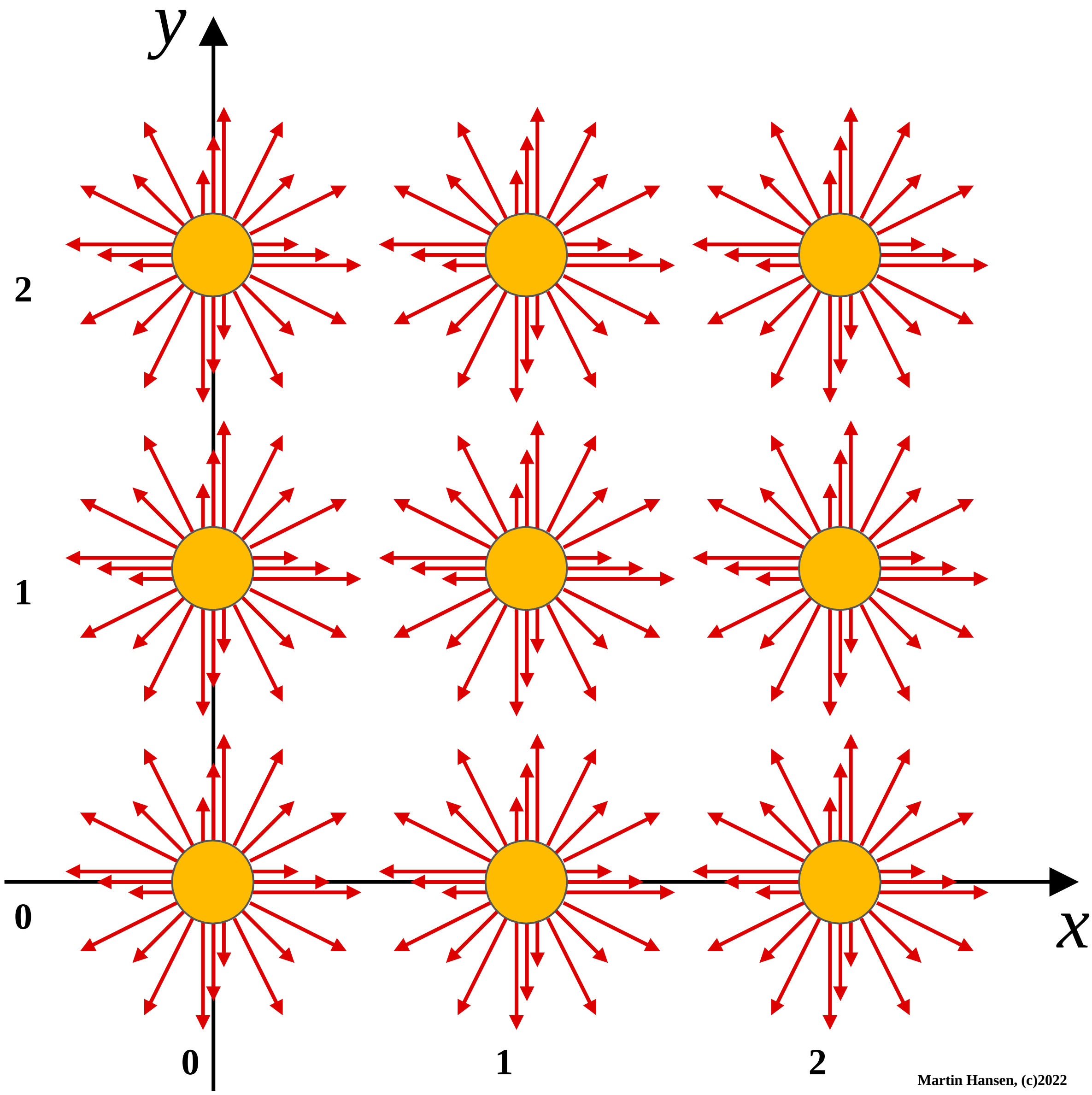
University Undergraduate Lectures in Mathematics
A Second Year Course

VECTOR SPACES



An illustration of the vector space $\mathbb{F}_3$

V E C T O R   S P A C E S

Lecture 1

University Undergraduate Lectures in Mathematics
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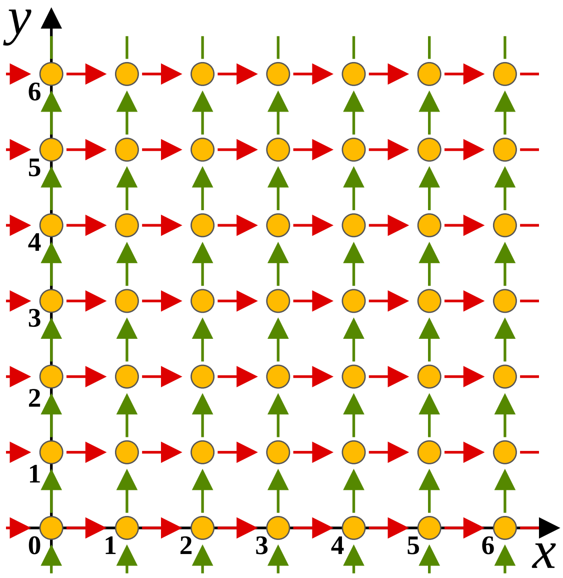
Vector Spaces

1.1 What is a Vector Space ?

A vector space consists of vectors and an arithmetic that acts upon those vectors.
It's specified over a field using a minimal set of vectors termed a basis.
For example, the set of vectors over the field of "integer 2-dimensional coordinates modulo seven" could be the definition of a particular vector space.

A suitable basis is $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and the points within the vector space are obtained from all possible linear combinations, modulo seven, of the basis vectors.

