2.6 Answers

Answer 1

Working through the three vector subspace conditions;

(1) The zero sequence can be thought of as a geometric progression with first term 0 and common ratio 3.

That is, $0 \times 3^0, \; 0 \times 3^1, \; 0 \times 3^2, \; 0 \times 3^3, \; ..., \; 0 \times 3^{n-1}, \; ...$

As required, the zero element of the parent is in the proposed vector subspace.

(2) Let two general geometric progressions with common ratio 3 and first terms *a* and *b* be,

$$\begin{array}{rl} & a \times 3^0, \quad a \times 3^1, \quad a \times 3^2, \quad a \times 3^3, \quad ..., \quad a \times 3^{n-1}, \; ... \\ \text{ADD} & b \times 3^0, \quad b \times 3^1, \quad b \times 3^2, \quad b \times 3^3, \quad ..., \quad b \times 3^{n-1}, \; ... \\ \hline & (a+b)3^0, \quad (a+b)3^1, \quad (a+b)\,3^2, \quad ..., \quad (a+b)3^{n-1}, \; ... \end{array}$$

which is a geometric progression with common ratio 3

As required, there is closure under addition.

(3) Given a general scalar, *s*, and geometric progression, GP, with common ratio 3 and first term *a*,

$$\begin{aligned} s\,\text{GP} &= s\left(a \times 3^0, \quad a \times 3^1, \quad a \times 3^2, \quad a \times 3^3, \quad ..., \quad a \times 3^{n-1}, \; ...\right) \\ &= sa \times 3^0, \quad sa \times 3^1, \quad sa \times 3^2, \quad sa \times 3^3, \; ..., \; sa \times 3^{n-1}, \; ... \end{aligned}$$

which is a geometric progression with common ratio 3

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of geometric progressions with common ratio 3 is indeed a vector subspace of the real vector space of all infinite sequences. □

[3 marks]

Answer 2

Working through the three vector subspace conditions;

(1) The zero vector, the origin, the point $(0,\ 0,\ 0)$, is in the proposed vector subspace, as required.

(2) Let two general points in the proposed vector subspace be,

$(p,\ q, -(p + q))$ and $(r,\ s, -(r + s))$ in which case the sum is,

$(p + r,\ q + s, -(p + q) - (r + s))$

This is a point on $x + y + z = 0$

As required, there is closure under addition.

(3) Given a general scalar, s, and a general point $(p,\ q, -(p + q))$,

$$s\big(p,\ q, -(p + q)\big) = \big(sp,\ sq, -s(p + q)\big)$$

which is a point on $x + y + z = 0$

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of points in 3-dimensional Euclidean space that satisfy the equation $x + y + z = 0$ is indeed a vector subspace. □

[3 marks]

Answer 3

Showing that any one of the three conditions necessary is not satisfied is sufficient to stop the proposed vector subspace being accepted as such.

Method 1

Observe that the zero vector, the origin, the point $(0,\ 0,\ 0)$, is not in the proposed vector subspace, as $0 + 0 + 0 \neq 1$

Method 2

Let two general points in the proposed vector subspace be,

$(p,\ q,\ 1 - (p + q))$ and $(r,\ s,\ 1 - (r + s))$ in which case the sum is,

$(p + r,\ q + s,\ 2 - (p + q) - (r + s))$

This is not a point on $x + y + z = 1$ because,

$$p + r + q + s + 2 - p - q - r - s \neq 1$$

$\therefore$ The closure under addition requirement is not satisfied.

(Or give any specific example rather than this generalised one)

Method 3

Given a general scalar, s, and a general point $(p,\ q,\ 1 - (p + q))$,

$$s\big(p,\ q,\ 1 - (p + q)\big) = \big(sp,\ sq, s - s(p + q)\big)$$

This is not a point on $x + y + z = 1$ because,

$$sp + sq + s - sp - sq \neq 1 \text{ for all } s$$

$\therefore$ The closure under scalar multiplication requirement is not satisfied.

