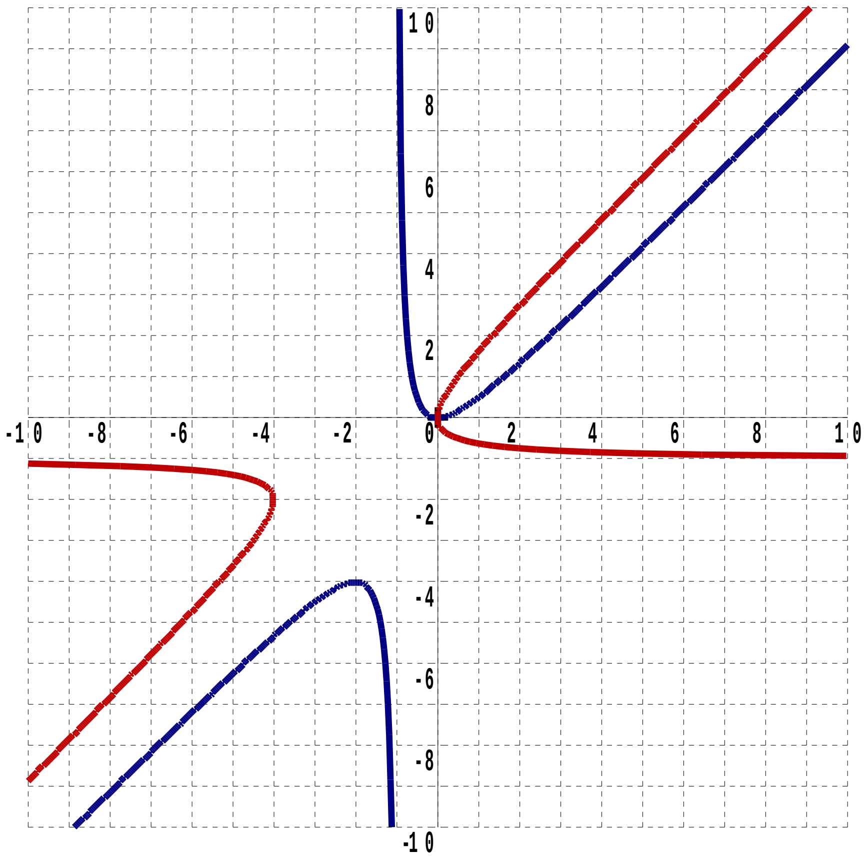
A-Level Pure Mathematics

Year 1
Graph Work
featuring
Graph Transformations

G • R • A • P • H W • O • R • K



The graphs of $y = \dfrac{x^2}{x+1}$ (in blue) and $x = \dfrac{y^2}{y+1}$ (in red)

Each is a reflection of the other in the line $y = x$

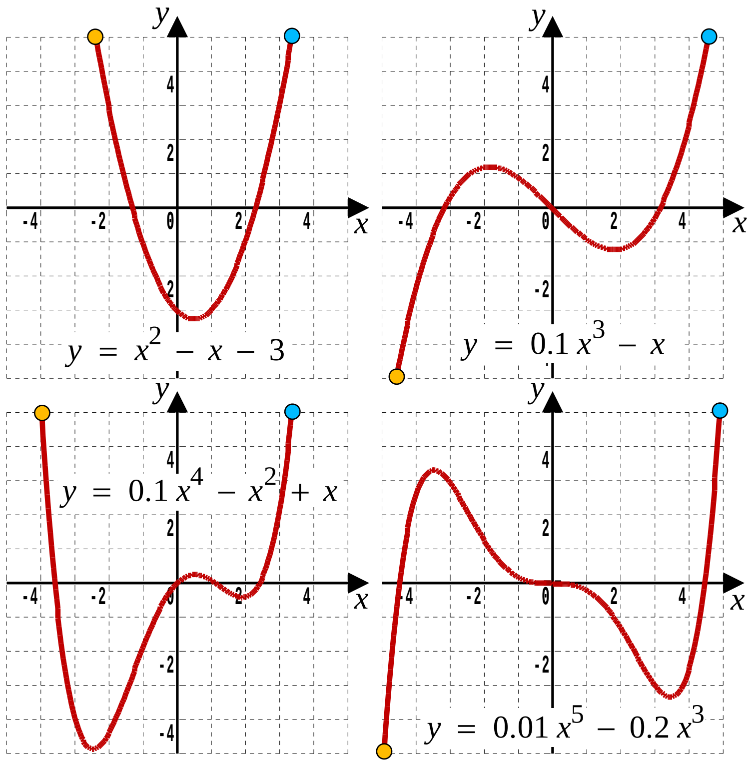
1.1 Graphing Polynomials

Faced with the task of quickly sketching a polynomial how best to proceed ?
It turns out that the degree of the polynomial is of crucial importance.

The Degree of a Polynomial
The degree of a polynomial the highest power of *x*.

1.2 Graphed Polynomials with Positive Highest Order Term

Here are a few examples of polynomials and their corresponding graphs.
For each, note the parity (odd or even) of the degree of the polynomial and, for large values of *x*, which quadrant the graph enters and exits.
(Remember that graphs are always read, like this sentence, from left to right)



Here is a summary of the key observations to be made,

Equation	Degree	Degree's Parity	Entry	Exit
$y = x^2 - x - 3$	2	even	2nd	1st
$y = 0.1x^3 - x$	3	odd	3rd	1st
$y = 0.1x^4 - x^2 + x$	4	even	2nd	1st
$y = 0.01x^5 - 0.2x^3$	5	odd	3rd	1st

The observations made suggest the first half of the following rule.

Large Values of *x* Rule for Polynomials

If the highest order term is positive then the exit is in the 1st quadrant.

Additionally,

- if the degree is even the entry is in the 2nd quadrant.
- if the degree is odd the entry is in the 3rd quadrant.

If the highest order term is negative then the exit is in the 4th quadrant.

Additionally,

- if the degree is even the entry is in the 3rd quadrant.
- if the degree is odd the entry is in the 2nd quadrant

1.3 Example

$$f(x) = x^4 + 4x^3 - 17x^2 - 60x$$

Given that $(x - 4)$ is a factor, factorise $f(x)$ completely and hence sketch the graph of $f(x)$ clearly marking all axis crossing points.

[8 marks]

1.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 74

Question 1

$$f(x) = x^4 + 2x^3 - 11x^2 - 12x$$

Given that $(x - 3)$ is a factor, factorise $f(x)$ completely and hence sketch the graph of $f(x)$ clearly marking all axis crossing points.

[8 marks]

Question 2

$$f(x) = 6x^3 + 7x^2 - 23x - 30$$

(i) Calculate the value of $f(1)$, the value of $f(2)$, and the value of $f(3)$

[3 marks]

The Factor Theorem

If, for a given polynomial function $p(x)$, $p(a) = 0$ (for some constant, a)

then $(x - a)$ is a factor of $p(x)$

(ii) Hence, using the factor theorem, factorise $f(x)$ completely and so sketch the graph of $f(x)$ clearly marking all axis crossing points.

[8 marks]

Question 3

$$p(x) = a^2 - x^2 \qquad \text{where } a \text{ is a positive constant}$$

Sketch the graph of $p(x)$ clearly making all axis crossing points in terms of a.

[3 marks]

Question 4

$$q(x) = x^2 + a^2 \qquad \text{where } a \text{ is a positive constant}$$

Sketch the graph of $q(x)$ clearly making all axis crossing points in terms of a.

[3 marks]

Question 5

$$r(x) = x^4 - a^4 \qquad \text{where } a \text{ is a positive constant}$$

Sketch the graph of $r(x)$ clearly making all axis crossing points in terms of a.

[3 marks]

Question 6

$$f(x) = 7x^3 - 36x - x^4$$

Given that $(x + 2)$ is a factor, factorise $f(x)$ completely and hence sketch the graph of $f(x)$ clearly marking all axis crossing points.

[8 marks]

Question 7

A-Level Examination Question from June 2019, Paper 1, Q1 (Edexcel)

$$f(x) = 3x^3 + 2ax^2 - 4x + 5a$$

Given that $(x + 3)$ is a factor of $f(x)$ find the value of the constant a

[3 marks]

Question 8

Continue the previous question by factorising $f(x)$ completely, using the value of a found, and so sketch the graph of $f(x)$ marking on its two axis crossing points.

[8 marks]

Question 9

Given that $f(x) = (x - 10)\left(x^2 - 2x\right) + 12x$

(i) Express $f(x)$ in the form $x\left(ax^2 + bx + c\right)$ where a, b and c are real constants

[3 marks]

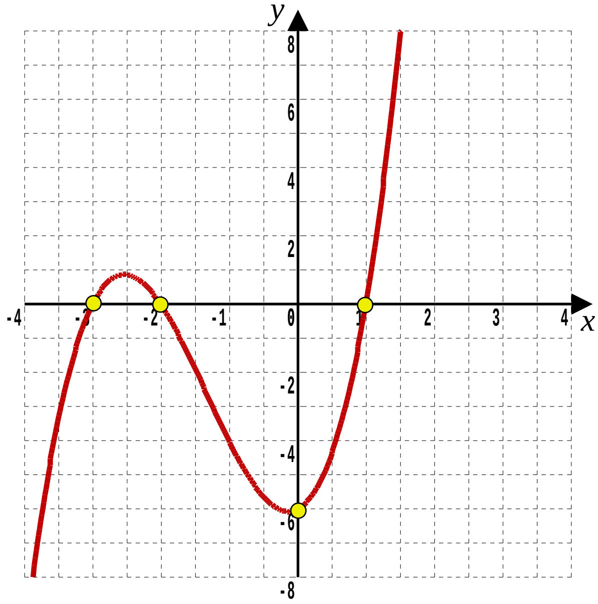
(ii) Hence factorise $f(x)$ completely.

[2 marks]

(iii) Sketch the graph of $y = f(x)$ showing clearly the points where the graph intersects the axes.

[3 marks]

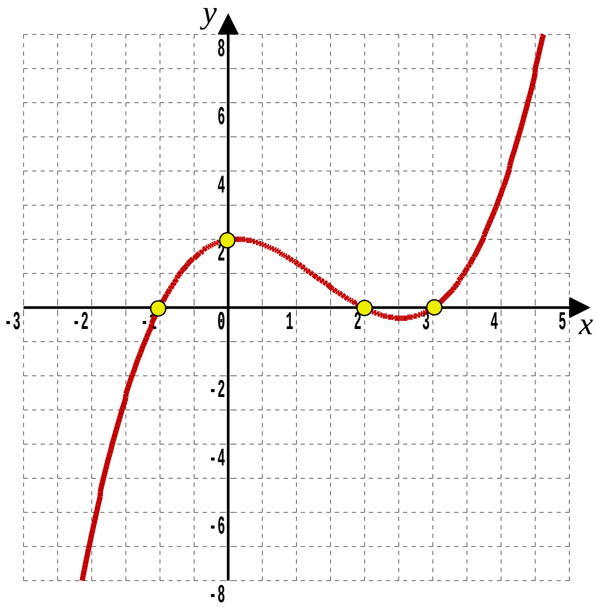
Question 10



The graph is of $y = x^3 + bx^2 + cx + d$, where b, c and d are real constants.
Find the values of b, c and d.

[3 marks]

Question 11



The graph is of $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are real constants.
Find the values of a, b, c and d.

[4 marks]

Question 12

$$f(x) = x^4 + 6x^3 + x^2 - 24x + 16$$

(i) Given that the roots of $f(x)$ have integer values, list the ten possible roots suggested by the term independent of x being 16.

[2 marks]

(ii) Use the factor theorem to find two roots of $f(x)$

[2 marks]

(iii) Hence, factorise $f(x)$ completely and so sketch the graph of $f(x)$ clearly marking all axis crossing points.