Colouring $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ red and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ green yields this visualisation:



There is an exploited ambiguity between thinking of a point as a point, or as a displacement vector (from the origin to the point).

The included points are derived from all possible values of $a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where a and b are modulo seven integers with additions and multiplications modulo seven.

In fact, given a couple of points within any vector space, (u, v) and (x, y), all calculations of the type, $a \begin{pmatrix} u \\ v \end{pmatrix} + b \begin{pmatrix} x \\ y \end{pmatrix}$ must yield an answer in the vector space.

This is termed "closure". In general, the arithmetic acting on the vector space is not specified separately; it's intimately tied to the vector space itself. Roughly speaking, "the normal rules of arithmetic apply". The technical detail of this will be given once some experience of working with vector spaces has been acquired.

1.2 What is a Suitable Basis ?

Let $\mathbb{Z}_7^2$ be the two-dimensional vector space of modulo seven coordinate points.
A basis has to comprise of linearly independent vectors.
That is, for any two vectors in the base, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, there must be no non-zero integer constant k for which $\boldsymbol{v}_1 = k\,\boldsymbol{v}_2$

As observed previously, a suitable basis is $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ because you can get to any point in the vector space in a unique way via a linear combination of those two basis vectors. For example, to get to the point $(1,\ 4)$,

$$1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 4 \end{pmatrix} \pmod 7$$

1.2.1 Problem

Show that another basis for $\mathbb{Z}_7^2$ is $B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ by first exhibiting a linear combination of those two basis vectors that gets to the point $(1,\ 4)$ and then a linear combination that gets to the generalised point (x, y).

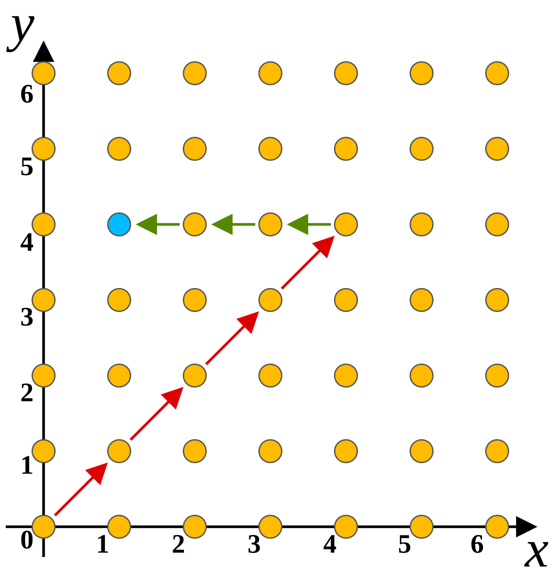
[6 marks]

1.2.2 Solution

It's possible to guess the answer but here's a methodical method.

We require, $a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 4 \end{pmatrix} \pmod 7 \;\Rightarrow\; \begin{cases} a - b \equiv 1 \pmod 7 \\ a \equiv 4 \pmod 7 \end{cases}$

Solve these equations simultaneously to get $a \equiv 4 \pmod 7$ and $b \equiv 3 \pmod 7$



An illustration of the solution $4\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 4 \end{pmatrix} \pmod 7$

In general, $a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} x \\ y \end{pmatrix} \pmod 7 \Rightarrow \begin{cases} a - b \equiv x \pmod 7 \\ a \equiv y \pmod 7 \end{cases}$

Solving the two resulting equations simultaneously this time yields that,

$$a \equiv y \pmod 7 \text{ and } b \equiv y - x \pmod 7$$

This shows that any point in the vector space can be reached.
Notice that for integer values of x and y, these give integer values of a and b.

$$\therefore\ B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} \text{ is a suitable basis.} \qquad \square$$

[6 marks]

1.3 A Polynomial Example

Let $\mathbb{F}_5$ be the field of integers modulo five.

Let V be the vector space of polynomials of the form,

$$p(x) \equiv a + bx + cx^2 \text{ where } a, b, c \in \mathbb{F}_5$$

(a) Write down,

(i) the scalar multiple $3\left(2 + 3x + 4x^2\right)$

(ii) the sum of $1 + 3x + 4x^2$ and $3 + 2x + 4x^2$

(iii) the zero vector in V

(iv) the inverse of the vector $2 + 3x + 4x^2$

Given that a suitable basis for V is $B = \left\{1 + x^2,\ x,\ x + x^2\right\}$

(b) Express $4 + x + 3x^2$ using B

Teaching Video: http://www.NumberWonder.co.uk/v9112/1a.mp4 (Part 1)
http://www.NumberWonder.co.uk/v9112/1b.mp4 (Part 2)

<= Part 1 Part 2 =>

[8 marks]

1.4 A Proof Example

In this section a proof will be developed to answer the following question;

For two bases of a vector space, one is a subset of the other.
Show that these bases are identical.

[8 marks]

1.4.1 Preliminary Thoughts #1

Suppose that $B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

If we try to take $B_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ as our subset basis then we can only move along the x-axis and get to integer points on that axis; B_3 does not span the entire space.

There are points in the vector space, such as $(1, \ 1)$ that we can't get to.

With reluctance it has to be accepted that B_3 has to have at least one other vector.

However, B_3 has to be a subset of B_1 : we're forced to extend B_3 to be identical to B_1.

1.4.2 Preliminary Thoughts #2

The same situation would arise if we started with $B_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ and tried for a subset basis of, say, $B_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. We could only move along and get to the integer points on the line with equation, $y = x$ and so B_4 does not span the space.

Once again there are points in the vector space, such as $(1, \ 4)$ that we can't get to.

To be able to move in the other dimension another vector has to be added to the basis that is not parallel to the existing vector. In order to extend B_4 whilst keeping it a subset of B_2 the only possible extension results in B_4 being identical to B_2.

