3.4.1 Solution to 3.2 Example (Teaching Video)

(i) From studying the list of vectors,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},\ \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\ \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

it is clear that there is no non-zero value of k such that,

$$\boldsymbol{v}_1 = k\boldsymbol{v}_2 \quad \text{or} \quad \boldsymbol{v}_1 = k\boldsymbol{v}_3 \quad \text{or} \quad \boldsymbol{v}_2 = k\boldsymbol{v}_3$$

However, we are told this list is not linearly independent and so there must be some combination of two that gives the other. To force the issue solve,

$$a\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} a - b = -1 \\ 2a + b = 4 \\ -a = -1 \end{cases}$$

From the lower equation $a = 1$ and then from the upper $b = 2$

Crucially, check that those two solutions satisfy the middle equation;

$$\begin{aligned} \text{LHS} &= 2a + b \\ &= 2 \times 1 + 2 \\ &= 4 \\ &= \text{RHS} \qquad \square \end{aligned}$$

Thus $\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and so list $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is not linearly independent.

[4 marks]

(ii) Now $\boldsymbol{v}_3$ can be removed from the list of vectors.

With a list of only two vectors, to show they are linearly independent simply observe that there is no non-zero value of k such that $\boldsymbol{v}_1 = k\boldsymbol{v}_2$.

Alternatively, show that the only value of a and b that makes $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = \boldsymbol{0}$ is $a = b = 0$. That is,

$$a\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a - b = 0 \\ 2a + b = 0 \\ -a = 0 \end{cases}$$

From the lower equation, $a = 0$, and then the upper equation gives $b = 0$

$\therefore$ $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. $\square$

[2 marks]

3.4.2 Solutions to 3.3 Exercise

Answer 1

(i) **(a)** The y-axis

(b) The line $y = x$ across the x-y plane (when $z = 0$)

As a vector straight line equation, $\boldsymbol{r} = \mu\begin{pmatrix}1\\1\\0\end{pmatrix}$

(c) The point (0, 0, 0)

[3 marks]

(ii) **(a)** $B = \left\{ \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$

To verify the two vectors used in the basis are linearly independent simply note that there is no non-zero value for k such that,

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = k\begin{pmatrix}1\\1\\0\end{pmatrix}$$

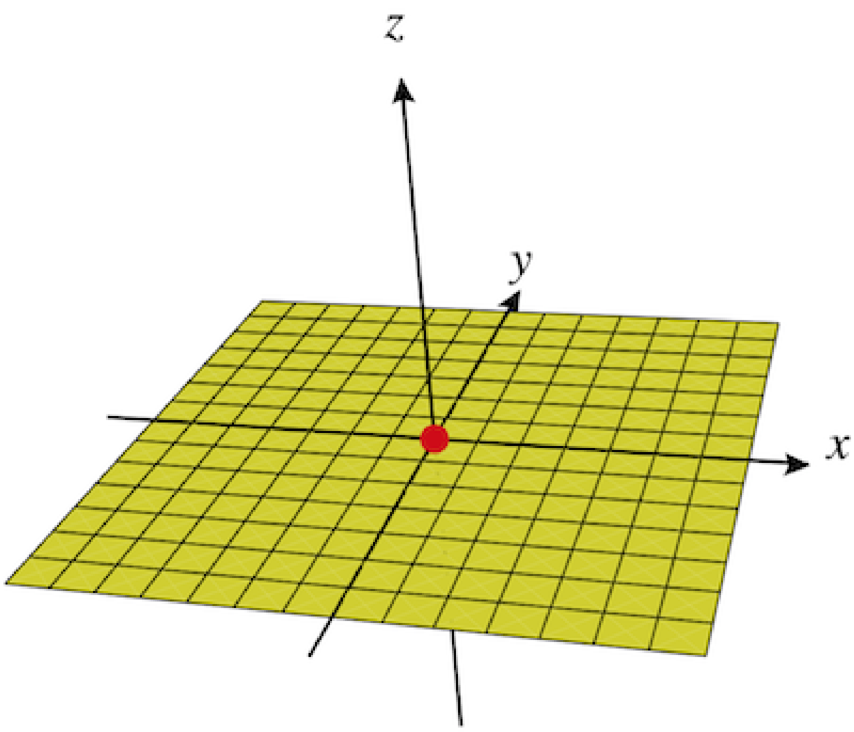
[2 marks]

