1.5 Solutions

1.5.1 Answer to 1.3 Example

Begin by factorising out an x,

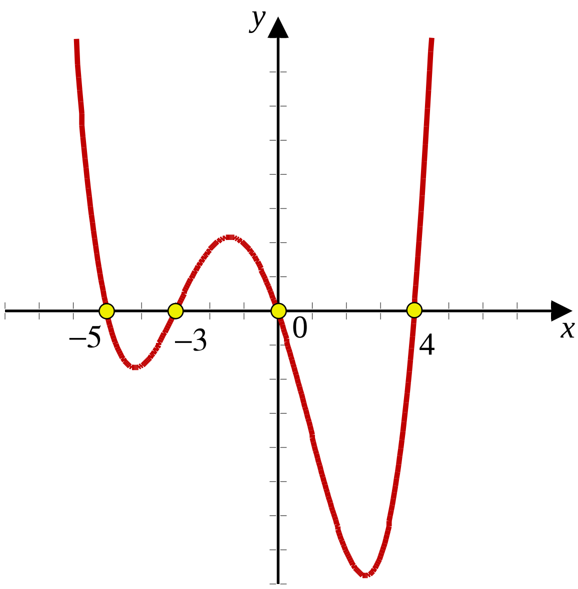
$$\begin{aligned} f(x) &= x^4 + 4x^3 - 17x^2 - 60x \\ &= x\left(x^3 + 4x^2 - 17x - 60\right) \end{aligned}$$

Then make use of the factor given in the question,

$$\begin{array}{r|rrrr} & x^2 & +\ 8x & +\ 15 & \\ \cline{2-5} x - 4 & x^3 + 4x^2 & -\ 17x & -\ 60 & \\ & x^3 - 4x^2 & & & \\ \cline{2-2} & 8x^2 & -\ 17x & & \\ & 8x^2 & -\ 32x & & \\ \cline{2-3} & & 15x & -\ 60 & \\ & & 15x & -\ 60 & \\ \cline{3-4} & & & 0 & \leftarrow \text{ As expected !} \end{array}$$

$$\begin{aligned} f(x) &= x\left(x^3 + 4x^2 - 17x - 60\right) \\ &= x(x-4)\left(x^2 + 8x + 15\right) \\ &= x(x-4)(x+3)(x+5) \end{aligned}$$

This has roots $x \in \{-5, -3, 0, 4\}$



[8 marks]

1.5.2 Answers to 5.3 Exercise

Answer 1

Begin by factorising out an x,

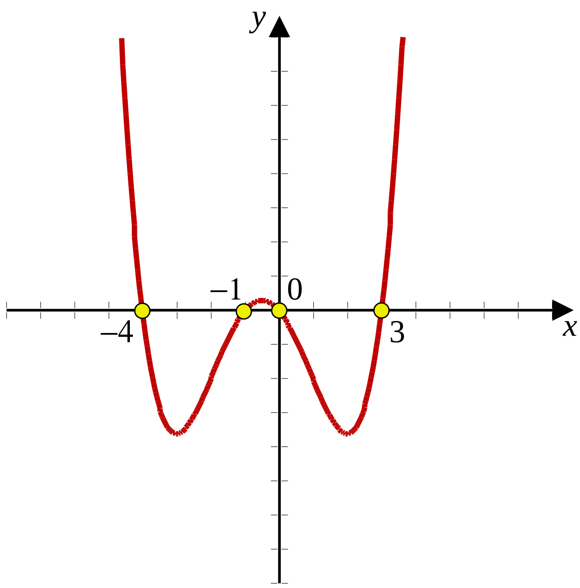
$$\begin{aligned} f(x) &= x^4 + 2x^3 - 11x^2 - 12x \\ &= x\left(x^3 + 2x^2 - 11x - 12\right) \end{aligned}$$