1.4.3 The Proof Technique

We are now set up to make the leap to a generalised proof to the question.

The proof technique to be used is to propose that two different versions of a “thing” exist but, via logical deduction, be forced to conclude that the proposed two different “things” have to actually be identical.

1.4.4 The Proof

Let a basis that spans an n-dimensional vector space, L^n, be,

$$L_1 = \{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \boldsymbol{v}_3, \ ..., \ \boldsymbol{v}_n \}$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, ..., \boldsymbol{v}_n$ are a set of linearly independent vectors.

Consider now a basis, L_2, that also spans L^n and which is a subset of L_1

Let us suppose that $L_2 = \{ \boldsymbol{v}_1 \}$. This proposal fails to span L^n because there are points in L^n of the form $a\boldsymbol{v}_1 + b\boldsymbol{v}_2$ that can not be reached by $k\boldsymbol{v}_1$ where a, b and k are integer constants.

In other words, there is no solution to the equation,

$$a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = k\boldsymbol{v}_1$$

We are thus forced to add the vector $\boldsymbol{v}_2$ to our proposed basis, L_2 which will then be $L_2 = \{ \boldsymbol{v}_1, \boldsymbol{v}_2 \}$. The logic now repeats with the observation that there are still points in L_2 that cannot be reached, this time of the form $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3$

L_2 still does not span L^n as there is no solution to the equation,

$a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3 = k\boldsymbol{v}_1 + l\boldsymbol{v}_2$ where a, b, c, k and l are integer constants.

This inductive argument repeats until we have been forced to add all the basis vectors of L_1 to L_2.

$$L_1 = L_2 = \{ \boldsymbol{v}_1, \ \boldsymbol{v}_2, \ \boldsymbol{v}_3, \ ..., \ \boldsymbol{v}_n \}$$

So, given one basis of a linear space, in trying to create another basis that is different to the first but also a subset of the first we have been forced to conclude that the two bases are, in fact, identical. $\square$

[8 marks]

1.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Tony proposes the following basis for the vector space $\mathbb{Z}_p^3$ where p is prime.

$$B = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

However, this set of vectors are not linearly independent.

Show this by finding integer values of a and b such that $a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 = \boldsymbol{v}_3$

[2 marks]

Question 2

Fred proposes the following basis for the vector space $\mathbb{Z}_p^4$ where p is prime.

$$B = \left\{ \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -4 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -3 \\ 6 \\ -4 \\ 1 \end{pmatrix} \right\}$$

However, this set of vectors are not linearly independent.

Show this by finding integer values of a and b such that $a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 = \boldsymbol{v}_3$

[2 marks]

Question 3

Show that the basis $B_3 = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$ is a suitable basis for $\mathbb{Z}_7^2$

Do this by first exhibiting a linear combination of the two basis vectors that gets to the point $(1,\ 1)$ and then a linear combination that gets to the point $(x,\ y)$.

[6 marks]

Question 4

Walter proposes $B = \left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ as a basis for $\mathbb{Z}_p^3$

Show that this is a valid proposal.

What are the coordinates of a general vector $\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ relative to the basis vectors ?

[6 marks]

Question 5

Let $\mathbb{F}_7$ be the field of integers modulo seven.

Let V be the vector space of polynomials of the form,

$$p(x) \equiv a + bx + c\,x^2 \text{ where } a, b, c \in \mathbb{F}_7$$

(a) Write down,

(i) the scalar multiple $6\left(5 + 4x + 6\,x^2\right)$

[1 mark]

(ii) the sum of $4 + 6\,x^2$ and $3 + 2x + 4\,x^2$

[1 mark]

(iii) the zero vector in V

[1 mark]

(iv) the inverse of the vector $5 + 2x + x^2$

[1 mark]

Given that a suitable basis for V is $B = \left\{1,\ 4x,\ 4x + x^2\right\}$

(b) express $3 + x + 5\,x^2$ using B

[4 marks]

Question 6

Let $V = \mathbb{P}_4$ be the vector space of polynomials of degree less than 4 over the field $\mathbb{R}$.

A general polynomial within this vector space is then

$$p(x) = ax^3 + bx^2 + cx + d \quad \text{for } a, b, c, d \in \mathbb{R}$$

Let W be the vector subspace of polynomials $p(x)$ such that $p(0) = p(1) = 0$

(i) Show that $p(x) = ax^3 + bx^2 + (-a - b)x$

[4 marks]

(ii) Hence, find a basis for W and state its dimension.

[3 marks]

(iii) Extend the basis for W to be a basis for V and state its dimension.

[2 marks]

Question 7
Consider the set A of all arithmetic progressions.
These form a vector space over the field $\mathbb{R}$.
Verify that, under all possible linear combinations of addition and scalar multiplication, the set A is closed.

[5 marks]

Question 8
Consider the set G of all geometric progressions.
These do not form a vector space over the field $\mathbb{R}$.
Show this by demonstrating that not all possible combinations of addition and scalar multiplication give a closed system.

[4 marks]

Question 9

Let U be the vector subspace of $\mathbb{R}^5$ defined by,

$$U = \left\{ \left(x_1,\ x_2,\ x_3,\ x_4,\ x_5 \right) \in \mathbb{R}^5 : x_1 = 3x_2 \ \text{ and } \ x_3 = 7x_4 \right\}$$

Find a basis of U

[4 marks]

Question 10

The basis $B = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ defines a vector subspace, a plane, within $\mathbb{R}^3$

Determine the Cartesian equation of the plane.