(b) As there are two vectors in the basis for $U + W$ it must be a 2-dimensional vector subspace, which will be a plane.

$$a\begin{pmatrix}0\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} b = x \\ a + b = y \\ 0 = z \end{cases}$$

As $z = 0$, that is the Cartesian equation of the plane,

[2 marks]

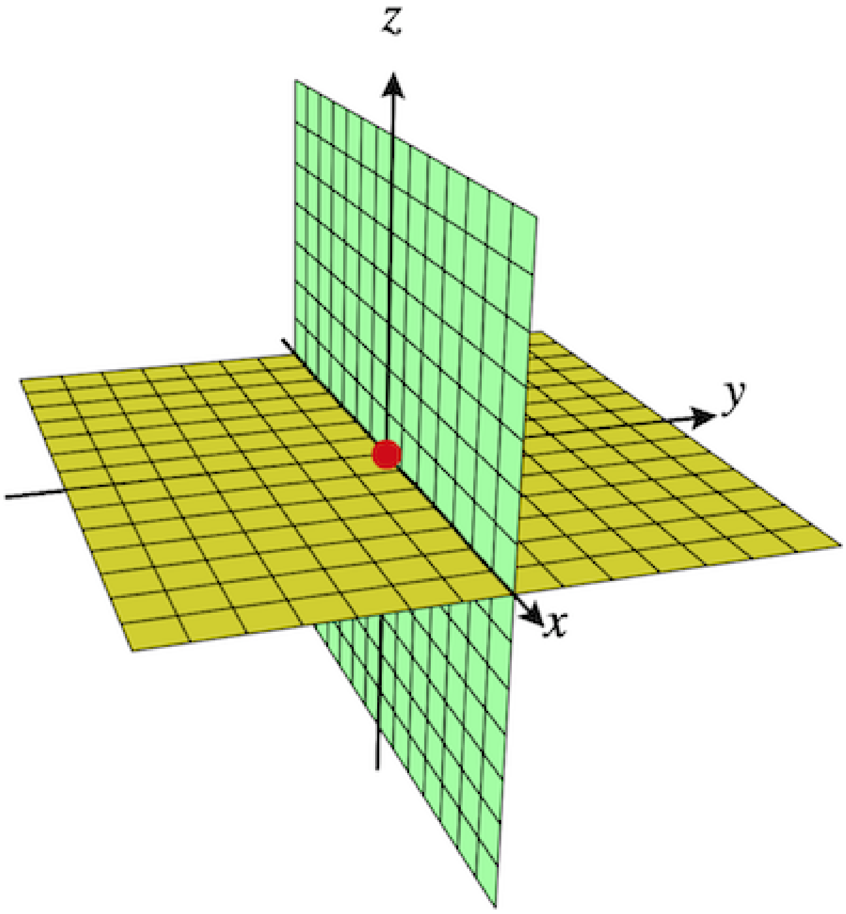


Answer 2

(i) **(a)** The plane $z = 0$

(b) The plane $y = 0$

(c) The x-axis



[3 marks]

(ii) **(a)** In merging together the basis for U with the basis for W we clearly do not need two copies of the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

The basis for $U + W$ is thus,

$$B_{U+W} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

which is immediately recognizable as the standard basis for $\mathbb{R}^3$ and so is linearly independent without any further checking.

[2 marks]

(b) Geometrically $U + W$ is all of the three dimensional vector space $\mathbb{R}^3$ which is also all of V

[2 marks]

Answer 3

(i) **(a)** $$a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} a = x \\ b = y \\ a = z \end{cases}$$