Then make use of the factor given in the question,

$$\begin{array}{r|rrrr} & & x^2 & +\ 5x & +\ 4 \\ \hline x - 3 & x^3 & +\ 2x^2 & -\ 11x & -\ 12 \\ & x^3 & -\ 3x^2 & & \\ \hline & & 5x^2 & -\ 11x & \\ & & 5x^2 & -\ 15x & \\ \hline & & & 4x & -\ 12 \\ & & & 4x & -\ 12 \\ \hline & & & & 0 \end{array} \quad \leftarrow \text{ As expected !}$$

$$\begin{aligned} f(x) &= x\left(x^3 + 2x^2 - 11x - 12\right) \\ &= x(x-3)\left(x^2 + 5x + 4\right) \\ &= x(x-3)(x+4)(x+1) \end{aligned}$$

This has roots $x \in \{-4, -1, 0, 3\}$



[8 marks]

Answer 2

$$f(x) = 6x^3 + 7x^2 - 23x - 30$$

(i) $f(1) = 6 \times 1^3 + 7 \times 1^2 - 23 \times 1 - 30 \Rightarrow f(1) = -40$

$f(2) = 6 \times 2^3 + 7 \times 2^2 - 23 \times 2 - 30 \Rightarrow f(2) = 0$

$f(3) = 6 \times 3^3 + 7 \times 3^2 - 23 \times 3 - 30 \Rightarrow f(3) = 126$

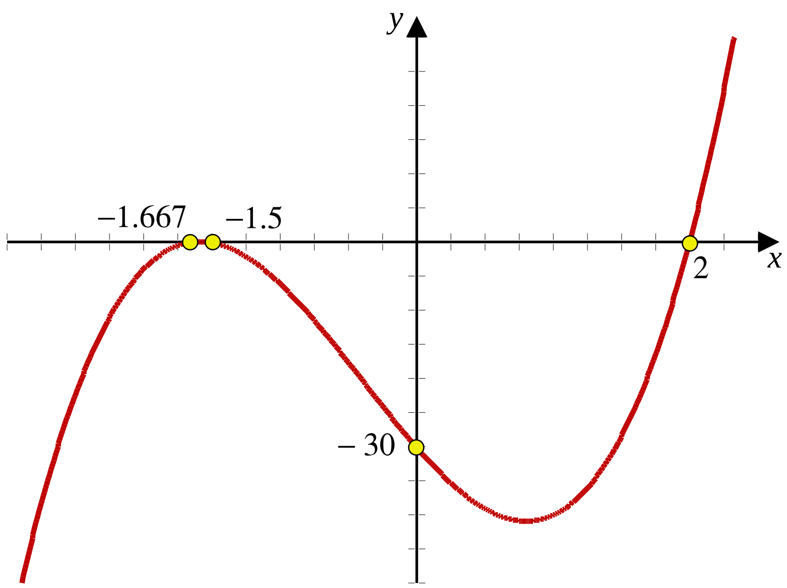
[3 marks]

(ii) From part (i) and the factor theorem, $(x - 2)$ is a factor of $f(x)$

$$
\begin{array}{r|l}
 & 6x^2 + 19x + 15 \\ \hline
x - 2 & 6x^3 + 7x^2 - 23x - 30 \\
 & \underline{6x^3 - 12x^2} \\
 & 19x^2 - 23x \\
 & \underline{19x^2 - 38x} \\
 & 15x - 30 \\
 & \underline{15x - 30} \\
 & 0 \leftarrow \text{As expected !}
\end{array}
$$

$$
\begin{aligned}
f(x) &= 6x^3 + 7x^2 - 23x - 30 \\
&= (x - 2)(6x^2 + 19x + 15) \\
&= (x - 2)(3x + 5)(2x + 3)
\end{aligned}
$$

This has roots $x \in \left\{ -\frac{5}{3}, -\frac{3}{2}, 2 \right\}$



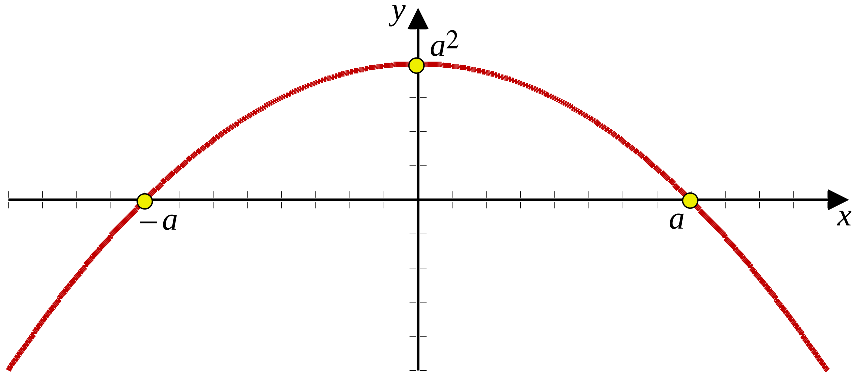
[8 marks]