[8 marks]

1.5 Solutions

1.5.1 Answer to 1.3 Example

Begin by factorising out an x,

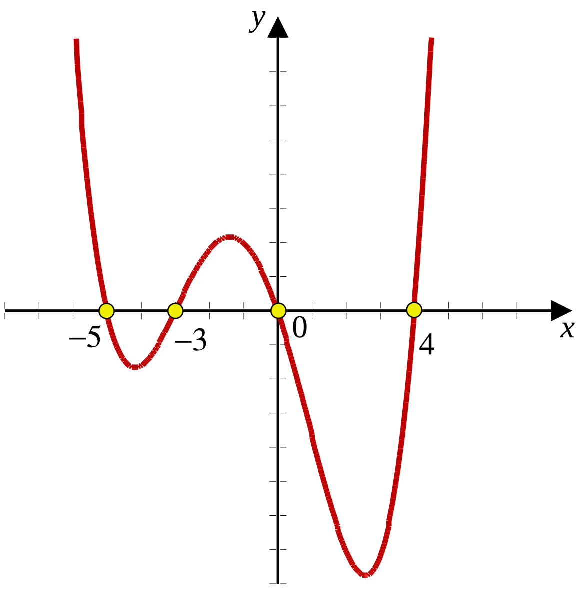
$$
\begin{aligned}
f(x) &= x^4 + 4x^3 - 17x^2 - 60x \\
&= x\left(x^3 + 4x^2 - 17x - 60\right)
\end{aligned}
$$

Then make use of the factor given in the question,

$$
\begin{array}{rrrrrrrrl}
 & & & x^2 & + & 8x & + & 15 & \\ \cline{2-8}
x - 4 & \multicolumn{1}{|r}{x^3} & + & 4x^2 & - & 17x & - & 60 & \\
 & x^3 & - & 4x^2 & & & & & \\ \cline{2-4}
 & & & 8x^2 & - & 17x & & & \\
 & & & 8x^2 & - & 32x & & & \\ \cline{4-6}
 & & & & & 15x & - & 60 & \\
 & & & & & 15x & - & 60 & \\ \cline{6-8}
 & & & & & & & 0 & \leftarrow \text{ As expected !}
\end{array}
$$

$$
\begin{aligned}
f(x) &= x\left(x^3 + 4x^2 - 17x - 60\right) \\
&= x(x-4)\left(x^2 + 8x + 15\right) \\
&= x(x-4)(x+3)(x+5)
\end{aligned}
$$

This has roots $x \in \{-5, -3, 0, 4\}$



[8 marks]

1.5.2 Answers to 5.3 Exercise

Answer 1

Begin by factorising out an x,

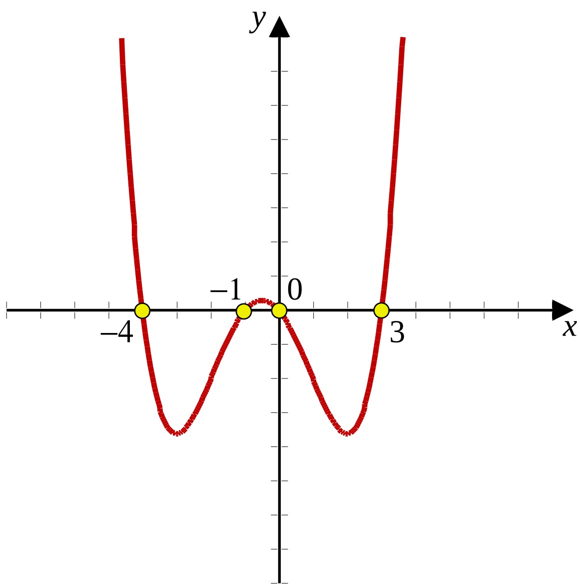
$$\begin{aligned} f(x) &= x^4 + 2x^3 - 11x^2 - 12x \\ &= x\left(x^3 + 2x^2 - 11x - 12\right) \end{aligned}$$

Then make use of the factor given in the question,

$$\begin{array}{r|l} & x^2 + 5x + 4 \\ \cline{2-2} x - 3 & x^3 + 2x^2 - 11x - 12 \\ & \underline{x^3 - 3x^2} \\ & 5x^2 - 11x \\ & \underline{5x^2 - 15x} \\ & 4x - 12 \\ & \underline{4x - 12} \\ & 0 \leftarrow \text{ As expected !} \end{array}$$

$$\begin{aligned} f(x) &= x\left(x^3 + 2x^2 - 11x - 12\right) \\ &= x\left(x - 3\right)\left(x^2 + 5x + 4\right) \\ &= x(x - 3)(x + 4)(x + 1) \end{aligned}$$

This has roots $x \in \{-4, -1, 0, 3\}$



[8 marks]

Answer 2

$$f(x) = 6x^3 + 7x^2 - 23x - 30$$

(i) $f(1) = 6 \times 1^3 + 7 \times 1^2 - 23 \times 1 - 30 \Rightarrow f(1) = -40$

$f(2) = 6 \times 2^3 + 7 \times 2^2 - 23 \times 2 - 30 \Rightarrow f(2) = 0$

$f(3) = 6 \times 3^3 + 7 \times 3^2 - 23 \times 3 - 30 \Rightarrow f(3) = 126$

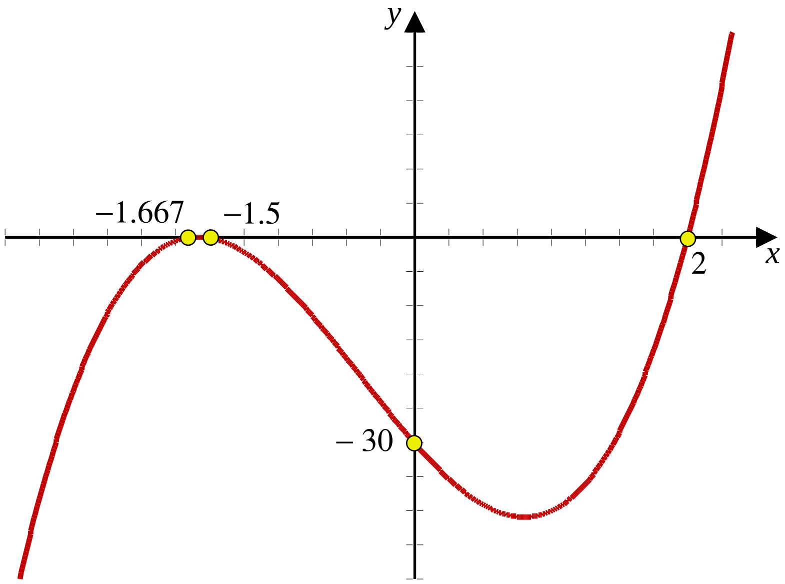
[3 marks]

(ii) From part (i) and the factor theorem, $(x - 2)$ is a factor of $f(x)$

$$\begin{array}{r|rrrr} & 6x^2 & +\ 19x & +\ 15 & \\ \hline x - 2 & 6x^3 + & 7x^2 & -\ 23x & -\ 30 \\ & 6x^3 - & 12x^2 & & \\ \hline & & 19x^2 & -\ 23x & \\ & & 19x^2 & -\ 38x & \\ \hline & & & 15x & -\ 30 \\ & & & 15x & -\ 30 \\ \hline & & & & 0 \end{array} \quad \leftarrow \text{As expected !}$$

$$\begin{aligned} f(x) &= 6x^3 + 7x^2 - 23x - 30 \\ &= (x-2)(6x^2 + 19x + 15) \\ &= (x-2)(3x+5)(2x+3) \end{aligned}$$

This has roots $x \in \left\{-\frac{5}{3}, -\frac{3}{2}, 2\right\}$



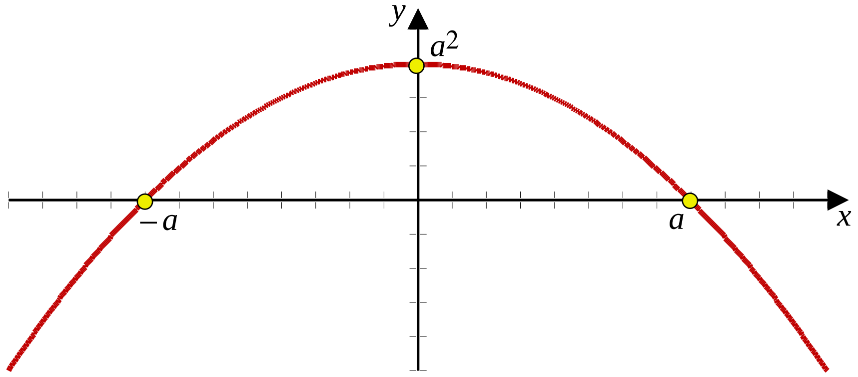
[8 marks]

Answer 3

As it's a difference of two squares,

$$\begin{aligned} p(x) &= a^2 - x^2 \\ &= (a + x)(a - x) \end{aligned}$$

This has roots $x \in \{-a,\ a\}$

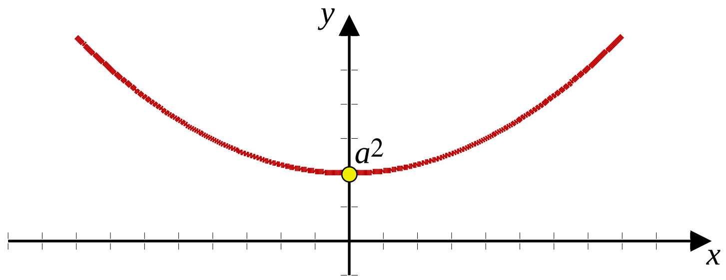


[3 marks]

Answer 4

$$q(x) = x^2 + a^2$$

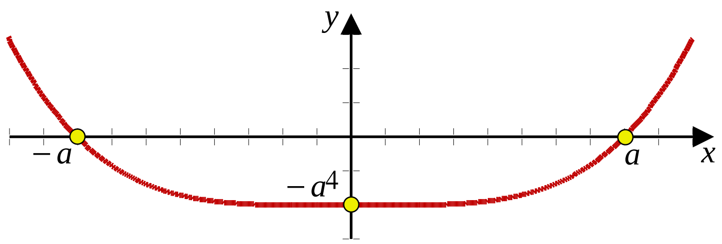
As the discriminant is $-4a^2$ (that is, negative) this has no real roots.
In other words, it does not cross the x-axis.



[3 marks]

Answer 5

$$\begin{aligned} r(x) &= x^4 - a^4 \\ &= \left(x^2 - a^2\right)\left(x^2 + a^2\right) \\ &= (x - a)(x + a)\left(x^2 + a^2\right) \end{aligned}$$



[3 marks]

Answer 6

Begin by factorising out $(-x)$,

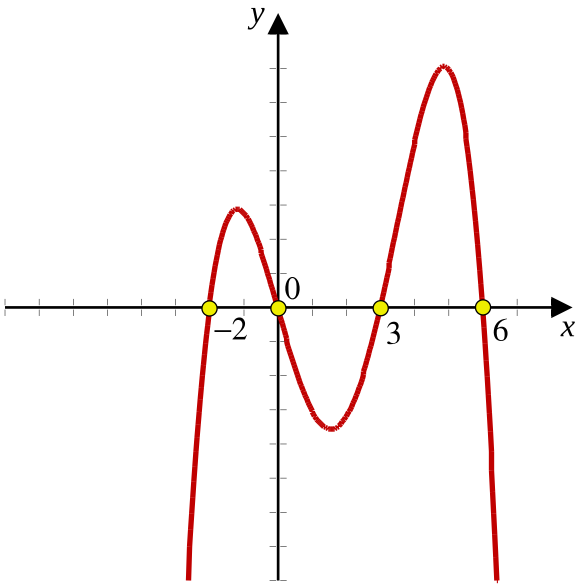
$$\begin{aligned} f(x) &= 7x^3 - 36x - x^4 \\ &= (-x)(-7x^2 + 36 + x^3) \\ &= (-x)(x^3 - 7x^2 + 36) \end{aligned}$$

Then make use of the factor given in the question,

$$\begin{array}{r|l} & \phantom{x^3 -{}} x^2 - 9x + 18 \\ \hline x + 2 & x^3 - 7x^2 + 0x + 36 \\ & x^3 + 2x^2 \\ \hline & - 9x^2 + 0x \\ & - 9x^2 - 18x \\ \hline & \phantom{x^3 - 9x^2 +{}} 18x + 36 \\ & \phantom{x^3 - 9x^2 +{}} 18x + 36 \\ \hline & 0 \quad \leftarrow \text{ As expected !} \end{array}$$

$$\begin{aligned} f(x) &= (-x)(x^3 - 7x^2 + 36) \\ &= (-x)(x + 2)(x^2 - 9x + 18) \\ &= (-x)(x + 2)(x - 3)(x - 6) \end{aligned}$$

This has roots $x \in \{-2,\ 0,\ 3,\ 6\}$



[8 marks]

Answer 7

As ($x + 3$) is a factor,

$$\begin{aligned} f(-3) &= 0 \\ 3(-3)^3 + 2a(-3)^2 - 4(-3) + 5a &= 0 \\ -81 + 18a + 12 + 5a &= 0 \\ 23a &= 69 \\ a &= 3 \end{aligned}$$

