1.6 Answers

University Undergraduate Lectures in Mathematics
A Second Year Course
Vector Spaces

1.6.1 Solution to 1.3 A Polynomial Example (Teaching Video)

(a) **(i)**

$$3\left(2+3x+4x^2\right) \equiv 6+9x+12x^2 \pmod 5$$
$$\equiv 1+4x+2x^2 \pmod 5$$

(ii)

$$1+3x+4x^2+3+2x+4x^2 \equiv 4+5x+8x^2 \pmod 5$$
$$\equiv 4+3x^2 \pmod 5$$

(iii)

$$-2-3x-4x^2 \equiv 3+2x+x^2 \pmod 5$$

(iv)

$$0+0x+0x^2 \pmod 5$$

[4 marks]

(b) **(i)** With all working modulo 5,

$$a\left(1+x^2\right)+b(x)+c\left(x+x^2\right) \equiv 4+x+3x^2$$
$$a+ax^2+bx+cx+cx^2 \equiv 4+x+3x^2$$
$$a+(b+c)x+(a+c)x^2 \equiv 4+x+3x^2$$
$$\therefore\ a \equiv 4$$
$$a+c \equiv 3$$
$$c \equiv 3-4$$
$$c \equiv -1$$
$$\therefore\ c \equiv 4$$
$$b+c \equiv 1$$
$$b \equiv 1-4$$
$$b \equiv -3$$
$$\therefore\ b \equiv 2$$

Thus,

$$4\left(1+x^2\right)+2(x)+4\left(x+x^2\right) \equiv 4+x+3x^2 \pmod 5$$

[4 marks]

1.6.2 Solutions to 1.5 Exercise

Answer 1

Let $a = -1$ and $b = -1$, in which case,

$$\begin{aligned} a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 &= (-1)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \boldsymbol{v}_3 \end{aligned}$$

$\therefore$ A basis cannot be formed from all of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ as they are not linearly independent. (A basis is a minimal set of linearly independent vectors)

[2 marks]

Answer 2

Focussing on the zero in $\boldsymbol{v}_1$ is the easy way into answering this question.

Let $a = 5$ and $b = -4$, in which case,

$$\begin{aligned} a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 &= (5)\begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix} + (-4)\begin{pmatrix} 2 \\ 1 \\ 1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 10 \\ 0 \\ -15 \end{pmatrix} + \begin{pmatrix} -8 \\ -4 \\ -4 \\ 16 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 6 \\ -4 \\ 1 \end{pmatrix} \\ &= \boldsymbol{v}_3 \end{aligned}$$

$\therefore$ A basis cannot be formed from all of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ as they are not linearly independent. (A basis is a minimal set of linearly independent vectors)

[2 marks]

Answer 3

$$a\begin{pmatrix}3\\2\end{pmatrix} + b\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}1\\1\end{pmatrix} \pmod 7 \;\Rightarrow\; \begin{cases} 3a + b \equiv 1 \;\;(\text{mod } 7 \\ 2a + 4b \equiv 1 \pmod 7 \end{cases}$$

$$\begin{aligned}
12a + 4b &\equiv 4 \pmod 7\\
\text{Subtract}\quad 2a + 4b &\equiv 1 \pmod 7\\
\hline
10a &\equiv 3 \pmod 7\\
3a &\equiv 3 \pmod 7\\
a &\equiv 1 \pmod 7\\
\text{Back substituting,}\;\; 3 \times 1 + b &\equiv 1 \pmod 7\\
b &\equiv -2 \pmod 7\\
b &\equiv 5 \pmod 7
\end{aligned}$$

The required linear combination is thus, $(1)\begin{pmatrix}3\\2\end{pmatrix} + 5\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}1\\1\end{pmatrix} \pmod 7$

[2 marks]

More generally,

$$a\begin{pmatrix}3\\2\end{pmatrix} + b\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}x\\y\end{pmatrix} \pmod 7 \;\Rightarrow\; \begin{cases} 3a + b \equiv x \pmod 7 \\ 2a + 4b \equiv y \pmod 7 \end{cases}$$

$$\begin{aligned}
12a + 4b &\equiv 4x \pmod 7\\
\text{Subtract}\quad 2a + 4b &\equiv y \pmod 7\\
\hline
10a &\equiv 4x - y \pmod 7\\
3a &\equiv 4x - y \pmod 7\\
3a &\equiv 4x + 14x - y - 14y \pmod 7\\
3a &\equiv 18x - 15y \pmod 7\\
a &\equiv 6x + 2y \pmod 7\\
\text{Back substituting,}\;\; 3(6x - 5y) + b &\equiv y \pmod 7\\
b &\equiv y - 18x + 15y \pmod 7\\
b &\equiv 16y - 18x \pmod 7\\
b &\equiv 3x + 2y \pmod 7
\end{aligned}$$

$$\therefore\; (6x + 2y)\begin{pmatrix}3\\2\end{pmatrix} + (3x + 2y)\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}x\\y\end{pmatrix} \pmod 7$$