[4 marks]

1.6.1 Solution to 1.3 A Polynomial Example (Teaching Video)

(a) **(i)**
$$3\left(2 + 3x + 4x^2\right) \equiv 6 + 9x + 12x^2 \pmod 5$$
$$\equiv 1 + 4x + 2x^2 \pmod 5$$

(ii)
$$1 + 3x + 4x^2 + 3 + 2x + 4x^2 \equiv 4 + 5x + 8x^2 \pmod 5$$
$$\equiv 4 + 3x^2 \pmod 5$$

(iii)
$$-2 - 3x - 4x^2 \equiv 3 + 2x + x^2 \pmod 5$$

(iv)
$$0 + 0x + 0x^2 \pmod 5$$

[4 marks]

(b) **(i)** With all working modulo 5,

$$a\left(1 + x^2\right) + b(x) + c\left(x + x^2\right) \equiv 4 + x + 3x^2$$
$$a + ax^2 + bx + cx + cx^2 \equiv 4 + x + 3x^2$$
$$a + (b + c)x + (a + c)x^2 \equiv 4 + x + 3x^2$$
$$\therefore\ a \equiv 4$$
$$a + c \equiv 3$$
$$c \equiv 3 - 4$$
$$c \equiv -1$$
$$\therefore\ c \equiv 4$$
$$b + c \equiv 1$$
$$b \equiv 1 - 4$$
$$b \equiv -3$$
$$\therefore\ b \equiv 2$$

Thus,

$$4\left(1 + x^2\right) + 2(x) + 4\left(x + x^2\right) \equiv 4 + x + 3x^2 \pmod 5$$

[4 marks]

1.6.2 Solutions to 1.5 Exercise

Answer 1

Let $a = -1$ and $b = -1$, in which case,

$$\begin{aligned} a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 &= (-1)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= \boldsymbol{v}_3 \end{aligned}$$

$\therefore$ A basis cannot be formed from all of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ as they are not linearly independent. (A basis is a minimal set of linearly independent vectors)

[2 marks]

Answer 2

Focussing on the zero in $\boldsymbol{v}_1$ is the easy way into answering this question.

Let $a = 5$ and $b = -4$, in which case,

$$\begin{aligned} a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 &= (5)\begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix} + (-4)\begin{pmatrix} 2 \\ 1 \\ 1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 10 \\ 0 \\ -15 \end{pmatrix} + \begin{pmatrix} -8 \\ -4 \\ -4 \\ 16 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 6 \\ -4 \\ 1 \end{pmatrix} \\ &= \boldsymbol{v}_3 \end{aligned}$$

$\therefore$ A basis cannot be formed from all of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ as they are not linearly independent. (A basis is a minimal set of linearly independent vectors)

[2 marks]

Answer 3

$$a\begin{pmatrix}3\\2\end{pmatrix} + b\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}1\\1\end{pmatrix} \pmod 7 \;\Rightarrow\; \begin{cases} 3a + \;\; b \equiv 1 \;\; (\text{mod } 7 \\ 2a + 4b \equiv 1 \pmod 7 \end{cases}$$

$$\begin{aligned}
&12a + 4b \equiv 4 \pmod 7\\
\text{Subtract}\quad &2a + 4b \equiv 1 \pmod 7\\
\hline
&10a \equiv 3 \pmod 7\\
&3a \equiv 3 \pmod 7\\
&a \equiv 1 \pmod 7\\
\text{Back substituting,}\quad &3 \times 1 + b \equiv 1 \pmod 7\\
&b \equiv -2 \pmod 7\\
&b \equiv 5 \pmod 7
\end{aligned}$$

The required linear combination is thus, $(1)\begin{pmatrix}3\\2\end{pmatrix} + 5\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}1\\1\end{pmatrix} \pmod 7$

[2 marks]

More generally,

$$a\begin{pmatrix}3\\2\end{pmatrix} + b\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}x\\y\end{pmatrix} \pmod 7 \;\Rightarrow\; \begin{cases} 3a + \;\; b \equiv x \pmod 7 \\ 2a + 4b \equiv y \pmod 7 \end{cases}$$

$$\begin{aligned}
&12a + 4b \equiv 4x \pmod 7\\
\text{Subtract}\quad &2a + 4b \equiv y \pmod 7\\
\hline
&10a \equiv 4x - y \pmod 7\\
&3a \equiv 4x - y \pmod 7\\
&3a \equiv 4x + 14x - y - 14y \pmod 7\\
&3a \equiv 18x - 15y \pmod 7\\
&a \equiv 6x + 2y \pmod 7\\
\text{Back substituting,}\quad &3(6x - 5y) + b \equiv y \pmod 7\\
&b \equiv y - 18x + 15y \pmod 7\\
&b \equiv 16y - 18x \pmod 7\\
&b \equiv 3x + 2y \pmod 7
\end{aligned}$$

$$\therefore\; (6x + 2y)\begin{pmatrix}3\\2\end{pmatrix} + (3x + 2y)\begin{pmatrix}1\\4\end{pmatrix} \equiv \begin{pmatrix}x\\y\end{pmatrix} \pmod 7$$

[4 marks]

Answer 4

We need to show that any vector $\boldsymbol{v}$, in the vector space can be written as a linear combination of the basis vectors, $\boldsymbol{v}_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

That is, $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 + c\boldsymbol{v}_3 = \boldsymbol{v}$ for some integers a, b and c to be determined.

With all of the workings being modulo p we have that;

$$a\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{cases} \quad -b \qquad \equiv x \\ -a \qquad + c \equiv y \\ \qquad b - c \equiv z \end{cases}$$

Solving simultaneously yields, $a \equiv -(x + y + z)$

$$b \equiv -(x)$$

$$c \equiv -(x + z)$$

Clearly, given any integer point in the vector space, the above three equations allow integer values of a, b and c to be calculated that will get to that point using a linear combination of the three basis vectors.