[3 marks]

Answer 8

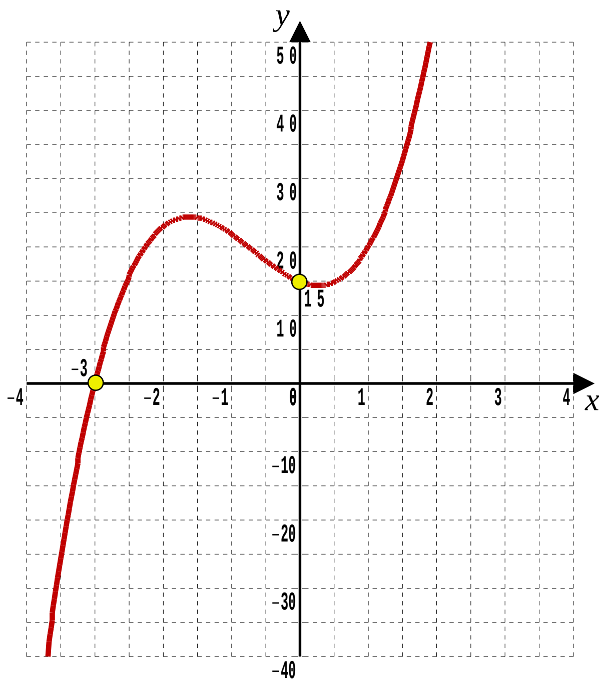
$$\begin{array}{r|rrrrrrr} & 3x^2 & & - & 3x & + & 5 & \\ \hline x + 3 & 3x^3 & + & 6x^2 & - & 4x & + & 15 \\ & 3x^3 & + & 9x^2 & & & & \\ \cline{2-4} & & & -3x^2 & - & 4x & & \\ & & & -3x^2 & - & 9x & & \\ \cline{4-6} & & & & & 5x & + & 15 \\ & & & & & 5x & + & 15 \\ \cline{6-8} & & & & & & & 0 \end{array} \quad \leftarrow \text{ As expected !}$$

$$f(x) = (x + 3)(3x^2 - 3x + 5)$$

The discriminant of the quadratic factor is, $D = (-3)^2 - 4 \times 3 \times 5$

$$= -51$$

As this is negative this quadratic factor adds no x-axis crossing points.



[8 marks]

Answer 9

(i) $f(x) = (x - 10)(x^2 - 2x) + 12x$

$= x^3 - 2x^2 - 10x^2 + 20x + 12x$

$= x^3 - 12x^2 + 32x$

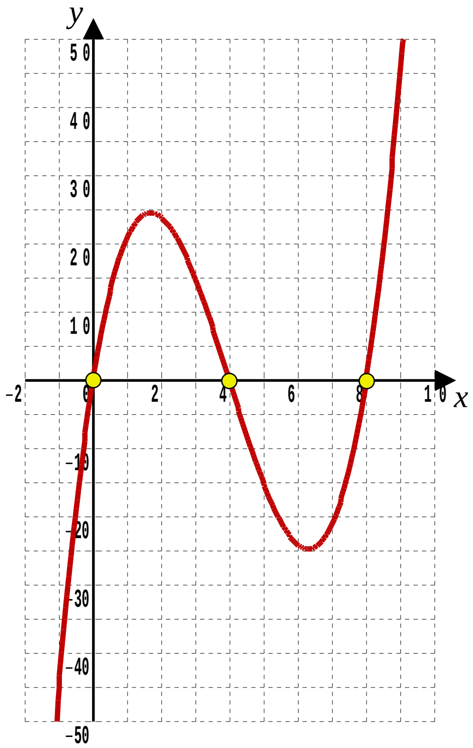
$= x(x^2 - 12x + 32)$ which is in the required form

[3 marks]

(ii) $f(x) = x(x - 4)(x - 8)$

[2 marks]

(iii)



[3 marks]

Answer 10

Observe that the graph crosses the x-axis when $x = -3$, $x = -2$ and $x = 1$

So the equation of the curve is of the form,

$$y = k(x+3)(x+2)(x-1) \text{ for some constant } k$$

The constant k need to be such that when $x = 0$, $y = -6$

$$-6 = k(3)(2)(-1)$$

$$k = 1$$

$$\begin{aligned} \therefore y &= (x+3)(x+2)(x-1) \\ &= \left(x^2 + 5x + 6\right)\left(x - 1\right) \\ &= x^3 + 5x^2 + 6x - x^2 - 5x - 6 \\ &= x^3 + 4x^2 + x - 6 \end{aligned}$$

[3 marks]

Answer 11

Observe that the graph crosses the x-axis when $x = -1$, $x = 2$ and $x = 3$

So the equation of the curve is of the form,

$$y = k(x+1)(x-2)(x-3) \text{ for some constant } k$$

The constant k need to be such that when $x = 0$, $y = 2$

$$2 = k(1)(-2)(-3)$$

$$k = \frac{1}{3}$$

$$\begin{aligned} \therefore y &= \frac{1}{3}(x+1)(x-2)(x-3) \\ &= \frac{1}{3}\left(x^2 - x - 2\right)\left(x - 3\right) \\ &= \frac{1}{3}\left(x^3 - x^2 - 2x - 3x^2 + 3x + 6\right) \\ &= \frac{1}{3}\left(x^3 - 4x^2 + x + 6\right) \\ &= \frac{1}{3}x^3 - \frac{4}{3}x^2 + \frac{1}{3}x + 2 \end{aligned}$$

[4 marks]

Answer 12

(i) Possible roots of $f(x) = x^4 + 6x^3 + x^2 - 24x + 16$ are

$$x = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16$$

[2 marks]

(ii) The full list is;

$f(+1) = 0, \quad f(2) = 36, \; f(4) = 576, \; f(8) = 7056, \; f(16) = 90000$

$f(-1) = 36, \; f(-2) = 36, \; f(-4) = 0, \quad f(-8) = 1296, \; f(-16) = 41616$

The two that can be found by the factor theorem are $x = 1$ and $x = -4$

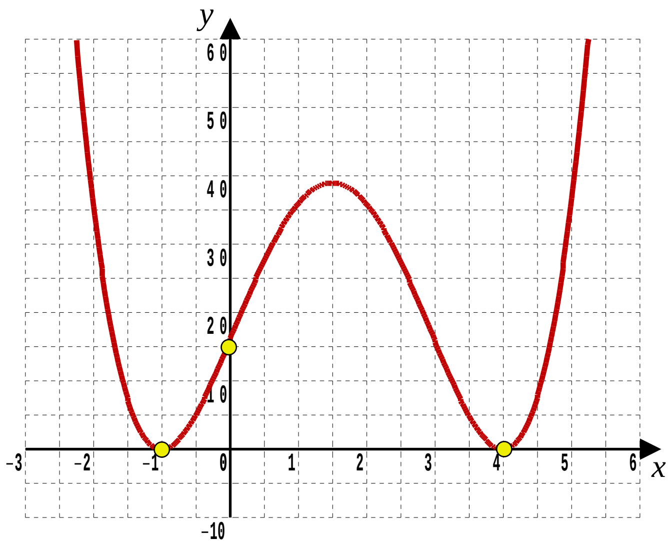
[2 marks]

(iii) Working out the quadratic factor,

$$(x - 1)(x + 4) = x^2 + 3x - 4$$

$$\begin{array}{r|rrrrr} & x^2 & + 3x & - 4 & & \\ \cline{2-6} x^2 + 3x - 4 & x^4 + 6x^3 & + x^2 & - 24x & + 16 & \\ & x^4 + 3x^3 & - 4x^2 & & & \\ \cline{2-3} & 3x^3 & + 5x^2 & - 24x & & \\ & 3x^3 & + 9x^2 & - 12x & & \\ \cline{2-4} & & - 4x^2 & - 12x & + 16 & \\ & & - 4x^2 & - 12x & + 16 & \\ \cline{3-5} & & & & 0 & \end{array}$$

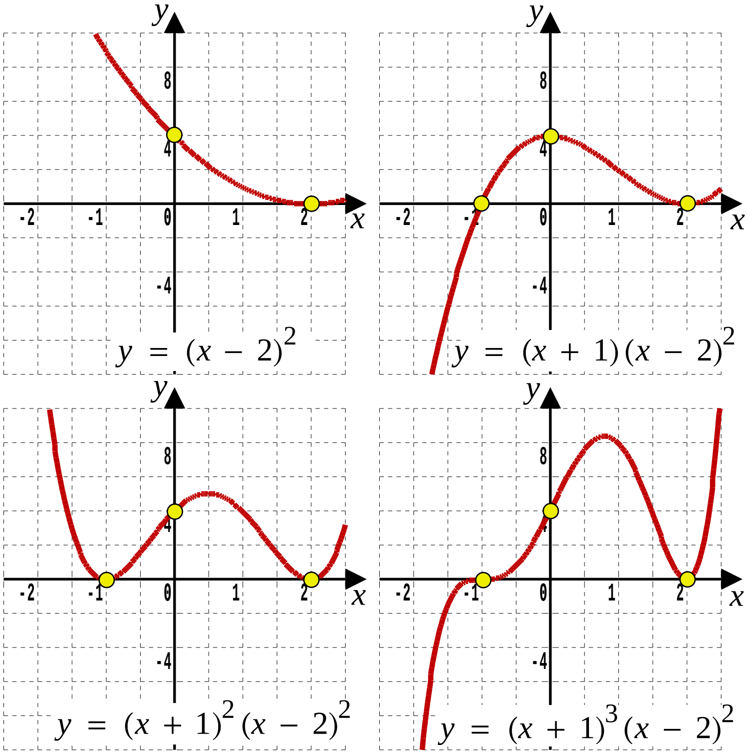
$$\begin{aligned} f(x) &= x^4 + 6x^3 + x^2 - 24x + 16 \\ &= \left(x^2 + 3x - 4\right)\left(x^2 + 3x - 4\right) \\ &= \left(x^2 + 3x - 4\right)^2 \\ &= (x - 1)^2 (x + 4)^2 \end{aligned}$$



Lesson 2

2.1 Multiplicity of Roots

Given the function, $f(x) = (x+1)^n (x-2)^2$ a mathematicain would say that that there is a root of multiplicity n at $x = -1$ and a root of multiplicity 2 at $x = 2$. The following graphs show what happens as the multiplicity of the root at $x = -1$, n, increments from 0 to 3.
In particular, observe the activity around the number – 1 on the x-axis.



Equation	n	parity of n	$x = -1$ activity	$x = 2$ activity
$y = (x-2)^2$	0	even	no root	touch from above
$y = (x+1)(x-2)^2$	1	odd	cross from below	touch from above
$y = (x+1)^2(x-2)^2$	2	even	touch from above	touch from above
$y = (x+1)^3(x-2)^2$	3	odd	cross from below	touch from above

The observations made suggest the following rule,

Cross or Touch Rule for Polynomial Roots

Given a polynomial of the form,

$$f(x) = (x - a)^n g(x)$$

the root at $x = a$ will be a *cross* if n is odd or a *touch* if n is even.

(Provided that $x = a$ is not also a root of the polynomial $g(x)$)
Whether the *cross* or *touch* is from above or below depends upon how the roots of $g(x)$ are distributed about $x = a$

2.2 The "Together" Sketches

Sketch each of the following curves, marking all intersections with the axes.

(i) $y = (x + 3)^3 (x - 1)^2 (x - 2)^2$

(ii) $y = x - x^3$

(iii) $y = (x + 3)^2 (4 - x)^2$

[9 marks]

2.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 70

Question 1

Sketch each of the following curves, marking all intersections with the axes.

(i) $y = (x)^2(x+2)^2(x+3)$

(ii) $y = x - x^5$

(iii) $y = (3-x)^3(x+2)^2$

[9 marks]

Question 2

(i) On the same axes sketch the curve $y = \left(x^2 - 1\right)\left(x - 2\right)$ and the line $y = 14x + 2$

[4 marks]

(ii) Use algebra to find the coordinates of the points of intersection.

[3 marks]

Question 3

(i) On the same axes sketch the curve with equations $y = (x - 2)(x + 2)^2$ and the curve with equation $y = -x^2 - 8$

[4 marks]

(ii) Use algebra to find the coordinates of the points of intersection.

[3 marks]

Question 4

A-Level Examination Question from January 2007, Paper C1, Q10 (Edexcel)

(a) On the same axes sketch the graphs of the curves with equations,

(i) $y = x^2(x - 2)$

(ii) $y = x(6 - x)$

and indicate on your sketches the coordinates of all points where the curves cross the x-axis.