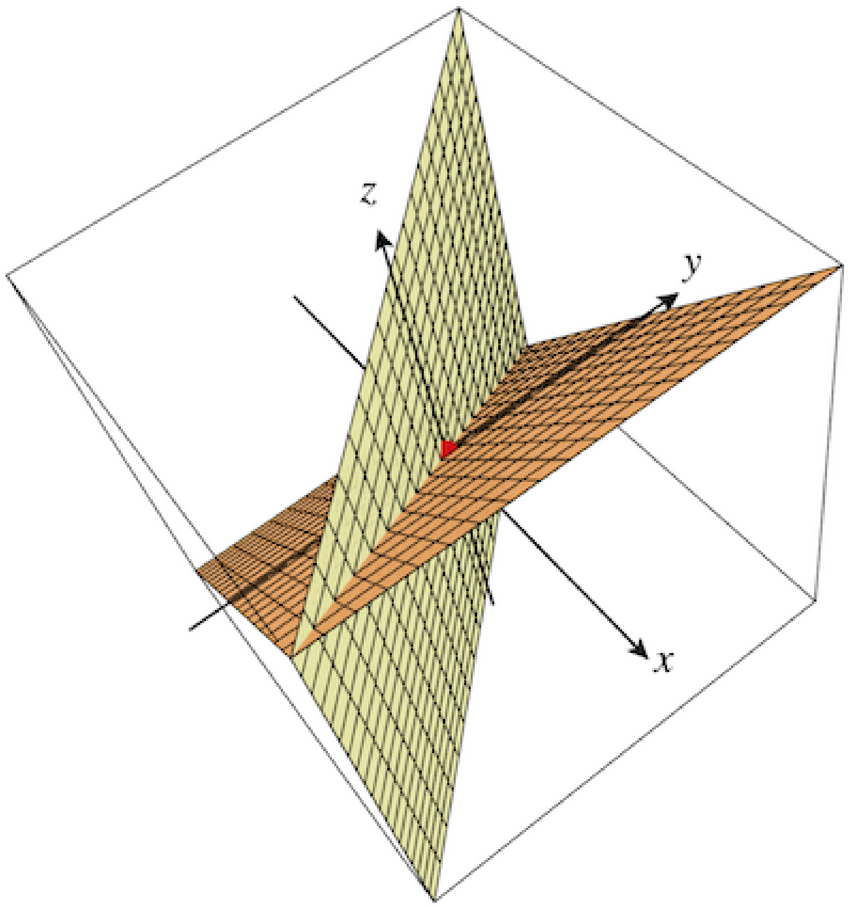
From the upper and lower equations, the equation of the 2-dimensional plane is $x - z = 0$

[3 marks]

(b) $$a\begin{pmatrix}1\\-1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} a = x \\ -a = y \\ b = z \end{cases}$$

From the two uppermost equations, the equation of the 2-dimensional plane is $x + y = 0$

[3 marks]



(c) The two planes will intersect in the line with equation

$x = -y = z$ (in Cartesian form)

$$\boldsymbol{r} = \mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$$ (in vector form)

[3 marks]

(ii) **(a)** The list of vectors from the merge of the two basis is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These cannot be linearly independent because the vector subspace has to have a dimension not greater than that of its parent.
A basis of four vectors is 4-dimensional, the parent is 3-dimensional.

That is, $dim(U + W) \leqslant dim(\mathbb{R}^3)$

$$dim(U + W) \leqslant 3$$

Notice that $\boldsymbol{v}_1 - \boldsymbol{v}_2 - \boldsymbol{v}_3 = \boldsymbol{v}_4$, so $\boldsymbol{v}_4$ can be dropped from the list.

If the remaining three vectors are linearly independent than the only solution to the following equation will be that $a = b = c = 0$.

$$a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a \quad\quad + c = 0 \\ \quad\quad b - c = 0 \\ a \quad\quad\quad\quad = 0 \end{cases}$$

The lower equation gives $a = 0$, which when substituted into the middle equation gives $b = 0$, and which when substituted into the upper equation gives $c = 0$.

$\therefore$ These three vectors are a suitable basis for $U + W$.

$$B_{U+W} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

(In fact any three chosen from the four would be suitable)

[4 marks]

(b) As the basis has three vectors it is three dimensional and so must be all of V. In other words $U + W$ is $\mathbb{R}^3$.

[2 marks]

Answer 4

(a) Working through the three vector subspace conditions;

(1) By definition, as we are told that U and W are vector subspaces, each must contain the zero vector.
Hence the zero vector is in the intersection of U and W.
$\therefore$ The requirement the zero be in the subspace $U \cap W$ is satisfied.

(2) Let two general elements in the proposed vector subspace $U \cap W$ be $\boldsymbol{x}$ and $\boldsymbol{y}$. Then $\boldsymbol{x}$ is in U and $\boldsymbol{y}$ is in U and so, because we are told that U is a vector subspace, $\boldsymbol{x} + \boldsymbol{y}$ must be in U.
By the same argument, $\boldsymbol{x}$ is in W and $\boldsymbol{y}$ is in W and so $\boldsymbol{x} + \boldsymbol{y}$ is in W.
Therefore, $\boldsymbol{x} + \boldsymbol{y}$ is in $U \cap W$.
As required, there is closure under addition.