Thus, Walter's proposed basis is a valid one.

[6 marks]

Answer 5

(a) **(i)** $6\left(5 + 4x + 6x^2\right) \equiv 30 + 24x + 36x^2 \pmod 7$

$\equiv 2 + 3x + x^2 \pmod 7$

(ii) $4 + 6x^2 + 3 + 2x + 4x^2 \equiv 7 + 2x + 10x^2 \pmod 7$

$\equiv 2x + 3x^2 \pmod 7$

(iii) $-5 - 2x - x^2 \equiv 2 + 5x + 6x^2 \pmod 7$

(iv) $0 + 0x + 0x^2 \pmod 7$

[4 marks]

(b) **(i)** With all working modulo 7,

$$a(1) + b(4x) + c\left(4x + x^2\right) \equiv 3 + x + 5x^2$$

$$a + 4bx + 4cx + cx^2 \equiv 3 + x + 5x^2$$

$$a + (4b + 4c)x + cx^2 \equiv 3 + x + 5x^2$$

$$\therefore\ a \equiv 3,\ c \equiv 5$$

$$4b + 4c \equiv 1$$

$$4b + 20 \equiv 1$$

$$4b \equiv -19$$

$$4b \equiv -12$$

$$b \equiv -3$$

$$\therefore\ b \equiv 4$$

Thus,

$$3(1) + 4(4x) + 5\left(4x + x^2\right) \equiv 3 + x + 5x^2 \pmod 7$$

[4 marks]

Answer 6

(i) $p(x) = ax^3 + bx^2 + cx + d$

$$p(0) = 0$$

$$a(0)^3 + b(0)^2 + c(0) + d = 0$$

$$\therefore\ d = 0$$

$$\text{Also, } p(1) = 0$$

$$a(1)^3 + b(1)^2 + c(1) + 0 = 0$$

$$a + b + c = 0$$

$$\therefore c = -a - b$$

$$\text{Thus, } p(x) = ax^3 + bx^2 + cx + d$$

$$= ax^3 + bx^2 + (-a - b)x + 0$$

[4 marks]

(ii)

$$= ax^3 - ax + bx^2 - bx$$

$$= a\left(x^3 - x\right) + b\left(x^2 - x\right)$$

$$\therefore\ \text{Basis is } \left\{ \left(x^3 - x\right), \left(x^2 - x\right) \right\}$$

$$\text{or } \left\{ x\left(x^2 - 1\right), x\left(x - 1\right) \right\}$$

The subspace W is those polynomials that are of degree less than three which are divisible by x and $(x - 1)$.

The dimension of a vector space is the same as the number of linearly independent vectors in the basis of that vector space.

Thus W is 2-dimensional.

[3 marks]

(iii) As W has a basis of dimension 2, and V is of dimension 4, two more vectors need to be added to W's basis to extend it to be suitable for V. Keeping in mind the need for all basis vectors to be linearly independent, the most obvious extensions involve adding in the constant 1 and the unity linear vector, x.

$$\therefore\ \text{Extended basis is } \left\{ 1,\ x,\ x\left(x^2 - 1\right), x\left(x - 1\right)\right\}$$

Other valid answers are possible (but why make it more complicated ?)

[2 marks]

Answer 7

In general an arithmetic progression is of the form,

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \dots,\ a + (n - 1)\,d$$

where a and d are real numbers.

Any given arithmetic progression is defined by the two parameters, a and d and this can be regarded as a point (a, d) in the plane $\mathbb{R}^2$.

So, does a linear combination of two different APs yield a third AP ?

Does $u\,\mathrm{AP}_1 + v\,\mathrm{AP}_2 = \mathrm{AP}_3$?

Let AP_1 be: $a,\ a + d,\ a + 2d,\ a + 3d,\ \dots,\ a + (n - 1)\,d,\ \dots$

and AP_2 be: $b,\ b + c,\ b + 2c,\ b + 3c,\ \dots,\ b + (n - 1)\,c,\ \dots$

$$u\,\mathrm{AP}_1 = ua,\ ua + ud,\ ua + 2ud,\ ua + 3ud,\ \dots,\ ua + u(n - 1)d,\ \dots$$

$$v\,\mathrm{AP}_2 = vb,\ vb + vc,\ vb + 2vc,\ vb + 3vc,\ \dots,\ vb + v(n - 1)c,\ \dots$$

$$\begin{aligned} u\,\mathrm{AP}_1 + v\,\mathrm{AP}_2 = {} & ua + vb, \\ & + ua + vb + ud + vc, \\ & + ua + vb + 2(ud + vc) \\ & + ua + vb + 3(ud + vc) \\ & + \dots \\ & + ua + vb + (n - 1)(ud + vc) \\ & + \dots \end{aligned}$$

which is of the form of an arithmetic progression.

So all such linear combinations of arithmetic progressions have closure under "the ordinary rules of arithmetic". ☐

[5 marks]

Answer 8

In general a geometric progression is of the form,

$$a,\ ar,\ a\,r^2,\ a\,r^3,\ \ldots,\ a\,r^{n-1}$$

when a and r are real numbers.

Any given geometric progression is defined by the two parameters, a and r, and this can be regarded as a point $(\,a, r\,)$ in $\mathbb{R}^2$.

However the resulting system is not closed.

It is possible to display a linear combination of two different GPs that do not yield a third GP, and a single example of this is sufficient to show that all GPs do not form a vector space.

As an example consider the sum of the following two GPs;

$$\begin{array}{rrrrrrl} & 1, & 2, & 4, & 8, & 16, & \ldots \\ \text{ADD} & 1, & 3, & 9, & 27, & 81, & \ldots \\ \hline & 2, & 5, & 13, & 35, & 97, & \ldots \end{array}$$

The resulting sequence is not a GP and so the system is not closed.