[6 marks]

(b) Use algebra to find the coordinates of the points where the graphs intersect.

[7 marks]

Question 5

$$f(x) = x^3 - 2x^2 + px + 36 \text{ where } p \in \mathbb{Z}$$

(i) Given that $x = 3$ is a root of $f(x)$ determine the value of p

[3 marks]

(ii) Factorise $f(x)$ completely.

[4 marks]

(iii) Sketch the graph of $f(x)$ marking on all axis intersections.

[3 marks]

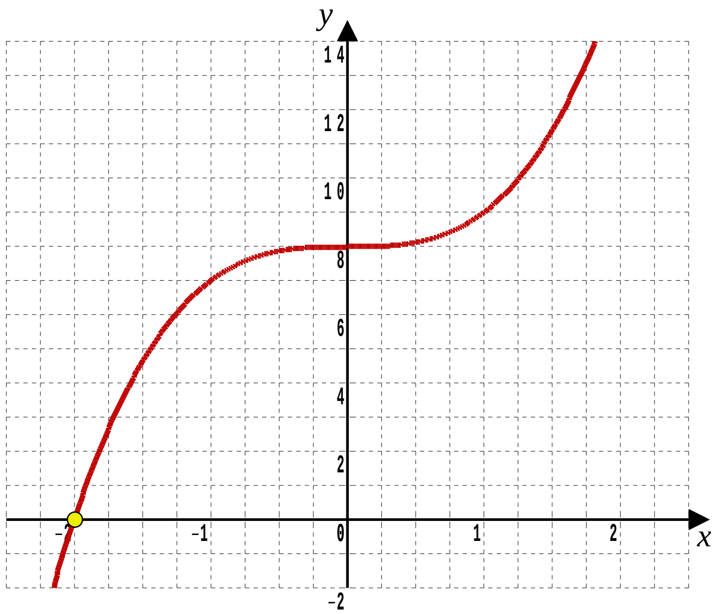
Question 6

The graph below, $y = f(x)$, is of the function $f(x) = x^3 + 8$

From the graph it is clear that $f(x)$ has only one root at $x = -2$

Mathematically, this root can be found by solving the equation $f(x) = 0$

This question looks at another mathematical way of showing there is only one root.



(i) Carry out a polynomial long division to show that,

$$x^3 + 8 = (x + 2)(x^2 + px + q)$$

where p and q are integers the values of which are to be found.

HINT: $x + 2 \overline{\big) x^3 + 0x^2 + 0x + 8}$

[3 marks]

(ii) By considering the discriminant of the factor $(x^2 + px + q)$ show that this quadratic factor yields no further roots of $f(x)$

[3 marks]

Question 7

$$f(x) = x^6 - 3x^5 - 15x^4 + 35x^3 + 90x^2 - 108x - 216$$

It is known that $f(x)$ has two distinct roots each of multiplicity three.

Thus, $f(x) = (x + p)^3 (x + q)^3$ where $p, q \in \mathbb{Z}$ with $p < q$

(i) Explain how the factor theorem proves that neither $(x - 1)$ nor $(x + 1)$ can be factors of $f(x)$

[3 marks]

(ii) By writing 216 as a product of primes, list two candidate possible factorisations of $f(x)$ keeping in mind that $p^3 q^3 = 216$.

[3 marks]

(iii) Use the factor theorem to determine which of the two candidates factorisations is the actual factorisation of $f(x)$

State the value of p and the value of q

[3 marks]

(iv) Sketch the graph of $f(x)$ marking on all axis intersections.

[3 marks]

Question 8

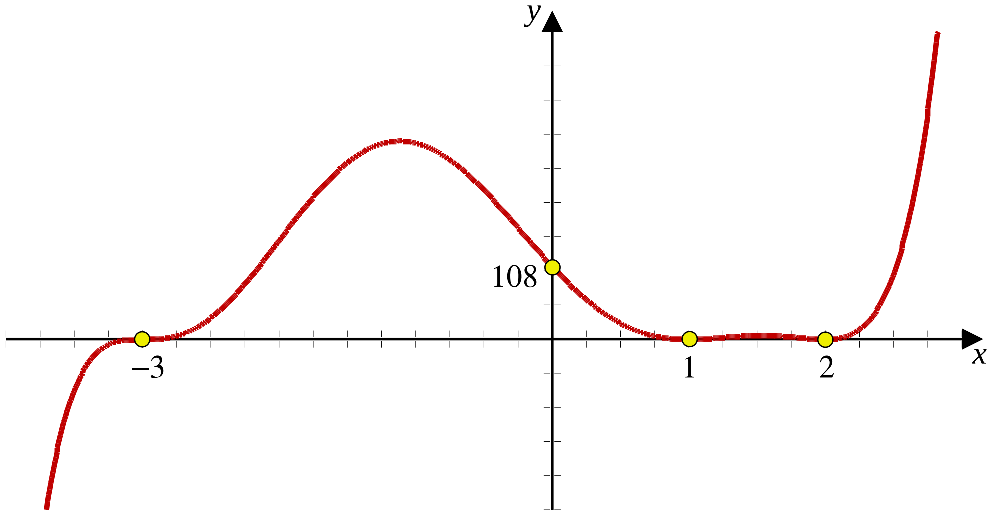
$$f(x) = 3x^3 + x^2 - x \quad \text{and} \quad g(x) = 2x(x - 1)(x + 1)$$

Show algebraically that the graphs of $f(x)$ and $g(x)$ have only one point of intersection, and hence find the coordinates of this point.

[6 marks]

2.4.1 Solutions to 2.2 The Together Sketches

(i) $y = (x+3)^3(x-1)^2(x-2)^2$



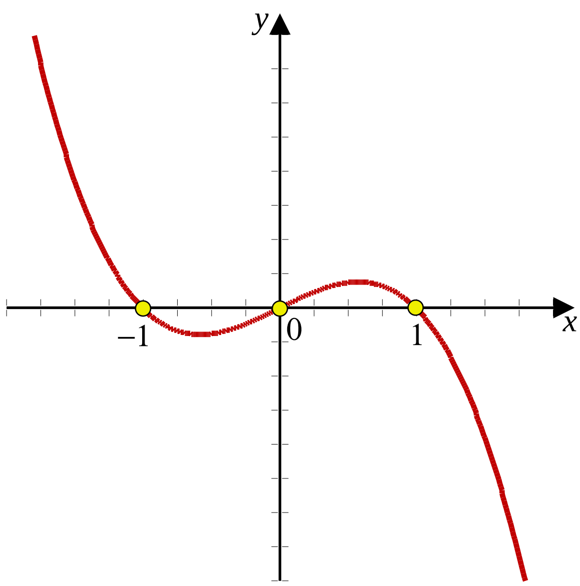
[3 marks]

(ii) $y = x - x^3$

$= x\left(1 - x^2\right)$

$= x(1-x)(1+x)$

$= -x(x-1)(x+1)$



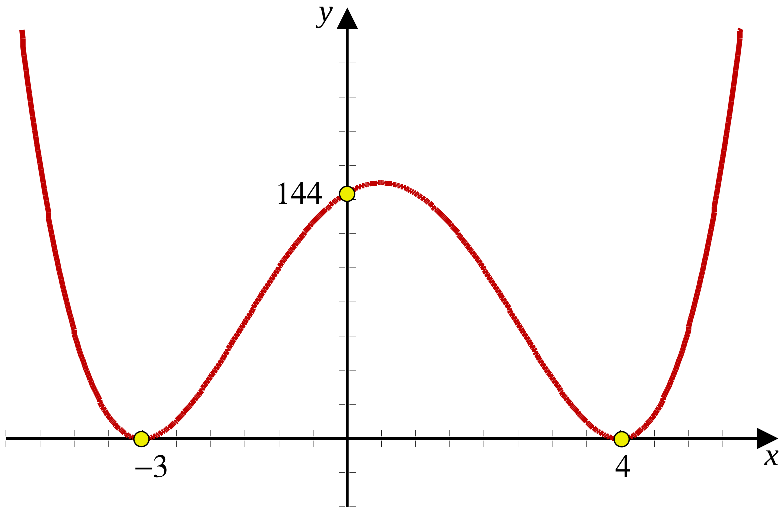
[3 marks]

(iii) $y = (x + 3)^2 (4 - x)^2$

$= (x + 3)^2 [(4 - x)]^2$

$= (x + 3)^2 [(-1)(x - 4)]^2$

$= (x + 3)^2 (x - 4)^2$

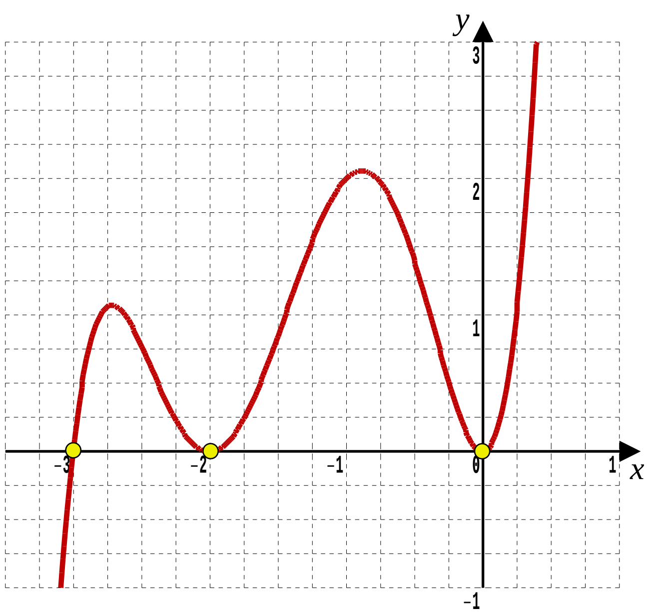


[3 marks]

2.4.2 Solutions to 2.3 Exercise

Answer 1

(i) $y = (x)^2 (x + 2)^2 (x + 3)$



[3 marks]

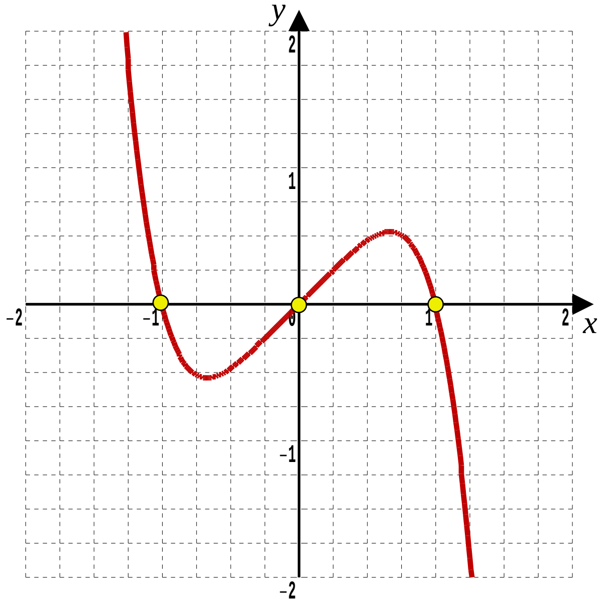
(ii) $y = x - x^5$

$$= x\left(1 - x^4\right)$$

$$= x\left(1 - x^2\right)\left(1 + x^2\right)$$

$$= x\left(1 - x\right)\left(1 + x\right)\left(1 + x^2\right)$$

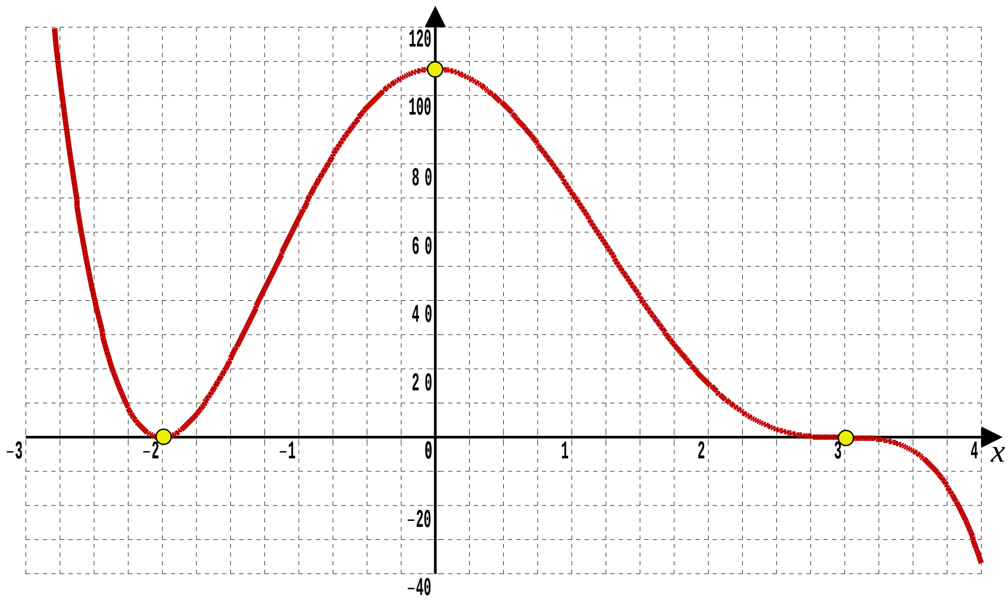
$$= (-1)\,x\,(x - 1)(x + 1)\left(x^2 + 1\right)$$