(3) Given a general scalar, s, and a general element $\boldsymbol{x}$ in $U \cap W$, then $\boldsymbol{x}$ is in U and so $s\boldsymbol{x}$ is in U as we are told U is a vector subspace.
By the same argument, $\boldsymbol{x}$ is in W and so $s\boldsymbol{x}$ is in W.
Therefore, $s\boldsymbol{x}$ is in $U \cap W$.
As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace $U \cap W$ is indeed a vector subspace. $\square$

[6 marks]

(b) Working through the three vector subspace conditions;

(1) By definition, as we are told that U and W are vector subspaces, each must contain the zero vector. Now, $\mathbf{0} + \mathbf{0} = \mathbf{0}$

Hence the zero vector is in the sum of U and W.

$\therefore$ The requirement the zero be in the subspace $U + W$ is satisfied.

(2) Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two general elements (vectors) in the proposed vector subspace $U + W$.

Each can be decomposed into a sum from which it was formed.

In other words,

$\boldsymbol{x} = \boldsymbol{u}_1 + \boldsymbol{w}_1$ and $\boldsymbol{y} = \boldsymbol{u}_2 + \boldsymbol{w}_2 \qquad \boldsymbol{u}_1, \boldsymbol{u}_2 \in U$ and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$

Consider now, the vector $\boldsymbol{x} + \boldsymbol{y}$,

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &= \boldsymbol{u}_1 + \boldsymbol{w}_1 + \boldsymbol{u}_2 + \boldsymbol{w}_2 \\ &= \left(\boldsymbol{u}_1 + \boldsymbol{u}_2 \right) + \left(\boldsymbol{w}_1 + \boldsymbol{w}_2 \right) \end{aligned}$$

(As a vector space over a field has commutativity and associativity)

Note that $\boldsymbol{u}_1 + \boldsymbol{u}_2$ is in U because $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U$ and U is a vector subspace in which, by definition, vector addition holds.

Similarly, $\boldsymbol{w}_1 + \boldsymbol{w}_2$ is in W.

Thus $\left(\boldsymbol{u}_1 + \boldsymbol{u}_2 \right) + \left(\boldsymbol{w}_1 + \boldsymbol{w}_2 \right) \in U + W$

$\therefore \; \boldsymbol{x} + \boldsymbol{y} \in U + W$

As required, there is closure under addition.

(3) Suppose that $\boldsymbol{x}$ is a general element in $U + W$ that, as before, can be decomposed into a sum from which it was formed,

$\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{w} \qquad \boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$

Let s be a general scalar from the field under the vector space

in which case $s\boldsymbol{x} = s\left(\boldsymbol{u} + \boldsymbol{w} \right) = s\boldsymbol{u} + s\boldsymbol{w}$

(As a vector space over a field is distributive)

Note that $s\boldsymbol{u}$ is in U because $\boldsymbol{u} \in U$ and U is a vector subspace in which, by definition, scalar multiplication holds.

Similarly, $s\boldsymbol{w}$ is in W.

Thus $s\boldsymbol{u} + s\boldsymbol{w} \in U + W$

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace $U + W$ is indeed a vector subspace. $\square$

[6 marks]

(c) Although not asked for, we can get a feel for this question by working out the equation of each of the two two-dimensional planes that are the geometric interpretation of the vector subspaces U and W.

