, a consequence is that $x \sim z$ which is what it means to be transitive. □

Thus, as the three defined axioms are all satisfied the relation is an equivalence relation. □

[8 marks]

Answer 5

$$x \sim y \ \text{ if } \ 3x^2 - y^2 \ \text{ is divisible by 2,} \qquad x, y \in \mathbb{Z}$$

(i) The relation is reflexive.

Proof :

$$x \sim x \Leftrightarrow 3x^2 - x^2 = 2x^2 \ \text{ which is divisible by 2}$$

Hence x is related to itself which is what it means to be reflexive. ☐

The relation is symmetric.

Proof :

If it is assumed that $x \sim y$ is divisible by 2 then, in consequence,

$$3x^2 - y^2 = 2k \qquad \text{for some integer } k$$

$$3x^2 - y^2 + 4y^2 = 2k + 4y^2 \qquad \text{adding } 4y^2 \text{ to both sides}$$

$$3x^2 + 3y^2 = 2k + 4y^2$$

$$3x^2 - 4x^2 + 3y^2 = 2k + 4y^2 - 4x^2 \qquad \text{subtract } 4x^2 \text{ from both sides}$$

$$3y^2 - x^2 = 2\left(k + 2y^2 - 2x^2\right)$$

and so $3y^2 - x^2$ is divisible by 2

$\therefore$ For any $x \sim y$, a consequence is that $y \sim x$
which is what it means to be symmetric. ☐

The relation is transitive.

Proof :

Assuming that $x \sim y$ and $y \sim z$ are divisible by 2 gives that,

$$3x^2 - y^2 = 2m \quad \text{and} \quad 3y^2 - z^2 = 2n \qquad \text{for } m, n \in \mathbb{Z}$$

$$\therefore \ 3x^2 - \left(2n + z^2\right) = 2m$$

$$3x^2 - z^2 = 2\left(m + n\right)$$

and so $3x^2 - z^2$ is divisible by 2

$\therefore$ For any $x \sim y$ and $y \sim x$, a consequence is that
$x \sim z$ which is what it means to be transitive. ☐

Thus, as the three defined axioms are all satisfied the relation
is an equivalence relation. ☐

[8 marks]

(ii) There are two equivalence classes, one containing the even numbers the other the odd

[2 marks]

Answer 6

$$x \sim y \text{ if } |x - y| \leqslant 3 \qquad x, y \in \mathbb{R}$$

(i) The relation is reflexive.

Proof :

$$x \sim x \Leftrightarrow |x - x| = 0 \text{ which is less than 3}$$

Hence x is related to itself which is what it means to be reflexive. □

The relation is symmetric.

Proof :

$$x \sim y \Leftrightarrow |x - y| \leqslant 3$$

is algebraically the same as

$$y \sim x \Leftrightarrow |y - x| \leqslant 3$$

∴ For all $x, y \in \mathbb{R}$, if $x \sim y$ then $y \sim x$ and so the relation is symmetric. □

The relation is not transitive.

Proof :

For example, let $x = 0$, $y = 2$ and $z = 4$

Then $x \sim y \Leftrightarrow |0 - 2| = 2 \leqslant 3$ as required,

and $y \sim z \Leftrightarrow |2 - 4| = 2 \leqslant 3$ as required,

but $x \sim z \Leftrightarrow |0 - 4| = 4$ which is not less than 3

∴ For any $x \sim y$ and $y \sim z$, the desired consequence that would yield transitivity is that $x \sim z$ which is not always the case for this relation. The relation has failed to be transitive. □

The relation is not an equivalence relation.

Proof :

Of the three axioms that would have to be satisfied only the reflexive and symmetric properties holds. The relation fails to be transitive. □

[6 marks]

Answer 7

Abelian Group's Singletons Theorem
In an Abelian group, each conjugacy class comprises a single element.

Proof :
Let G be an Abelian group, and let x be any element of G.
The conjugacy class containing x is the set,

$$\{g\,x\,g^{-1} \;:\; g \in G\}$$

However, for any $g \in G$, the property of being Abelian necessitates $gx = xg$ with the consequence that,

$$\begin{aligned} g\,x\,g^{-1} &= (g\,x)\,g^{-1} && \text{Associativity} \\ &= (xg)\;g^{-1} && \text{Abelian property} \\ &= x\left(g\,g^{-1}\right) && \text{Associativity} \\ &= x\,e && \text{Inverses} \\ &= x && \text{Identity} \end{aligned}$$

In other words, the conjugacy class of x is $\{g\,x\,g^{-1} \;:\; g \in G\} = \{x\}$ which shows that each conjugacy class comprises a single element. □

In many respects, what this proof is saying is that the Abelian groups are the most straight forward and, perhaps, the least interesting. Non-Abelian groups are going to have richer and more diverse structures and progress in Group Theory is going to have to, sooner or later, be focussing on them.

[4 marks]

Answer 8

Conjugate Powers Theorem

If g conjugates x to y then g conjugates x^n to y^n for any integer n.

In other words, if $y = g\,x\,g^{-1}$ then $y^n = g\,x^n\,g^{-1}$ for $\mathbb{Z}$

Proof for positive integers:

When $n = 1$, $y^1 = g\,x^1\,g^{-1}$

$$\Leftrightarrow\ y = g\,x\,g^{-1}$$

which is the definition of conjugation and so, by definition, true

Assume that when $n = k$, $y^k = g\,x^k\,g^{-1}$
in which case...

$$\begin{aligned} y^{k+1} &= \left(y^k\right)\left(y^1\right) \\ &= \left(g\,x^k\,g^{-1}\right)\left(g\,x\,g^{-1}\right) \\ &= g\,x^k\left(g^{-1}\,g\right)x\,g^{-1} && \text{Associativity} \\ &= g\,x^k\,e\,x\,g^{-1} && \text{Inverses} \\ &= g\,x^{k+1}\,g^{-1} && \text{Identity} \end{aligned}$$

$\therefore$ if $y^k = g\,x^k\,g^{-1}$ is true when $n = k$, then it is also true when $n = k + 1$

Consequently, as $y^k = g\,x^k\,g^{-1}$ when $n = 1$, it is true for all $n \in \mathbb{Z}^+$

by mathematical induction. □

[6 marks]