[3 marks]

(iii) $y = (3 - x)^3 (x + 2)^2$

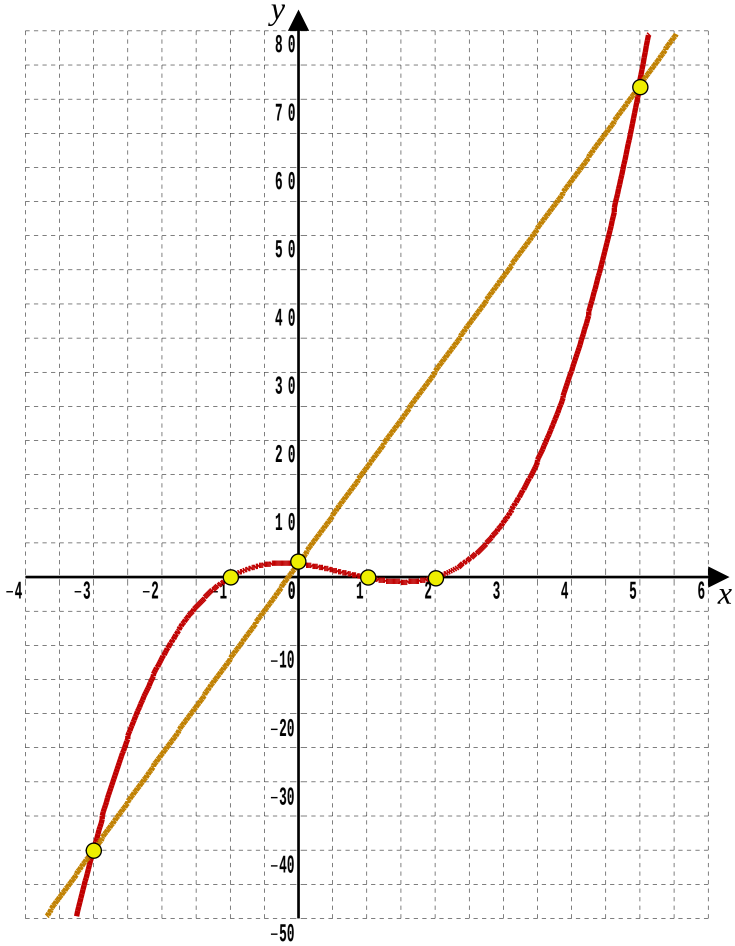
$$= (-1)(x - 3)^3 (x + 2)^2$$



[3 marks]

Answer 2

(i) The curve $y = \left(x^2 - 1\right)\left(x - 2\right)$ and the line $y = 14x + 2$



[4 marks]

(ii) To find the points of intersection, using algebra,

$$\left(x^2 - 1\right)\left(x - 2\right) = 14x + 2$$

$$x^3 - 2x^2 - x + 2 - 14x - 2 = 0$$

$$x^3 - 2x^2 - 15x = 0$$

$$x\left(x^2 - 2x - 15\right) = 0$$

$$x(x - 5)(x + 3) = 0$$

Either $x = -3$ or $x = 0$ or $x = 5$

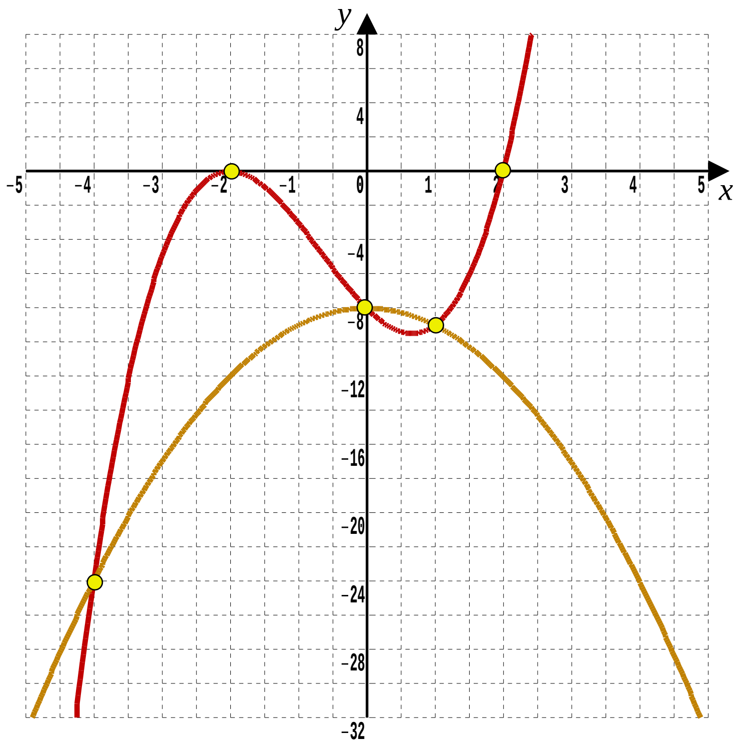
Now use $y = 14x + 2$ to get the intersection points...

$$(-3,\ 40),\ \left(0,\ 2\right),\ \left(5,\ 72\right)$$

[3 marks]

Answer 3

(i) The cubic curve $y = (x - 2)(x + 2)^2$ and quadratic $y = -x^2 - 8$



[4 marks]

(ii) To find the points of intersection, using algebra,

$$(x - 2)(x + 2)^2 = -x^2 - 8$$

$$\left(x^2 - 4\right)(x + 2) = -x^2 - 8$$

$$x^3 + 2x^2 - 4x - 8 = -x^2 - 8$$

$$x^3 + 3x^2 - 4x = 0$$

$$x\left(x^2 + 3x - 4\right) = 0$$

$$x(x + 4)\ (x - 1) = 0$$

Either $x = -4$ or $x = 0$ or $x = 1$

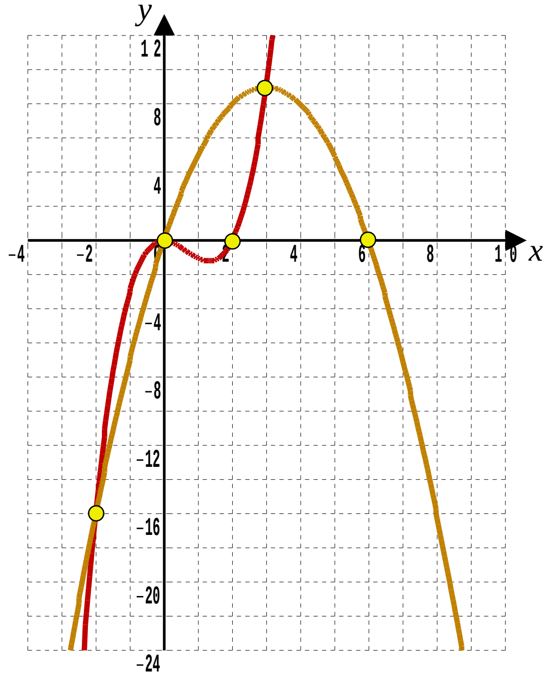
Now use $y = (x - 2)(x + 2)^2$ to get the intersection points...

$$(-4, -24),\ \left(0, -8\right),\ \left(1, -9\right)$$

[3 marks]

Answer 4

(i) The cubic curve $y = x^2(x - 2)$ and quadratic $y = x(6 - x)$



[6 marks]

(b) To find the points of intersection, using algebra,

$$x^2(x - 2) = x(6 - x)$$

$$x^3 - 2x^2 = 6x - x^2$$

$$x^3 - x^2 - 6x = 0$$

$$x\left(x^2 - x - 6\right) = 0$$

$$x(x + 2)(x - 3) = 0$$

Either $x = -2$ or $x = 0$ or $x = 3$

Now use $y = x(6 - x)$ to get the intersection points...

$$(-2, -16), \ (0, 0), \ (3, 9)$$

[7 marks]

Answer 5

$$f(x) = x^3 - 2x^2 + px + 36$$

(i) By the factor theorem, $f(3) = 0$

$$(3)^3 - 2(3)^2 + p(3) + 36 = 0$$

$$27 - 18 + 3p + 36 = 0$$

$$3p = -45$$

$$p = -15$$

[3 marks]

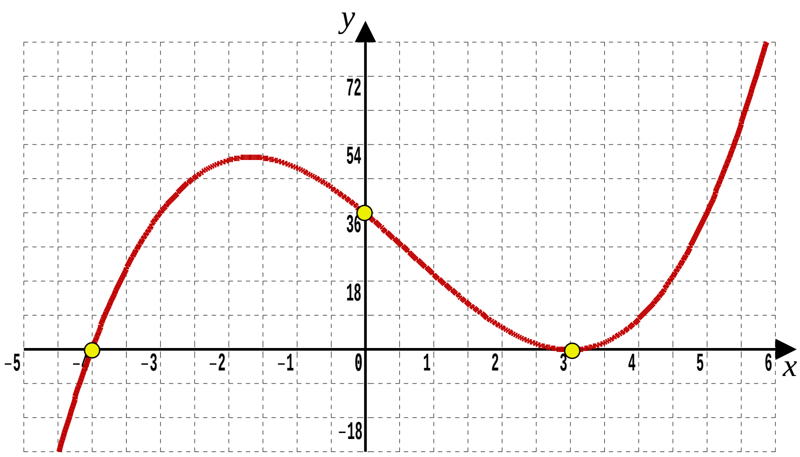
(ii)

$$\begin{array}{r|l}
 & \phantom{x^3 - {}} x^2 + x - 12 \\
\cline{2-2}
x - 3 & x^3 - 2x^2 - 15x + 36 \\
 & \underline{x^3 - 3x^2} \\
 & \phantom{x^3 - {}} x^2 - 15x \\
 & \phantom{x^3 - {}} \underline{x^2 - 3x} \\
 & - 12x + 36 \\
 & \underline{- 12x + 36} \\
 & 0 \ \leftarrow \text{As expected !}
\end{array}$$

$$\begin{aligned}
f(x) &= x^3 - 2x^2 - 15x + 36 \\
&= (x - 3)\left(x^2 + x - 12\right) \\
&= (x - 3)(x - 3)(x + 4) \\
&= (x - 3)^2(x + 4)
\end{aligned}$$

[4 marks]

(iii)



[3 marks]

Answer 6

(i)

$$
\begin{array}{r|l}
 & \quad\quad\quad x^2 \quad\quad 2x + 4 \\
\cline{2-2}
x + 2 & x^3 + {\color{red}0}x^2 + {\color{red}0}x + 8 \\
 & x^3 + 2x^2 \\
\cline{2-2}
 & \quad\quad - 2x^2 + 0x \\
 & \quad\quad - 2x^2 - 4x \\
\cline{2-2}
 & \quad\quad\quad\quad\quad\quad 4x + 8 \\
 & \quad\quad\quad\quad\quad\quad 4x + 8 \\
\cline{2-2}
 & \quad\quad\quad\quad\quad\quad\quad\quad 0 \leftarrow \text{ As expected !}
\end{array}
$$

$$x^3 + 8 = (x + 2)\left(x^2 + 2x + 4\right)$$

$$\therefore p = 2 \text{ and } q = 4$$

(ii) For the quadratic factor, $\left(x^2 + 2x + 4\right)$

$$
\begin{aligned}
D &= b^2 - 4ac \\
&= 2^2 - 4 \times 1 \times 4 \\
&= -12
\end{aligned}
$$

As the discriminant is zero the quadratic factor contributes no further roots. □

[3 marks]