[3 marks]

Answer 4

Showing that any one of the three conditions is not satisfied is sufficient to stop the proposed vector subspace being accepted as such.

Method 1

The zero polynomial, which is the number 0, is not a polynomial of degree three as the definition of a degree three polynomial, a cubic, is that it is of the form, $p(x) = a x^3 + b x^2 + cx + d$ where $a, b, c, d \in \mathbb{R},\ a \neq 0$.

Method 2

Let one polynomial of degree three be, $p(x) = x - x^3$, and another be $q(x) = x^3$, in which case $p(x) + q(x) = x$ which is not a polynomial of degree three. $\therefore$ The closure under addition requirement is not satisfied.

[3 marks]

Answer 5

(i) Symmetric Matrix Part + Antisymmetric Matrix Part

$$= \frac{1}{2}\left(\mathbf{M} + \mathbf{M}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathbf{M} - \mathbf{M}^{\mathrm{T}}\right)$$

$$= \frac{1}{2}\mathbf{M} + \frac{1}{2}\mathbf{M}^{\mathrm{T}} + \frac{1}{2}\mathbf{M} - \frac{1}{2}\mathbf{M}^{\mathrm{T}}$$

$$= \mathbf{M} \qquad \square$$

[2 marks]

(ii) • A square $n \times n$ matrix, $\mathbf{S}$, is antisymmetric iff $\mathbf{S}^{\mathrm{T}} = -\mathbf{S}$

$$\therefore \begin{pmatrix} s_{11} & s_{21} & \dots & s_{n1} \\ s_{12} & s_{22} & \dots & s_{n2} \\ \dots & \dots & \dots & \dots \\ s_{1n} & s_{2n} & \dots & s_{nn} \end{pmatrix} = \begin{pmatrix} -s_{11} & -s_{12} & \dots & -s_{1n} \\ -s_{21} & -s_{22} & \dots & -s_{2n} \\ \dots & \dots & \dots & \dots \\ -s_{n1} & -s_{n2} & \dots & -s_{nn} \end{pmatrix}$$

In other words, $s_{ij} = -s_{ji}$ for $1 \leqslant i, j \leqslant n$

On the diagonal, let $k = i = j$, and the requirement is that $s_{kk} = -s_{kk}$

Therefore, $s_{kk} = 0$ so an antisymmetric matrix has zeros on its diagonal. $\square$

[2 marks]

(iii) Let a general 3×3 antisymmetric matrix be, $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$

$$\therefore \begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix} - a\begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix}$$

$$= 0 - abc + abc$$

$$= 0 \qquad \text{and so is singular.} \qquad \square$$

[3 marks]

Answer 6

(i) $$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots + i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so a basis for the vector space is,

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

which is 9-dimensional.

[2 marks]

(ii) Working through the three vector subspace conditions;

(1) The zero vector is the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which can be considered antisymmetric and so is in the vector subspace, as required.

(2) Let two general antisymmetric matrices be,

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix}$$

in which case the sum is, $\begin{pmatrix} 0 & a+d & b+e \\ -(a+d) & 0 & c+f \\ -(b+e) & -(c+f) & 0 \end{pmatrix}$

which is an antisymmetric matrix.

As required, there is closure under addition.

(3) Given a general scalar, s, and a general antisymmetric matrix,

$$s\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \begin{pmatrix} 0 & sa & sb \\ -sa & 0 & sc \\ -sb & -sc & 0 \end{pmatrix}$$

which is an antisymmetric matrix.

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of real 3×3 antisymmetric matrices is indeed a vector subspace. □

[3 marks]

(iii)

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

A suitable basis is, for example,

$$B = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

[2 marks]

(iv) The vector subspace is 3-dimensional.

[1 mark]

(v) The number of vectors in the basis of an $n \times n$ antisymmetric matrix can be explored for a few low values of n, as in part (iii) and (iv),

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

2×2	3×3	4×4
1-dimensional	3-dimensional	6-dimensional

The dimensionality is clearly given by triangular numbers and a triangle can even be drawn around the linearly independent entries as shown.