$\therefore$ The set of all geometric progressions do not form a vector space.

[4 marks]

Answer 9

Without restrictions,

$$x_1\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

With the restrictions,

$$3x_2\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + 7x_4\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

$$x_2\begin{pmatrix}3\\0\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\7\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

$$x_2\begin{pmatrix}3\\1\\0\\0\\0\end{pmatrix} + x_4\begin{pmatrix}0\\0\\7\\1\\0\end{pmatrix} + x_5\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

$$\therefore \text{ A basis is, } B = \left\{\begin{pmatrix}3\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\7\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\}$$

[4 marks]

Answer 10

Let the coordinates of a point on the plane be $(\,x,\,y,\,z\,)$

Using the basis provided,

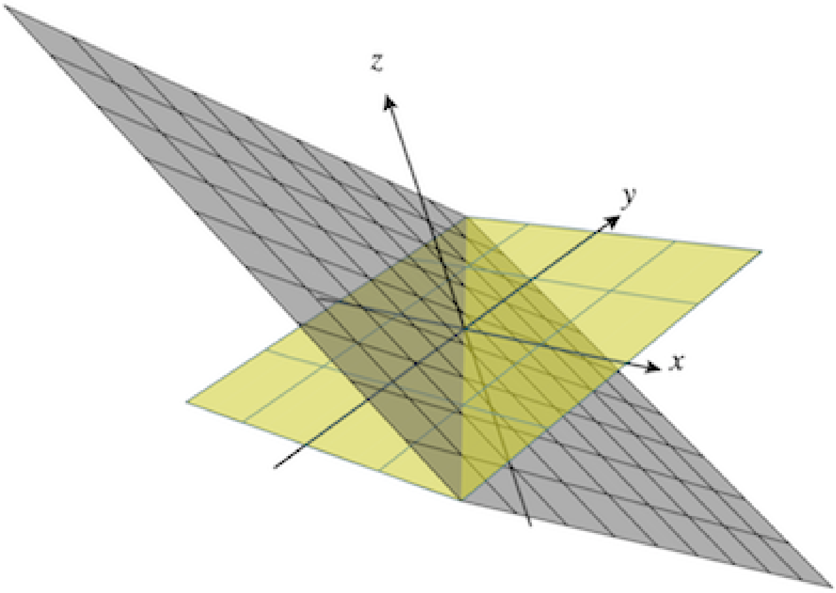
$$a\begin{pmatrix}0\\-1\\1\end{pmatrix} + b\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} \quad\; - b = x \\ -a \quad\;\; = y \\ \; a + b = z \end{cases}$$

Substitution the upper two equations into the lower gives,

$$a + b = z$$

$$(-y) + (-x) = z$$

$$x + y + z = 0 \text{ as the equation of the plane}$$



[4 marks]

2.1 A Formal Definition

The first lecture informally introduced vector spaces and subspaces. In particular the fact that a basis underpinned the structure was established, along with an emphasis on the additive nature of scalar multiples of the basis vectors. This was presented as the requirement that linear combinations of the basis vectors formed a system with the property of closure and that "the normal rules of arithmetic" applied. However, a formal definition is now needed to tighten up on exactly what a vector space is.

Definition: Vector Space

A vector space consists of a set V of elements over a field $\mathbb{F}$ such that,

- $(V, +)$ is an Abelian group
- given any vectors $\boldsymbol{v}. \boldsymbol{w} \in V$ and $a, b \in \mathbb{F}$
  - $a\boldsymbol{v} \in V$ (multiplicative closure)
  - $1\boldsymbol{v} = \boldsymbol{v}$ (multiplicative identity)
  - $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$ (multiplicative associativity)
  - $a(\boldsymbol{v} + \boldsymbol{w}) = a\boldsymbol{v} + a\boldsymbol{w}$ (multiplicative left distributivity)
  - $(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$ (multiplicative right distributivity)

Definition: Group

If V is a set and $+$ is a binary operation defined on V, then $(V, +)$ is a group if the following four axioms hold:

- **Closure:** For all $\boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{v} + \boldsymbol{w} \in V$
- **Identity:** There exists an identity element, a "zero vector", $\boldsymbol{0} \in V$

  This is such that, for all $\boldsymbol{v} \in V$, $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v}$
- **Inverses:** For each $\boldsymbol{v} \in V$, there exists an inverse element $-\boldsymbol{v} \in V$

  This is such that $\boldsymbol{v} + -\boldsymbol{v} = -\boldsymbol{v} + \boldsymbol{v} = 0$
- **Associativity:** For all $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z} \in V$, $\boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{z}) = (\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{z}$

Definition: Abelian Group

If V is a set and $+$ is a binary operation defined on V, and $(V, +)$ is a group, then $(V, +)$ is an Abelian group if,

- **Commutativity:** For all $\boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$

2.2 Examples of Vector Spaces

Working through all of the conditions to determine if a proposed vector space satisfies them all is tedious. Fortunately the following can be quoted as "known":

(i) $\mathbb{R}^n$: The set of all n-dimensional column vectors with real elements forms a real vector space under vector addition and scalar multiplication.

(ii) $\mathbb{C}^n$: The set of all n-dimensional column vectors with complex elements forms a complex vector space under vector addition and scalar multiplication.

(iii) $\mathbb{P}_n$: The set of all polynomials whose degree is less than n with polynomial addition and scalar multiplication. This vector space is n-dimensional.

(iv) $\mathbb{Z}_p^n$: The set of all n-dimensional column vectors with integer elements forms an integer vector space under addition and scalar multiplication modulo p where p is a prime number.