Answer 7

$$f(x) = x^6 - 3x^5 - 15x^4 + 35x^3 + 90x^2 - 108x - 216$$

(i) $f(1) = 1^6 - 3 \times 1^5 - 15 \times 1^4 + 35 \times 1^3 + 90 \times 1^2 - 108 \times 1 - 216$

$= 1 - 3 - 15 + 35 + 90 - 108 - 216$

$= -216 \quad \therefore \ (x - 1)$ is not a factor of $f(x)$

$f(-1) = 1 + 3 - 15 - 35 + 90 + 108 - 216$

$= -64 \quad \therefore \ (x + 1)$ is not a factor of $f(x)$

[3 marks]

(ii) Using the FACT button on a calculator, $216 = 2^3 \times 3^3$ and as $(x \pm 1)$ can not be factors (from part (i)) it would seem that the factorisation is either $f(x) = (x+2)^3(x-3)^3$ or $f(x) = (x-2)^3(x+3)^3$

[3 marks]

(iii) Students need only do one of,

$$f(2) = -64$$

$$f(-2) = 0$$

$$f(3) = 0$$

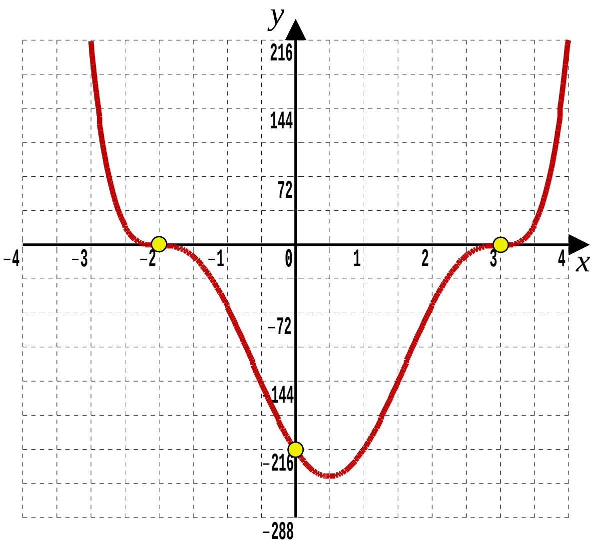
$$f(-3) = 216$$

as any one of these four results is enough to deduce that,

$$f(x) = (x+2)^3(x-3)^3$$

[3 marks]

(iv)



[3 marks]

Answer 8

Points of intersection must satisfy,

$$f(x) = g(x)$$
$$3x^3 + x^2 - x = 2x(x - 1)(x + 1)$$
$$3x^3 + x^2 - x = 2x\left(x^2 - 1\right)$$
$$3x^3 + x^2 - x - 2x^3 + 2x = 0$$
$$x^3 + x^2 + x = 0$$
$$x\left(x^2 + x + 1\right) = 0$$

Either $x = 0$ or $x^2 + x + 1 = 0$

However, $x^2 + x + 1 = 0$ has a discriminant of – 3

As this is negative there are no further roots.

The only point of intersection is thus $(0,\ 0)$

[6 marks]

3.1 A New Transformation

A transformation not encountered at GCSE level is that of inversion.

Given a function, $f(x)$, the inversion of this function is $\dfrac{1}{f(x)}$.

Although not recommended, this could be written as $[\,f(x)\,]^{-1}$ but notice that, in general, this is not the same as $f^{-1}(x)$ which is the inverse function.

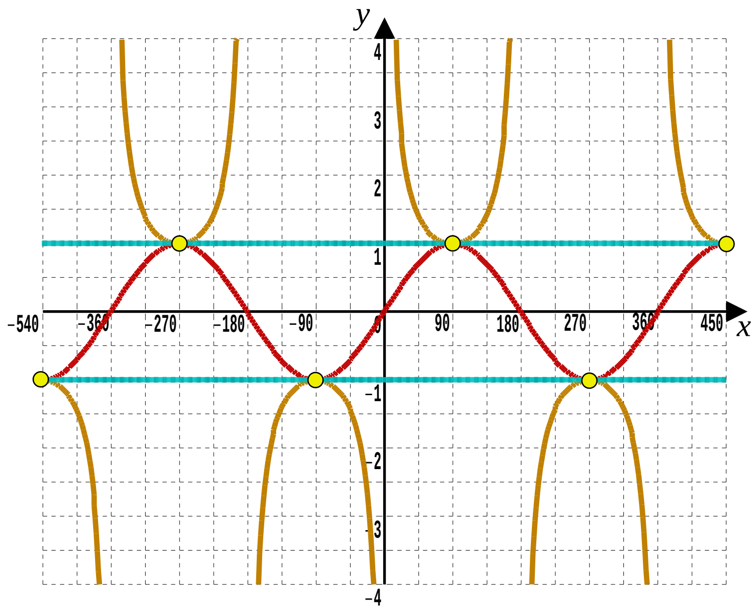
Inversion

The transformation inversion is the taking of the reciprocal of a given function.

Under inversion, $f(x) \;\to\; \dfrac{1}{f(x)}$

3.2 Example (Sine Function)

On the graph below, in red, is the familiar sine function, $f(x) = \sin x$ and, in gold, the inversion of this function $\dfrac{1}{f(x)} = \dfrac{1}{\sin x}$



Notice that,

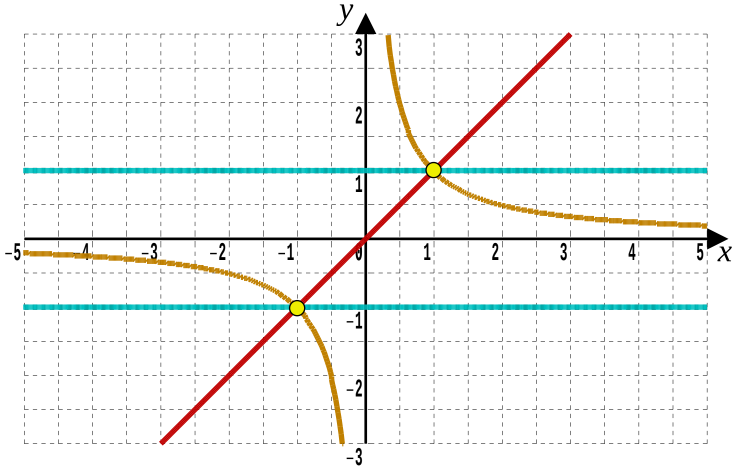
- as the reciprocal of ±1 is also ±1, points on the lines $y = \pm 1$ are invariant.
- the inversion has a vertical asymptote each time the sine function is zero.

☞ Draw in the vertical asymptotes on the graph.

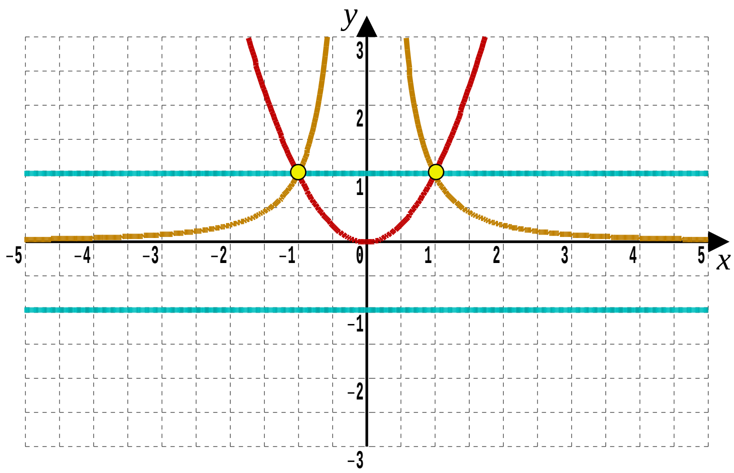
[1 mark]

3.3 Example (Power Function)

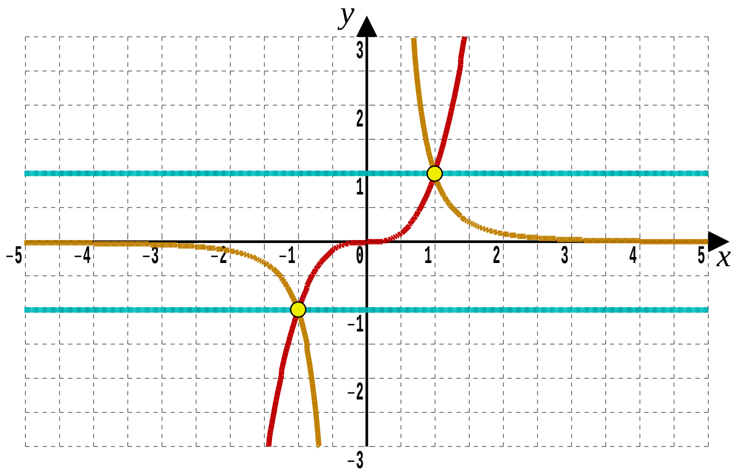
The power function, $f(x) = x^n$ has inversion, $\dfrac{1}{f(x)} = \dfrac{1}{x^n}$



$$n = 1 \ : \ y = x \ : \ y = \frac{1}{x}$$



$$n = 2 \ : \ y = x^2 \ : \ y = \frac{1}{x^2}$$



$$n = 3 \ : \ y = x^3 \ : \ y = \frac{1}{x^3}$$

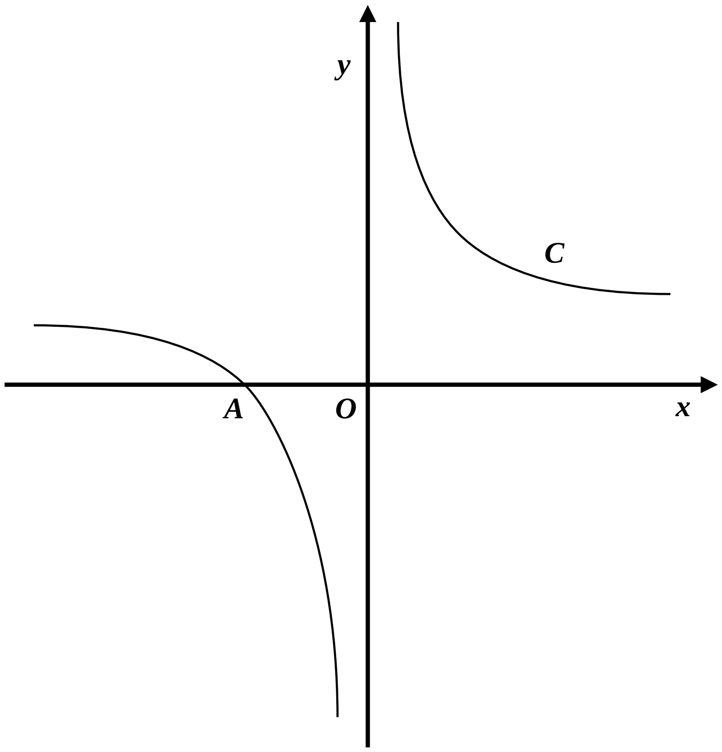
3.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 72

Question 1

A-Level C1 Examination Question from May 2014, Q4



The diagram shows a sketch of the curve *C* with equation

$$y = \frac{1}{x} + 1 \qquad x \neq 0$$

The curve *C* crosses the *x*-axis at the point *A*.

(a) State the *x* coordinate of the point *A*.

[1 mark]

The curve *D* has equation $y = x^2(x - 2)$ for all real values of *x*.

(b) Add a sketch of curve *D* to the diagram, above.
Show the coordinates of each point where the curve *D* crosses the axes.