[4 marks]

Answer 4

We need to show that any vector $\boldsymbol{v}$, in the vector space can be written as a linear combination of the basis vectors, $\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

That is, $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3 = \boldsymbol{v}$ for some integers a, b and c to be determined.
With all of the workings being modulo p we have that;

$$a\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{cases} \quad - b \phantom{{}-c} \equiv x \\ -a \phantom{{}-b} \quad + c \equiv y \\ \quad\ \ b - c \equiv z \end{cases}$$

Solving simultaneously yields,

$$a \equiv -(x + y + z)$$

$$b \equiv -(x)$$

$$c \equiv -(x + z)$$

Clearly, given any integer point in the vector space, the above three equations allow integer values of a, b and c to be calculated that will get to that point using a linear combination of the three basis vectors.
Thus, Walter's proposed basis is a valid one.

[6 marks]

Answer 5

(a) **(i)** $6\left(5 + 4x + 6x^2\right) \equiv 30 + 24x + 36x^2 \pmod 7$

$\equiv 2 + 3x + x^2 \pmod 7$

(ii) $4 + 6x^2 + 3 + 2x + 4x^2 \equiv 7 + 2x + 10x^2 \pmod 7$

$\equiv 2x + 3x^2 \pmod 7$

(iii) $-5 - 2x - x^2 \equiv 2 + 5x + 6x^2 \pmod 7$

(iv) $0 + 0x + 0x^2 \pmod 7$

[4 marks]

(b) **(i)** With all working modulo 7,

$$a(1) + b(4x) + c\left(4x + x^2\right) \equiv 3 + x + 5x^2$$

$$a + 4bx + 4cx + cx^2 \equiv 3 + x + 5x^2$$

$$a + (4b + 4c)x + cx^2 \equiv 3 + x + 5x^2$$

$$\therefore\ a \equiv 3,\ c \equiv 5$$

$$4b + 4c \equiv 1$$

$$4b + 20 \equiv 1$$

$$4b \equiv -19$$

$$4b \equiv -12$$

$$b \equiv -3$$

$$\therefore\ b \equiv 4$$

Thus,

$$3(1) + 4(4x) + 5\left(4x + x^2\right) \equiv 3 + x + 5x^2 \pmod 7$$

[4 marks]

Answer 6

(i) $p(x) = ax^3 + bx^2 + cx + d$

$$p(0) = 0$$

$$a(0)^3 + b(0)^2 + c(0) + d = 0$$

$$\therefore\ d = 0$$

$$\text{Also, } p(1) = 0$$

$$a(1)^3 + b(1)^2 + c(1) + 0 = 0$$

$$a + b + c = 0$$

$$\therefore c = -a - b$$

$$\begin{aligned} \text{Thus, } p(x) &= ax^3 + bx^2 + cx + d \\ &= ax^3 + bx^2 + (-a - b)x + 0 \end{aligned}$$

[4 marks]

(ii)

$$\begin{aligned} &= ax^3 - ax + bx^2 - bx \\ &= a\left(x^3 - x\right) + b\left(x^2 - x\right) \end{aligned}$$

$$\therefore\ \text{Basis is } \left\{ \left(x^3 - x\right), \left(x^2 - x\right) \right\}$$

$$\text{or } \left\{ x\left(x^2 - 1\right), x\left(x - 1\right) \right\}$$

The subspace W is those polynomials that are of degree less than three which are divisible by x and $(x - 1)$.

The dimension of a vector space is the same as the number of linearly independent vectors in the basis of that vector space.

Thus W is 2-dimensional.

[3 marks]

(iii) As W has a basis of dimension 2, and V is of dimension 4, two more vectors need to be added to W's basis to extend it to be suitable for V. Keeping in mind the need for all basis vectors to be linearly independent, the most obvious extensions involve adding in the constant 1 and the unity linear vector, x.

$$\therefore\ \text{Extended basis is } \left\{ 1,\ x,\ x\left(x^2 - 1\right), x\left(x - 1\right) \right\}$$

Other valid answers are possible (but why make it more complicated ?)

[2 marks]

Answer 7

In general an arithmetic progression is of the form,

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots,\ a + (n - 1)\, d$$

where a and d are real numbers.

Any given arithmetic progression is defined by the two parameters, a and d and this can be regarded as a point (a, d) in the plane $\mathbb{R}^2$.

So, does a linear combination of two different APs yield a third AP ?