(v) $\mathbb{R}^{m \times n}$: The set of all $m \times n$ matrices with real entries forms a real mn-dimensional vector space with matrix addition and scalar multiplication.

(vi) $\mathbb{C}^{m \times n}$: The set of all $m \times n$ matrices with complex entries forms a complex mn-dimensional vector space with matrix addition and scalar multiplication.

(vii) The set of all infinite sequences forms a complex vector space under corresponding term-by-term addition and scalar multiplication.

2.3 Vector Subspaces

Given the above list, exam questions tend to focus on deciding if a non-empty subset of one of the above is itself a vector space.

Testing for a Vector Subspace

Let V be a vector space over a field $\mathbb{F}$ and let S be a non-empty subset of V.

S is a vector subspace of V if and only if S satisfies the following conditions,

(1) Additive Identity : $0 \in S$

(2) Closed under Addition : $s, t \in S$ implies $s + t \in S$

(3) Closed under scalar multiplication : $a \in \mathbb{F}$ and $s \in S$ implies $as \in S$

2.4 Example

Consider a proposed vector space of 2×2 symmetric matrices $\mathbf{M}$ with real entries.

That is, matrices satisfying $\mathbf{M}_{ij} = \mathbf{M}_{ji}$

(i) Show that the proposed vector space is indeed a vector space.

[3 marks]

(ii) Write down an explicit basis for this vector space.

[2 marks]

(iii) What is the dimension of this vector space ?

[1 mark]

Teaching Video : http://www.NumberWonder.co.uk/v9112/2.mp4

<= The video will talk through the solution to the example

(i) It is known that the set of all $m \times n$ matrices with real entries forms a real mn-dimensional vector space with matrix addition and scalar multiplication. Therefore, it only needs to be shown that the vector space of 2×2 symmetric matrices with real entries is a vector subspace of this larger vector space.

(1) Observe that, $\mathbf{0} \;=\; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which is symmetric.

As required, the zero element of the parent is in the vector subspace.

(2) Let two general 2×2 symmetric matrices be,

$$\mathbf{A} \;=\; \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \text{ and } \mathbf{B} \;=\; \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}$$

in which case, $\mathbf{A} + \mathbf{B} \;=\; \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ a_2 + b_2 & a_3 + b_3 \end{pmatrix}$ which is symmetric.

As required, there is closure under addition.

(3) Let a general 2×2 symmetric matrix and a general scalar be,

$$\mathbf{A} \;=\; \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \text{ and } s$$

in which case, $s\mathbf{A} \;=\; s\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \;=\; \begin{pmatrix} sa_1 & sa_2 \\ sa_2 & sa_3 \end{pmatrix}$ which is symmetric.

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of 2×2 symmetric matrices with real entries is indeed a vector subspace of the set of all $m \times n$ matrices with real entries.

It can be considered to be a vector space in its own right. □

[3 marks]

(ii) A suitable basis for the vector space of 2×2 symmetric matrices with real entries could be,

$$B \;=\; \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

[2 marks]

(iii) The vector space of 2×2 symmetric matrices with real entries is a 3-dimensional vector space, because the basis contains three vectors.

[1 mark]

2.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 50

Question 1

The set of all infinite sequences forms a real vector space over the field of real numbers under corresponding term-by-term addition and scalar multiplication. Consider a proposed vector subspace of geometric progressions with common ratio of 3. Show that the proposed vector subspace is indeed a vector subspace.

[3 marks]

Question 2

The set of points $(x, y, z) \in \mathbb{R}^3$ forms a real vector space over the field of reals.
Consider a proposed vector subspace defined by the restriction $x + y + z = 0$
Show that this proposed vector subspace is indeed a vector subspace.

[3 marks]

Question 3

The set of points $\left(x, y, z \right) \in \mathbb{R}^3$ forms a real vector space over the field of reals.

Consider a proposed vector subspace defined by the restriction $x + y + z = 1$
Show that the proposed vector subspace is not a vector subspace.

[3 marks]

Question 4

Show that all polynomials whose degree is equal to three do not form a vector space over the field of reals with polynomial addition and scalar multiplication.

[3 marks]

Question 5

Given a matrix **M**, the transpose of **M**, written $\mathbf{M}^{\mathrm{T}}$, is an operator which flips the matrix over on its diagonal: rows become columns and columns become rows.

$$\text{For example, } \begin{pmatrix} 1 & 5 & -6 \\ -2 & -4 & 3 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 \\ 5 & -4 \\ -6 & 3 \end{pmatrix}$$

Properties of the Transpose

- A square matrix, **S**, is considered orthogonal iff $\mathbf{S}^{\mathrm{T}} \times \mathbf{S} = \mathbf{S} \times \mathbf{S}^{\mathrm{T}} = \mathbf{I}$
- A square matrix, **S**, is symmetric iff $\mathbf{S}^{\mathrm{T}} = \mathbf{S}$
- A square matrix, **S**, is antisymmetric iff $\mathbf{S}^{\mathrm{T}} = -\mathbf{S}$
- The symmetric part of any square matrix **M** is $\frac{1}{2}\left(\mathbf{M} + \mathbf{M}^{\mathrm{T}}\right)$
- The antisymmetric part of any square matrix **M** is $\frac{1}{2}\left(\mathbf{M} - \mathbf{M}^{\mathrm{T}}\right)$
- $(\mathbf{A} \pm \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} \pm \mathbf{B}^{\mathrm{T}}$
- $\left(\mathbf{M}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{M}^{-1}\right)^{\mathrm{T}}$ provided $det(\mathbf{M}) \neq 0$

Explain why,

(i) Any square matrix is the sum of its symmetric and antisymmetric parts.