[3 marks]

(c) Using your sketch, state, giving a reason, the number of real solutions to the equation

$$x^2(x - 2) = \frac{1}{x} + 1$$

[1 mark]

Question 2

A-Level Examination Question from January 2009, Paper C1, Q8 (Edexcel)

The point P (1, a) lies on the curve with equation

$$y = (x + 1)^2 (2 - x)$$

(a) Find the value of a

[1 mark]

(b) On the axes below sketch the curves with the following equations;

(i) $y = (x + 1)^2 (2 - x)$ **(ii)** $y = \dfrac{2}{x}$

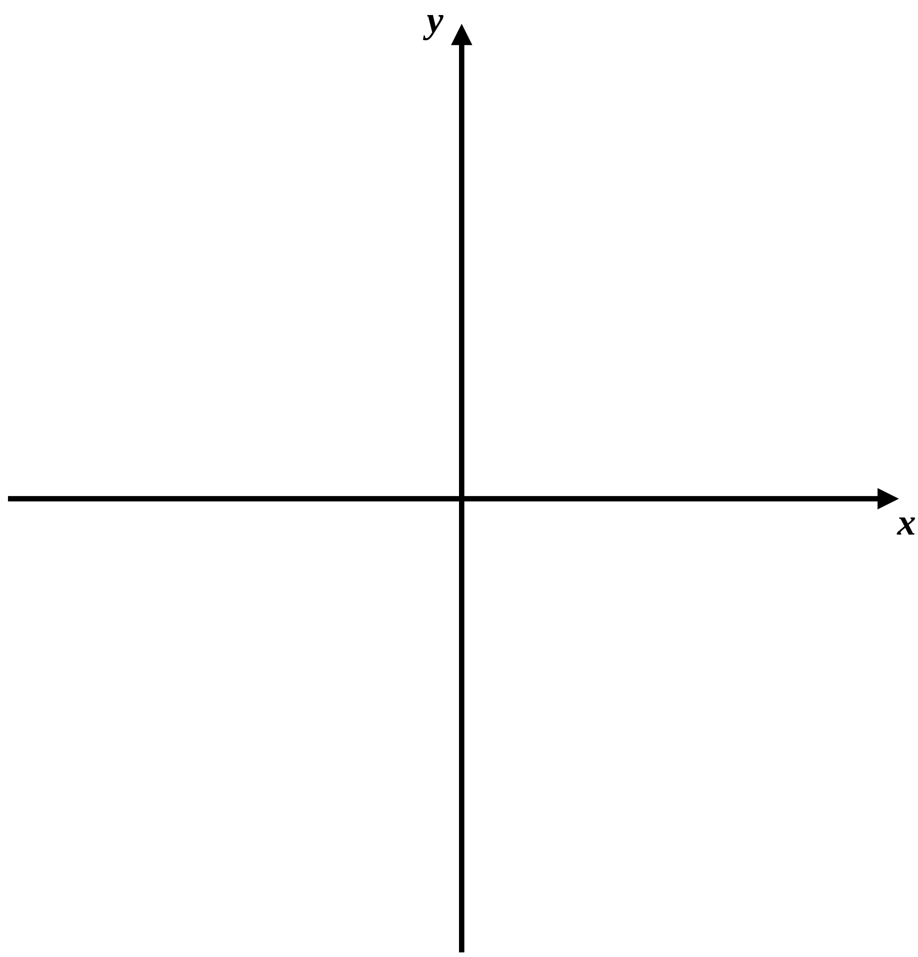
On your diagram show clearly the coordinates of any points at which the curves meet the axes.

[5 marks]

(c) With reference to your diagram in part (b) state the number of real solutions to the equation

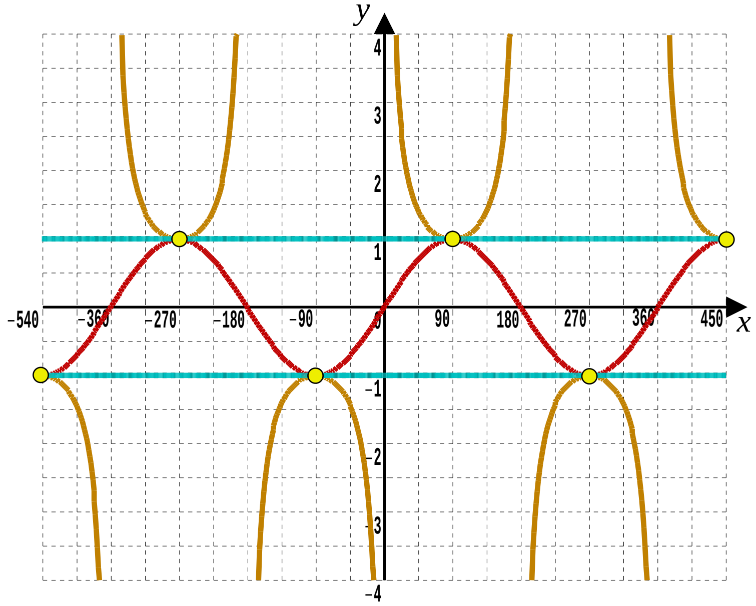
$$(x + 1)^2 (2 - x) = \frac{2}{x}$$

[1 mark]



Question 3

Graphed is the inversion of the sine function, $f(x) = \sin x \;\rightarrow\; \dfrac{1}{f(x)} = \dfrac{1}{\sin x}$



Produce similar graphs for the inversion for,

(i) the cosine function, $f(x) = \cos x \;\rightarrow\; \dfrac{1}{f(x)} = \dfrac{1}{\cos x}$

(ii) the tangent function, $f(x) = \tan x \;\rightarrow\; \dfrac{1}{f(x)} = \dfrac{1}{\tan x}$

These inversions have names, $\dfrac{1}{\sin x} = \csc x, \quad \dfrac{1}{\cos x} = \sec x\,, \quad \dfrac{1}{\tan x} = \cot x$

[3, 3 marks]

Question 4
For the curve with equation

$$y = (x - 1)^2 - 16$$

(i) write down the coordinates of the minimum point,

[1 mark]

(ii) expand the brackets, and hence write down the coordinates of where the curve crosses the y-axis,

[1 mark]

(iii) factorise your part (ii) answer, and hence write down the coordinates of where the curve crosses the x-axis.

[1 mark]

(iv) Sketch the curve.

[2 marks]

Question 5
For the curve with equation

$$y = (x - 4)^2 + 1$$

explain, using mathematics, why it does not cross the x-axis.

[2 marks]

Question 6

A-Level Examination Question from January 2011, Paper C1, Q10 (Edexcel)

(a) Sketch the graphs of

(i) $y = x(x+2)(3-x)$ **(ii)** $y = -\dfrac{2}{x}$

Show clearly the coordinates of all points where the curves cross the coordinate axes.

[6 marks]

(b) Using your sketch state, giving a reason, the number of real solutions to the equation

$$x(x+2)(3-x) = -\frac{2}{x}$$

[2 marks]

Question 7

A-level Examination Question from January 2006, Paper C1, Q10 (Edexcel)

$$x^2 + 2x + 3 \equiv (x + a)^2 + b$$

(a) Find the values of the constants a and b

[2 marks]

(b) In the space provided below, sketch the graph of

$$y = x^2 + 2x + 3$$

Indicate clearly the coordinates of any intersections with the coordinate axes.

[3 marks]

(c) Find the value of the discriminant of $x^2 + 2x + 3$

Explain how the sign of the discriminant relates to your sketch in part (b)

[2 marks]

The equation $x^2 + kx + 3 = 0$, where k is a constant, has no real roots.

(d) Find the set of possible values of k, giving your answer in surd form.

[4 marks]

Question 8

A-level Examination Question from January 2005, Paper C1, Q10

Given that

$$f(x) = x^2 - 6x + 18, \qquad x \geqslant 0$$

(a) express $f(x)$ in the form $(x - a)^2 + b$, where a and b are integers.

[3 marks]

The curve C with equation $y = f(x)$, $x \geqslant 0$, meets the y-axis at P and has a minimum point at Q

(b) Sketch the graph of C, showing the coordinates of P and Q

[4 marks]

The line $y = 41$ meets C at the point R

(c) Find the coordinates of R, giving your answer in the form $p + q\sqrt{2}$ where p and q are integers.

[5 marks]

Question 9

A-Level Examination Question from May 2010, Paper C1, Q10 (Edexcel)

(a) On the same axes sketch the graphs of the curves with equations

(i) $y = x(4 - x)$ **(ii)** $y = x^2(7 - x)$

showing clearly the coordinates of the points where the curves cross the coordinate axes.

[5 marks]

(b) Show that the x-coordinate of the points of intersection of

$y = x(4 - x)$ and $y = x^2(7 - x)$

are given by the solutions to the equation

$$x\left(x^2 - 8x + 4\right) = 0$$

[3 marks]

The point A lies on both the curve and the x and y coordinate of A are both positive.

(c) Find the exact coordinates of A, leaving your answer in the form

$$\left(p + q\sqrt{3},\ r + s\sqrt{3} \right)$$

where p, q, r and s are integers.

[7 marks]

Answer 1

(a) Let $y = 0$ in which case,

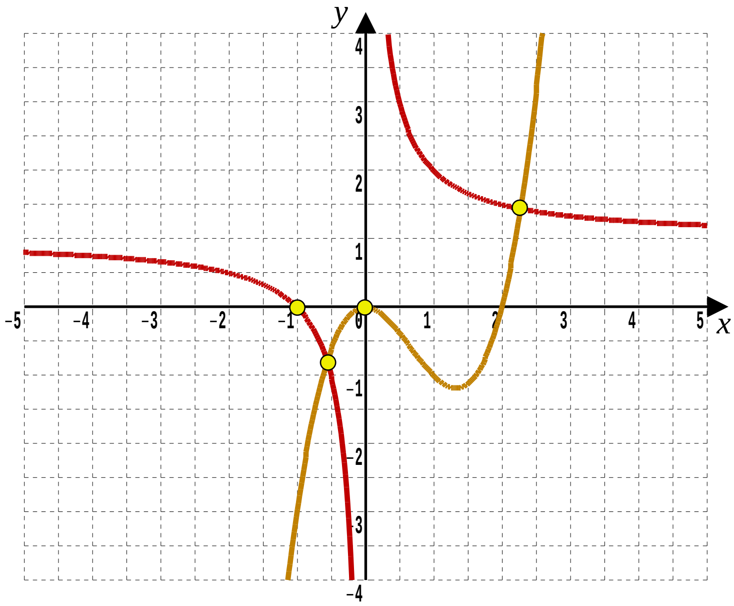
$$\frac{1}{x} + 1 = 0$$

$$1 + x = 0$$

$$x = -1$$

[1 mark]

(b)



Curve D has intercepts with the x-axis at (0, 0) or (2, 0)

[3 marks]

(c) The equation,

$$x^2 (x - 2) = \frac{1}{x} + 1$$

will have two solutions as it will find where the two curves, graphed above intersect.

[1 mark]

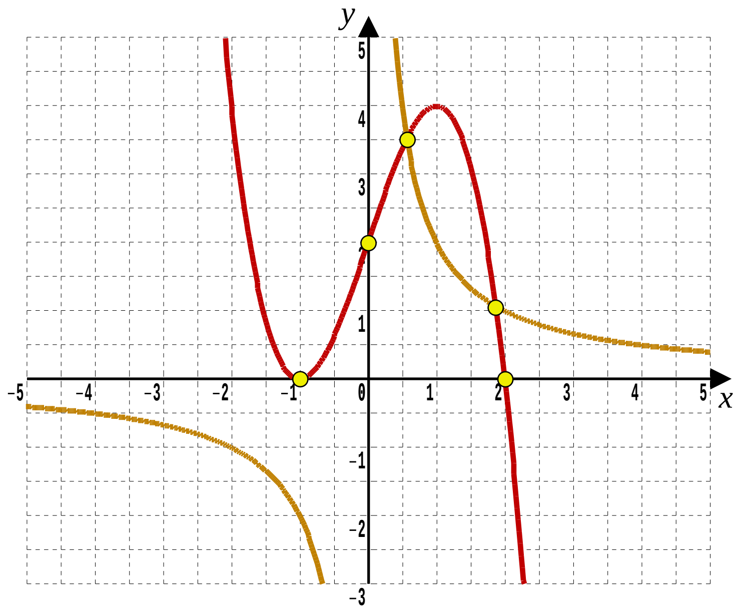
Answer 2

$$y = (x+1)^2(2-x)$$

(a) When $x = 1$, $y = 4$ $\quad \therefore\ a = 4$

[1 mark]

(b)



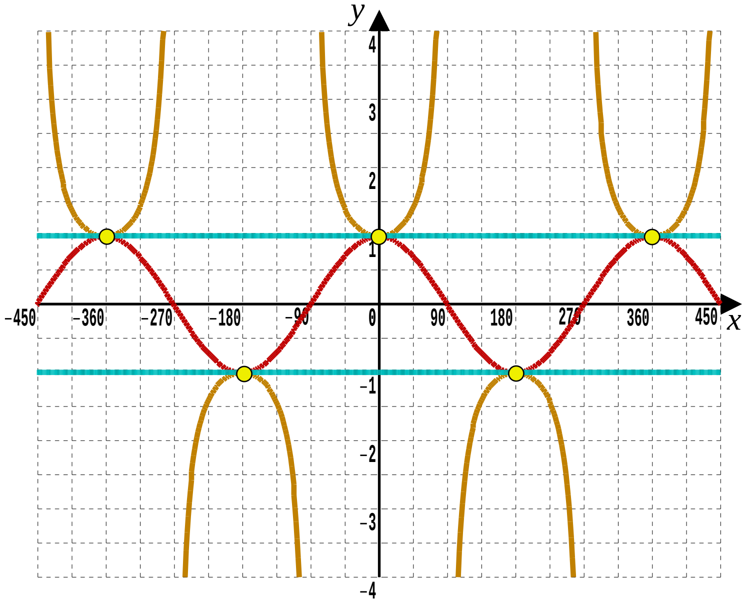
[5 marks]

(c) Two solutions as the two curves intercept in two places.