$\therefore$ Number of dimensions of the vector subspace is $\dfrac{n(n-1)}{2}$

[2 marks]

Answer 7

(i) This question is about showing that any vector $\boldsymbol{v} \in \mathbb{R}^3$ can be written as a linear combination of the basis vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$

That is, $a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 + c\,\boldsymbol{v}_3 = \boldsymbol{v}$ for some a, b, and c to be determined.

$$a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \begin{cases} a \phantom{{}+b} + 2c = x \\ -a + b \phantom{{}+2c} = y \\ - b + c = z \end{cases}$$

These simultaneous equations can be expressed in matrix form as,

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

From a calculator,

$$\text{If } \mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ then } \mathbf{M}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

(Classwiz fx911ex: “MENU 4”, “1” to input matrix, “Option 3”, “x^{-1}”, “=”)

$$\therefore \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore \quad a = \frac{x - 2y - 2z}{3}, \quad b = \frac{x + y - 2z}{3} \text{ and } c = \frac{x + y + z}{3}$$

This shows that, for any specified x. y and z, suitable values can be found for a, b and c such that a linear combination of the basis vectors yields that specified point.

[5 marks]

(ii)
$$\left(\frac{x - 2y - 2z}{3}\right)\boldsymbol{v}_1 + \left(\frac{x + y - 2z}{3}\right)\boldsymbol{v}_2 + \left(\frac{x + y + z}{3}\right)\boldsymbol{v}_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

[2 marks]

Answer 8

Statement : The set of all orthogonal matrices do not form a vector space.

Proof,

The zero vector is not orthogonal because $\mathbf{O}^{\mathrm{T}}\mathbf{O} = \mathbf{O} \neq \mathbf{I}$

$\therefore$ At least one of the necessary conditions to be a vector subspace is not satisfied. ☐

Alternatively, it could be shown that under either addition or scalar multiplication the closure property does not hold. For example,

$\mathbf{I}^{\mathrm{T}}\mathbf{I} = \mathbf{I}$ so $\mathbf{I}$ is orthogonal but $2\mathbf{I}$ is not orthogonal as $(2\mathbf{I})^{\mathrm{T}}(2\mathbf{I}) = 4\mathbf{I} \neq \mathbf{I}$

[5 marks]

Answer 9

If $B = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is a suitable basis for a vector space it will be possible to get to any point in that vector space via a linear combination of the basis vectors. As any vector space for which B is suitable will be two dimensional (because there are two vectors in B) let a point in that vector space be (x, y) in which case,

$$a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where a and b and the addition and multiplication are from the field over which the vector space acts.

$$\begin{array}{rr} & a + b = x \\ \text{ADD} & -a + b = y \\ \hline & 2b = x + y \end{array}$$

$$b = \frac{x + y}{2} \quad \text{and so} \quad a = \frac{x - y}{2}$$

This shows that,

(i) The proposed basis is not suitable for the vector space $\mathbb{Z}_p^2$ because this vector space is over a field of integers modulo p and integer values of x and y will not necessarily result in integer values of a and b.

(ii) The proposed basis is suitable for the vector space $\mathbb{R}^2$ because this vector space is over a field of real numbers and real values of x and y will result in real values for a and b.

[5 marks]

Answer 10

Clearly, any vector in S can be written as,

$$\begin{pmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{pmatrix} = a\begin{pmatrix} 3 \\ 6 \\ -9 \\ -3 \end{pmatrix} + b\begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} + c\begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix}$$

However, these three vectors are not linearly independent because,

$\begin{pmatrix} 3 \\ 6 \\ -9 \\ -3 \end{pmatrix} = (-3)\begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix}$ so a suitable basis is, for example, $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} \right\}$

which means the vector subspace spanned by this basis is 2-dimensional.

[4 marks]