Answer 3

As it's a difference of two squares,

$$\begin{aligned} p(x) &= a^2 - x^2 \\ &= (a + x)(a - x) \end{aligned}$$

This has roots $x \in \{-a,\ a\}$

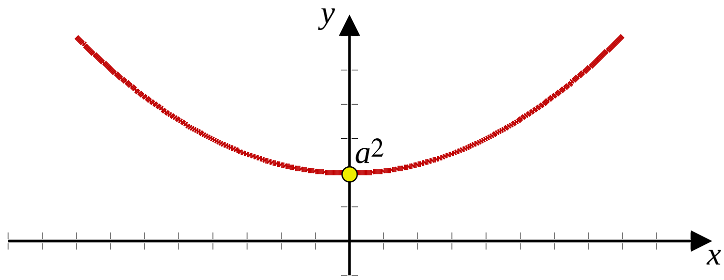


[3 marks]

Answer 4

$$q(x) = x^2 + a^2$$

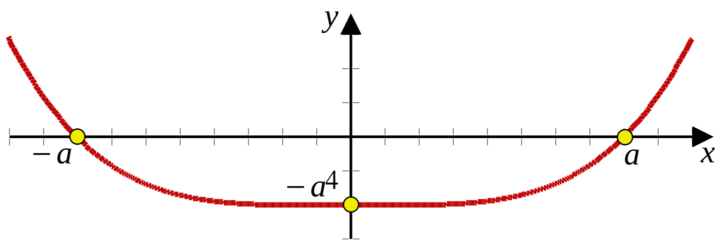
As the discriminant is $-4a^2$ (that is, negative) this has no real roots.
In other words, it does not cross the x-axis.



[3 marks]

Answer 5

$$\begin{aligned} r(x) &= x^4 - a^4 \\ &= \left(x^2 - a^2\right)\left(x^2 + a^2\right) \\ &= (x - a)(x + a)\left(x^2 + a^2\right) \end{aligned}$$



[3 marks]

Answer 6

Begin by factorising out $(-x)$,

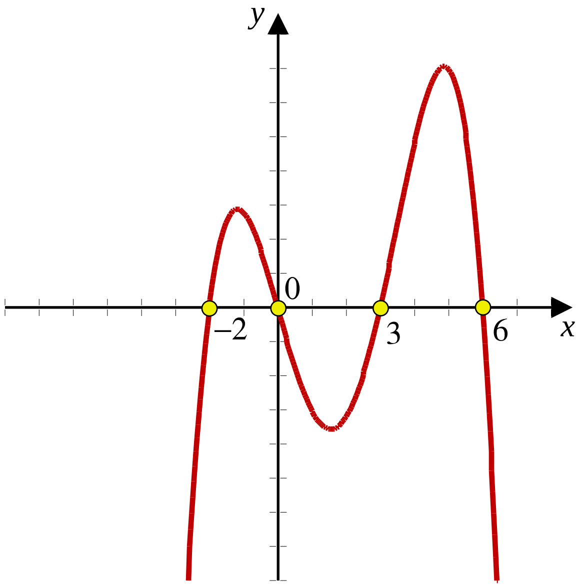
$$\begin{aligned} f(x) &= 7x^3 - 36x - x^4 \\ &= (-x)(-7x^2 + 36 + x^3) \\ &= (-x)(x^3 - 7x^2 + 36) \end{aligned}$$