For U,

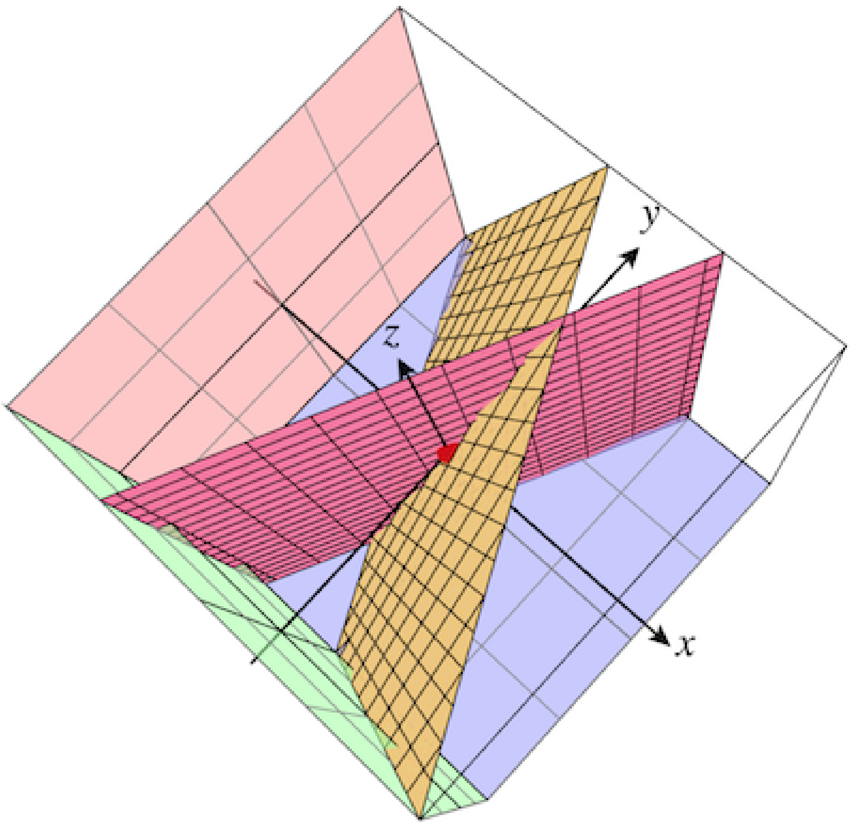
$$a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \left\{\begin{array}{rcrcl} a & & & = & x\\ 2a & & & = & y\\ & & b & = & z\end{array}\right.$$

This subspace is the plane with Cartesian equation $y = 2x$

For W,

$$a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \left\{\begin{array}{rcrcl} & - & b & = & x\\ a & + & 2b & = & y\\ a & & & = & z\end{array}\right.$$

This subspace is the plane with Cartesian equation $2x + y - z = 0$



The intersection of two planes that are not parallel is a line (one-dimensional)

Its equation is,

$$2x = y = \frac{z}{2} \quad \text{(in Cartesian form)}$$

$$\boldsymbol{r} = \mu\begin{pmatrix}1\\2\\4\end{pmatrix} \quad \text{(in vector form)}$$

So far, it has been established that

$$dim(U) = 2,\ dim(W) = 2 \text{ and } dim(U \cap W) = 1$$

The remaining issue is to determine $dim(U + W)$

The list of vectors from the merge of the two basis is

$$\boldsymbol{v}_1 = \begin{pmatrix}1\\2\\0\end{pmatrix},\ \boldsymbol{v}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix}-1\\2\\0\end{pmatrix}$$

These cannot be linearly independent because the vector subspace has to have a dimension not greater than that of its parent.
A basis of four vectors is 4-dimensional, the parent is 3-dimensional.

That is, $dim\left(U + W\right) \leqslant dim\left(\mathbb{R}^3\right)$

$$dim\left(U + W\right) \leqslant 3$$

Notice that $-\boldsymbol{v}_1 - 4\boldsymbol{v}_2 + 4\boldsymbol{v}_3 = \boldsymbol{v}_4$, so $\boldsymbol{v}_4$ can be dropped from the list.
If the remaining three vectors are linearly independent than the only solution to the following equation will be that $a = b = c = 0$.

$$a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix} + c\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{cases} a \qquad\qquad = 0 \\ 2a \qquad + c = 0 \\ \qquad b + c = 0 \end{cases}$$

The upper equation gives $a = 0$, which when substituted into the middle equation gives $c = 0$. Substituted $c = 0$ into the lower equation gives $b = 0$.
$\therefore$ These three vectors are a suitable basis for $U + W$.

$$B_{U+W} = \left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

(In fact any three chosen from the four would be suitable)

$$\therefore dim\ (U + W) = 3$$

Verification: The claim is that, for this example,

$$\begin{aligned} dim(U + W) &= dim(U) + dim(W) - dim(U \cap W) \\ \text{RHS} &= dim(U) + dim(W) - dim(U \cap W) \\ &= 2 + 2 - 1 \\ &= 3 \\ &= \text{LHS} \end{aligned}$$

$\therefore$ The proposed formula holds in this case. ☐

[6 marks]