[1 mark]

Answer 3

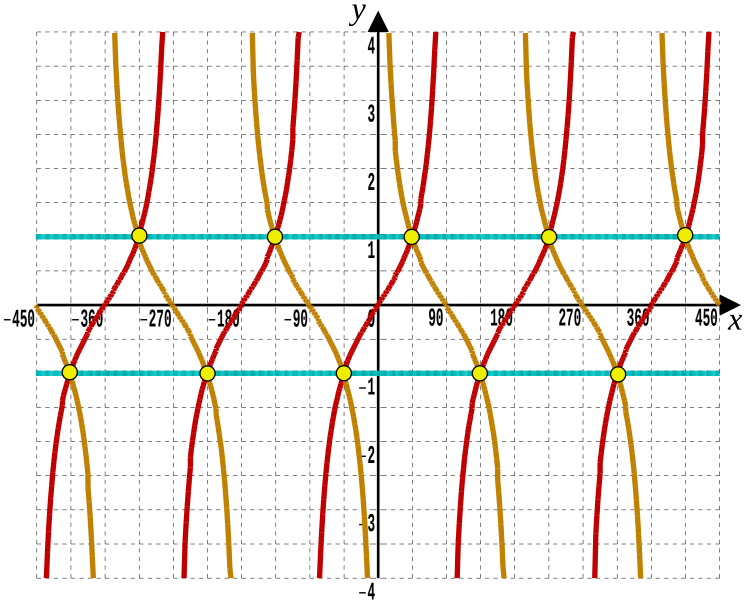
(i)



Red : $f(x) = cos(x)$ Gold : $\dfrac{1}{f(x)} = \dfrac{1}{cos(x)}$

[3 marks]

(ii)



Red : $f(x) = tan(x)$ Gold : $\dfrac{1}{f(x)} = \dfrac{1}{tan(x)}$

[3 marks]

Answer 4

(i) The minimum point is $(1, -16)$

[1 mark]

(ii) $(x - 1)^2 - 16 = x^2 - 2x - 15$

This will cross the y-axis at $(0, -15)$

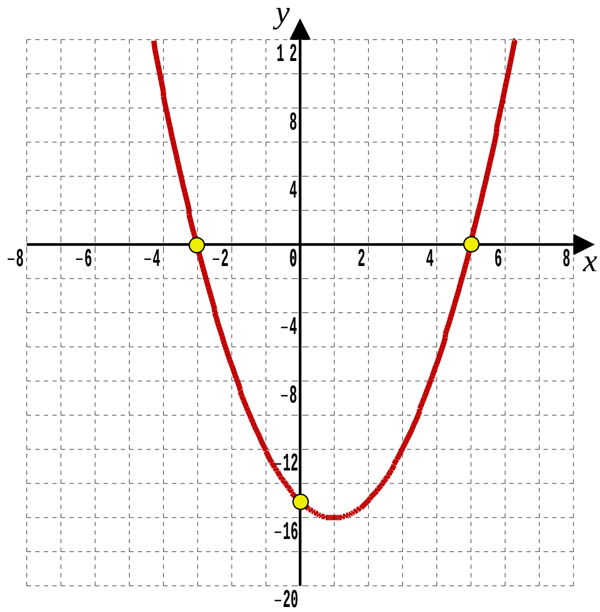
[1 mark]

(iii) $(x + 3)(x - 5)$

This will cross the x-axis at $(-3, 0)$ and $(5, 0)$

[1 mark]

(iv)



[2 marks]

Answer 5

$$y = (x - 4)^2 + 1$$

The curve of this equation would cross the x-axis when $y = 0$ and this would require that the following equation be satisfied,

$$(x - 4)^2 + 1 = 0$$

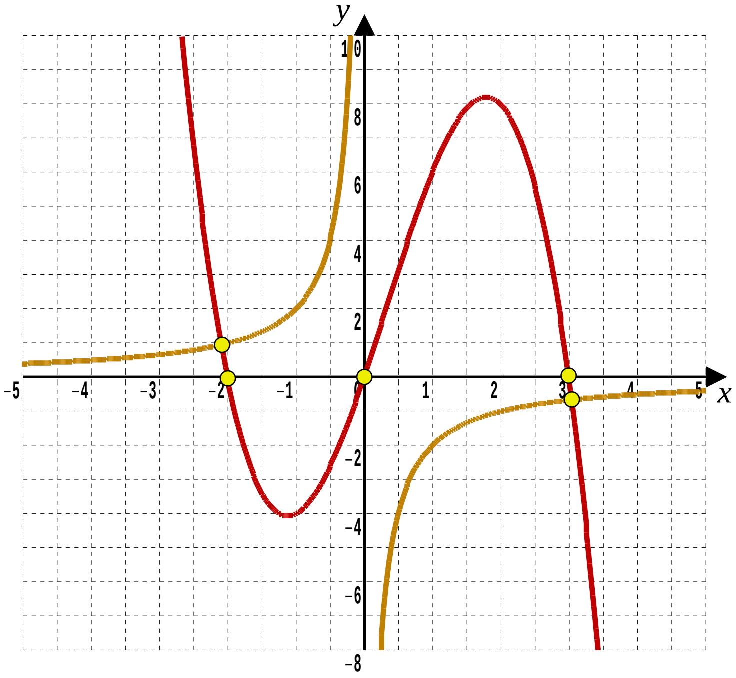
This clearly has no solutions for the least value of $(x - 4)^2$ is zero but, even then, 1 is added on.

$\therefore$ The curve of this equation does not cross the x-axis. ☐

[2 marks]

Answer 6

(a)



Red: $y = x(x+2)(3-x)$ Gold: $y = -\dfrac{2}{x}$

[6 marks]

(b) There will be two solutions to the equation

$$x(x+2)(3-x) = -\frac{2}{x}$$

because the left hand side is the red graph and the right hand side is the gold graph and, from the sketch, the two graphs intersect in two places.

[2 marks]

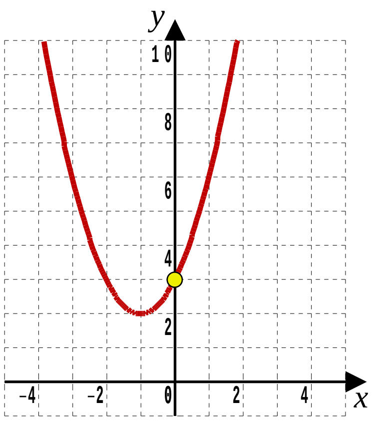
Answer 7

(a)

$$x^2 + 2x + 3 = \left[x^2 + 2x \right] + 3$$
$$= \left[(x + 1)^2 - 1 \right] + 3$$
$$= (x + 1)^2 + 2$$
$$\therefore\ a = 1 \text{ and } b = 2$$

[2 marks]

(b)



[3 marks]

(c)

$$D = 2^2 - 4 \times 1 \times 3$$
$$= -8$$

As this is negative, the equation has no roots.

Graphically, this corresponds to there being no x-axis interceptions.

(d)

$$D = k^2 - 4 \times 1 \times 3$$
$$= k^2 - 12$$

$\therefore$ For no real roots, $k^2 - 12 < 0$

$$k^2 < 12$$
$$-2\sqrt{3} < k < 2\sqrt{3}$$

[4 marks]

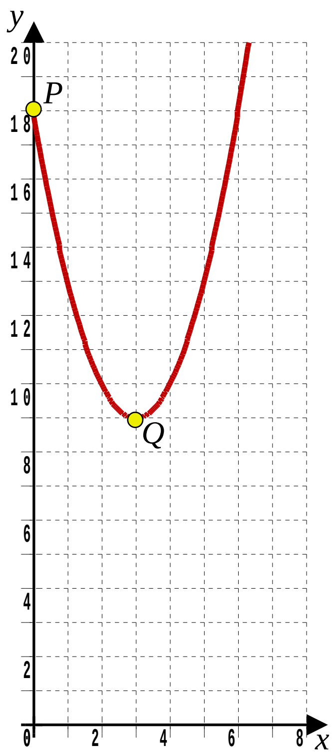
Answer 8

(a)

$$x^2 - 6x + 18 = \left[x^2 - 6x \right] + 18$$
$$= \left[(x - 3)^2 - 9 \right] + 18$$
$$= (x - 3)^2 + 9$$
$$\therefore \; a = 3 \text{ and } b = 9$$

[3 marks]

(b)



[3 marks]

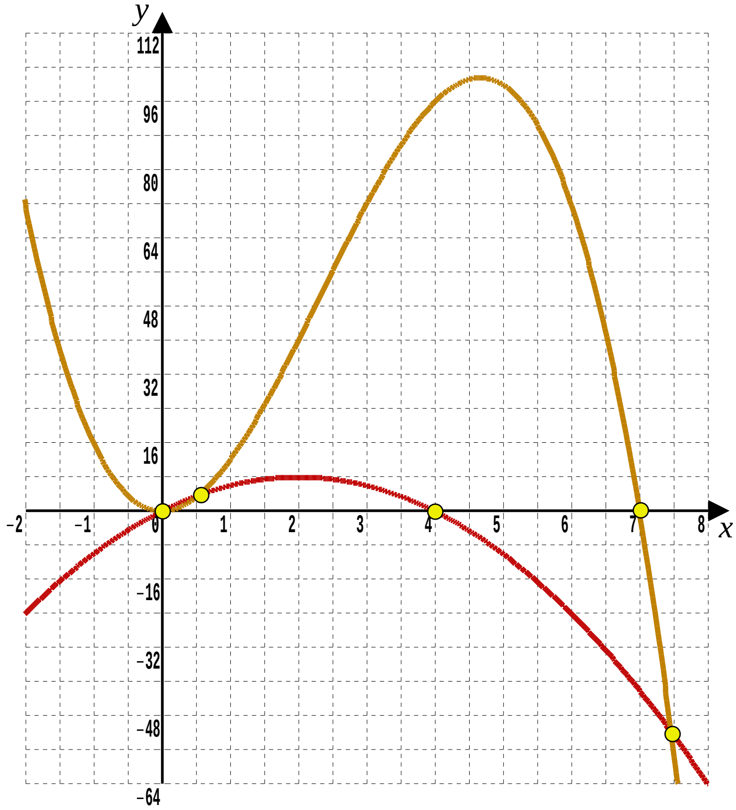
(c)

$$(x - 3)^2 + 9 = 41$$
$$(x - 3)^2 = 32$$
$$x - 3 = \pm 4\sqrt{2}$$
$$x = 3 \pm 4\sqrt{2}$$
$$\therefore \; p = 3 \text{ and } r = 4 \text{ (reject } r = -4 \text{ as told } x \geqslant 0)$$

[5 marks]

Answer 9

(a)



[5 marks]

(b) Points of intersection will be found from,

$$x(4-x) = x^2(7-x)$$

$$4x - x^2 = 7x^2 - x^3$$

$$x^3 - 8x^2 + 4x = 0$$

$$x\left(x^2 - 8x + 4\right) = 0$$

[3 marks]

(c) The solution, $x = 0$ is rejected as the coordinates of A are both positive.

$$\therefore\ x^2 - 8x + 4 = 0$$

$$\left[x^2 - 8x \right] + 4 = 0$$

$$\left[(x - 4)^2 - 16 \right] + 4 = 0$$

$$(x - 4)^2 = 12$$

$$x = 4 - 2\sqrt{3} \quad \text{(must be lesser of two)}$$

$$y = \left(4 - 2\sqrt{3}\right)\left(4 - 4 + 2\sqrt{3}\right)$$

$$= \left(4 - 2\sqrt{3}\right)\left(2\sqrt{3}\right)$$

$$= 8\sqrt{3} - 12$$

$$= -12 + 8\sqrt{3}$$

$\therefore$ Intersection sought is $\left(4 - 2\sqrt{3}, -12 + 8\sqrt{3}\right)$

$\therefore p = 4,\ q = -2,\ r = -12$ and $s = 8$

[7 marks]