Then make use of the factor given in the question,

$$\begin{array}{r|rrrr} & & x^2 & -\ 9x & +\ 18 \\ \hline x + 2 & x^3 & -\ 7x^2 & +\ 0x & +\ 36 \\ & x^3 & +\ 2x^2 & & \\ \hline & & -\ 9x^2 & +\ 0x & \\ & & -\ 9x^2 & -\ 18x & \\ \hline & & & 18x & +\ 36 \\ & & & 18x & +\ 36 \\ \hline & & & & 0 \end{array} \quad \leftarrow \text{As expected !}$$

$$\begin{aligned} f(x) &= (-x)(x^3 - 7x^2 + 36) \\ &= (-x)(x + 2)(x^2 - 9x + 18) \\ &= (-x)(x + 2)(x - 3)(x - 6) \end{aligned}$$

This has roots $x \in \{-2,\ 0,\ 3,\ 6\}$



[8 marks]

Answer 7

As ($x + 3$) is a factor,

$$\begin{aligned} f(-3) &= 0 \\ 3(-3)^3 + 2a(-3)^2 - 4(-3) + 5a &= 0 \\ -81 + 18a + 12 + 5a &= 0 \\ 23a &= 69 \\ a &= 3 \end{aligned}$$

[3 marks]

Answer 8

$$\begin{array}{r|l} & 3x^2 - 3x + 5 \\ \hline x + 3 & 3x^3 + 6x^2 - 4x + 15 \\ & \underline{3x^3 + 9x^2} \\ & \quad -3x^2 - 4x \\ & \quad \underline{-3x^2 - 9x} \\ & \qquad 5x + 15 \\ & \qquad \underline{5x + 15} \\ & \qquad\qquad 0 \end{array}$$

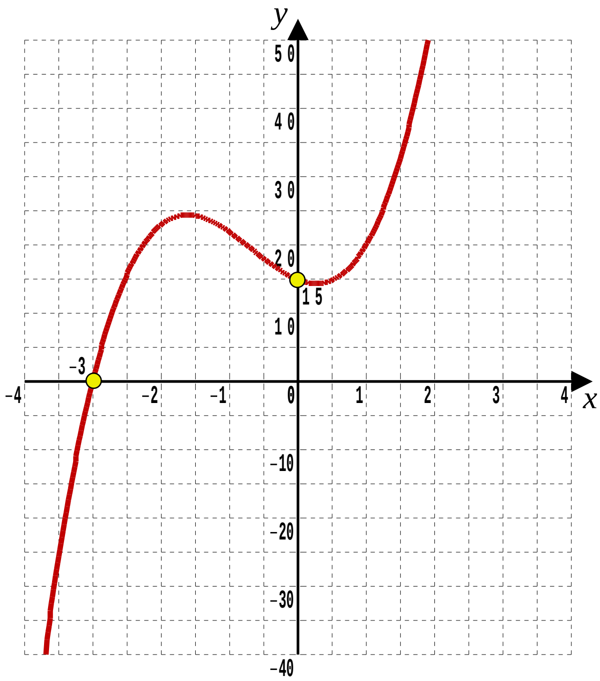
$\leftarrow$ As expected !

$$f(x) = (x + 3)(3x^2 - 3x + 5)$$

The discriminant of the quadratic factor is, $D = (-3)^2 - 4 \times 3 \times 5$

$= -51$

As this is negative this quadratic factor adds no x-axis crossing points.



[8 marks]

Answer 9

(i) $f(x) = (x - 10)(x^2 - 2x) + 12x$

$= x^3 - 2x^2 - 10x^2 + 20x + 12x$

$= x^3 - 12x^2 + 32x$

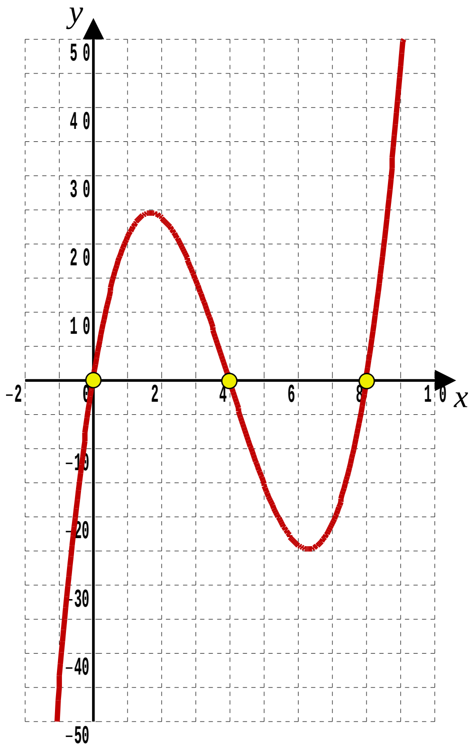
$= x(x^2 - 12x + 32)$ which is in the required form