Does $u\,\mathrm{AP}_1 + v\,\mathrm{AP}_2 = \mathrm{AP}_3$?

Let AP_1 be: $a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots,\ a + (n - 1)\, d,\ \ldots$

and AP_2 be: $b,\ b + c,\ b + 2c,\ b + 3c,\ \ldots,\ b + (n - 1)\, c,\ \ldots$

$$u\,\mathrm{AP}_1 = ua,\ ua + ud,\ ua + 2ud,\ ua + 3ud,\ \ldots,\ ua + u(n - 1)d,\ \ldots$$

$$v\,\mathrm{AP}_2 = vb,\ vb + vc,\ vb + 2vc,\ vb + 3vc,\ \ldots,\ vb + v(n - 1)c,\ \ldots$$

$$\begin{aligned} u\,\mathrm{AP}_1 + v\,\mathrm{AP}_2 = {} & ua + vb, \\ & + ua + vb + ud + vc, \\ & + ua + vb + 2(ud + vc) \\ & + ua + vb + 3(ud + vc) \\ & + \ldots \\ & + ua + vb + (n - 1)(ud + vc) \\ & + \ldots \end{aligned}$$

which is of the form of an arithmetic progression.

So all such linear combinations of arithmetic progressions have closure under "the ordinary rules of arithmetic". □

[5 marks]

Answer 8

In general a geometric progression is of the form,

$$a,\ ar,\ a\,r^2,\ a\,r^3,\ \ldots,\ a\,r^{n-1}$$

when a and r are real numbers.

Any given geometric progression is defined by the two parameters, a and r, and this can be regarded as a point (a, r) in $\mathbb{R}^2$.

However the resulting system is not closed.

It is possible to display a linear combination of two different GPs that do not yield a third GP, and a single example of this is sufficient to show that all GPs do not form a vector space.

As an example consider the sum of the following two GPs;

$$\begin{array}{rrrrrrl} & 1, & 2, & 4, & 8, & 16, & \ldots \\ \text{ADD} & 1, & 3, & 9, & 27, & 81, & \ldots \\ \hline & 2, & 5, & 13, & 35, & 97, & \ldots \end{array}$$

The resulting sequence is not a GP and so the system is not closed.

$\therefore$ The set of all geometric progressions do not form a vector space.

[4 marks]

Answer 9

Without restrictions,

$$x_1\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

With the restrictions,

$$3x_2\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + 7x_4\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

$$x_2\begin{pmatrix}3\\0\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\7\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

$$x_2\begin{pmatrix}3\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\7\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

$$\therefore \text{ A basis is, } B = \left\{\begin{pmatrix}3\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\7\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$$

[4 marks]

Answer 10

Let the coordinates of a point on the plane be (x, y, z)

Using the basis provided,

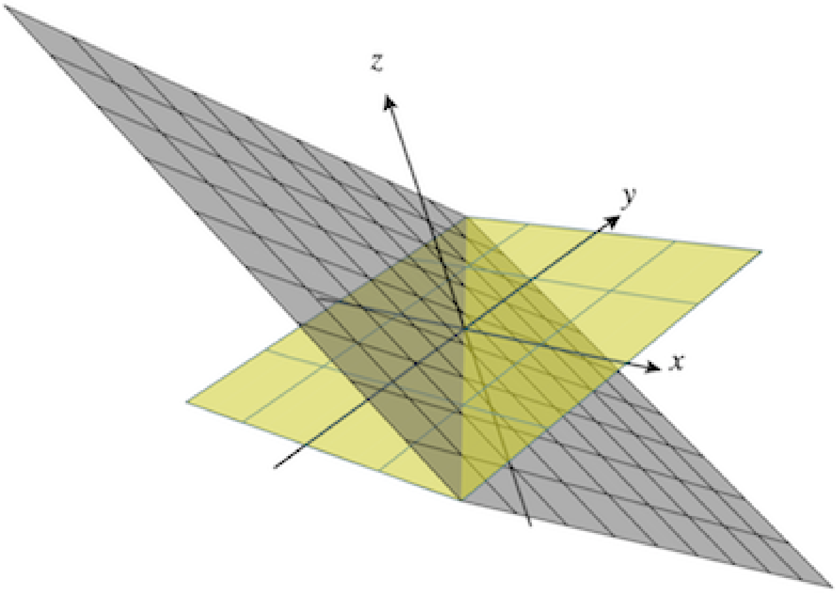
$$a\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} \quad -b = x \\ -a \quad = y \\ a + b = z \end{cases}$$

Substitution the upper two equations into the lower gives,

$$a + b = z$$

$$(-y) + (-x) = z$$

$$x + y + z = 0 \text{ as the equation of the plane}$$



[4 marks]