[3 marks]

(ii) $f(x) = x(x - 4)(x - 8)$

[2 marks]

(iii)



[3 marks]

Answer 10

Observe that the graph crosses the x-axis when $x = -3$, $x = -2$ and $x = 1$

So the equation of the curve is of the form,

$$y = k(x+3)(x+2)(x-1) \text{ for some constant } k$$

The constant k need to be such that when $x = 0$, $y = -6$

$$-6 = k(3)(2)(-1)$$

$$k = 1$$

$$\begin{aligned} \therefore y &= (x+3)(x+2)(x-1) \\ &= \left(x^2 + 5x + 6\right)\left(x - 1\right) \\ &= x^3 + 5x^2 + 6x - x^2 - 5x - 6 \\ &= x^3 + 4x^2 + x - 6 \end{aligned}$$

[3 marks]

Answer 11

Observe that the graph crosses the x-axis when $x = -1$, $x = 2$ and $x = 3$

So the equation of the curve is of the form,

$$y = k(x+1)(x-2)(x-3) \text{ for some constant } k$$

The constant k need to be such that when $x = 0$, $y = 2$

$$2 = k(1)(-2)(-3)$$

$$k = \frac{1}{3}$$

$$\begin{aligned} \therefore y &= \frac{1}{3}(x+1)(x-2)(x-3) \\ &= \frac{1}{3}\left(x^2 - x - 2\right)\left(x - 3\right) \\ &= \frac{1}{3}\left(x^3 - x^2 - 2x - 3x^2 + 3x + 6\right) \\ &= \frac{1}{3}\left(x^3 - 4x^2 + x + 6\right) \\ &= \frac{1}{3}x^3 - \frac{4}{3}x^2 + \frac{1}{3}x + 2 \end{aligned}$$

[4 marks]

Answer 12

(i) Possible roots of $f(x) = x^4 + 6x^3 + x^2 - 24x + 16$ are

$$x = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16$$

[2 marks]

(ii) The full list is;

$f(+1) = 0, \quad f(2) = 36, \quad f(4) = 576, \quad f(8) = 7056, \quad f(16) = 90000$

$f(-1) = 36, \quad f(-2) = 36, \quad f(-4) = 0, \quad f(-8) = 1296, \quad f(-16) = 41616$

The two that can be found by the factor theorem are $x = 1$ and $x = -4$

[2 marks]

(iii) Working out the quadratic factor,

$$(x - 1)(x + 4) = x^2 + 3x - 4$$

$$
\begin{array}{r|rrrrr}
 & & & x^2 & +\ 3x & -\ 4 \\
\cline{2-6}
x^2 + 3x - 4 & x^4 & +\ 6x^3 & +\ x^2 & -\ 24x & +\ 16 \\
 & x^4 & +\ 3x^3 & -\ 4x^2 & & \\
\cline{2-4}
 & & 3x^3 & +\ 5x^2 & -\ 24x & \\
 & & 3x^3 & +\ 9x^2 & -\ 12x & \\
\cline{3-5}
 & & & -\ 4x^2 & -\ 12x & +\ 16 \\
 & & & -\ 4x^2 & -\ 12x & +\ 16 \\
\cline{4-6}
 & & & & & 0
\end{array}
$$

$$
\begin{aligned}
f(x) &= x^4 + 6x^3 + x^2 - 24x + 16 \\
&= \left(x^2 + 3x - 4\right)\left(x^2 + 3x - 4\right) \\
&= \left(x^2 + 3x - 4\right)^2 \\
&= (x - 1)^2 (x + 4)^2
\end{aligned}
$$