[2 marks]

(ii) Any antisymmetric matrix must have zeros on its diagonal.

[2 marks]

(iii) Any 3×3 antisymmetric matrix is singular.

[3 marks]

Question 6

Consider the vector space of 3×3 matrices $\mathbf{W}$ with real entries $\mathbf{W}_{ij}$

(i) What is the dimension of this vector space ?
Give a reason for your answer.

[2 marks]

A real antisymmetric matrix $\mathbf{W}$ has real entries such that $\mathbf{W}_{ij} = -\mathrm{W}_{ji}$

(ii) Show that the real antisymmetric 3×3 matrices form a vector subspace.

[3 marks]

(iii) Write down an explicit basis for this vector subspace.

[2 marks]

(iv) What is the dimension of this vector subspace ?

[1 mark]

(v) Given that the real antisymmetric $n \times n$ matrices form a vector subspace for any given positive integer value of n, what is the dimension of the vector subspace. Give your answer as a formula in terms of n.

[2 marks]

Question 7

(i) Use matrix methods and a calculator (or other device) that can handle matrices to show that any vector in $\mathbb{R}^3$ can be written as a linear combination of the following basis vectors,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

[5 marks]

(ii) Write a general vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of this basis in terms of x, y and z.

[2 marks]

Question 8

Let V be the vector space of all real 2×2 square matrices.

Let $\mathbf{A}$ be an orthogonal real 2×2 matrix; one where $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$

Consider the subset, $W := \{ \mathbf{A} \in V \mid \mathbf{A} \text{ is an orthogonal matrix} \}$

Is the set of all orthogonal matrices, W, a vector subspace of V ?

Justify your answer.

[5 marks]

Question 9

Show that the proposed basis, $B = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is not suitable for the vector space $\mathbb{Z}_p^2$, where p is a prime, but is suitable for the vector space $\mathbb{R}^2$.

[5 marks]

Question 10

A proposed vector subspace of $\mathbb{R}^4$ is based upon the following set,

$$S = \left\{ \begin{pmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{pmatrix} \middle| \; a, b, c \in \mathbb{R} \right\}$$

Find a suitable basis for S and state the dimension of the resulting vector subspace.

[4 marks]

2.6 Answers

Answer 1

Working through the three vector subspace conditions;

(1) The zero sequence can be thought of as a geometric progression with first term 0 and common ratio 3.

That is, $0 \times 3^0,\ 0 \times 3^1,\ 0 \times 3^2,\ 0 \times 3^3,\ \ldots,\ 0 \times 3^{n-1},\ \ldots$

As required, the zero element of the parent is in the proposed vector subspace.

(2) Let two general geometric progressions with common ratio 3 and first terms a and b be,

$$\begin{array}{lllllll} & a \times 3^0, & a \times 3^1, & a \times 3^2, & a \times 3^3, & \ldots, & a \times 3^{n-1}, \ \ldots \\ \text{ADD} & b \times 3^0, & b \times 3^1, & b \times 3^2, & b \times 3^3, & \ldots, & b \times 3^{n-1}, \ \ldots \\ \hline \end{array}$$

$$(a + b)3^0,\ (a + b)3^1,\ (a + b)\ 3^2,\ \ldots,\ (a + b)3^{n-1},\ \ldots$$

which is a geometric progression with common ratio 3

As required, there is closure under addition.

(3) Given a general scalar, s, and geometric progression, GP, with common ratio 3 and first term a,

$$\begin{aligned} s\,\text{GP} &= s\left(a \times 3^0,\ a \times 3^1,\ a \times 3^2,\ a \times 3^3,\ \ldots,\ a \times 3^{n-1},\ \ldots\right) \\ &= sa \times 3^0,\ sa \times 3^1,\ sa \times 3^2,\ sa \times 3^3,\ \ldots,\ sa \times 3^{n-1},\ \ldots \end{aligned}$$

which is a geometric progression with common ratio 3

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of geometric progressions with common ratio 3 is indeed a vector subspace of the real vector space of all infinite sequences. □

[3 marks]

Answer 2

Working through the three vector subspace conditions;

(1) The zero vector, the origin, the point $(0,\ 0,\ 0)$, is in the proposed vector subspace, as required.

(2) Let two general points in the proposed vector subspace be,

$(p,\ q, -(p+q))$ and $(r,\ s, -(r+s))$ in which case the sum is,

$(p+r,\ q+s, -(p+q)-(r+s))$

This is a point on $x+y+z=0$

As required, there is closure under addition.

(3) Given a general scalar, s, and a general point $(p,\ q, -(p+q))$,

$$s\big(p,\ q, -(p+q)\big) = \big(sp,\ sq, -s(p+q)\big)$$

which is a point on $x+y+z=0$

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of points in 3-dimensional Euclidean space that satisfy the equation $x+y+z=0$ is indeed a vector subspace. □

[3 marks]

Answer 3

Showing that any one of the three conditions necessary is not satisfied is sufficient to stop the proposed vector subspace being accepted as such.

Method 1

Observe that the zero vector, the origin, the point $(0,\ 0,\ 0)$, is not in the proposed vector subspace, as $0+0+0 \neq 1$

Method 2

Let two general points in the proposed vector subspace be,

$(p,\ q,\ 1-(p+q))$ and $(r,\ s,\ 1-(r+s))$ in which case the sum is,

$(p+r,\ q+s,\ 2-(p+q)-(r+s))$

This is not a point on $x+y+z=1$ because,

$$p+r+q+s+2-p-q-r-s \neq 1$$

$\therefore$ The closure under addition requirement is not satisfied.

(Or give any specific example rather than this generalised one)

Method 3

Given a general scalar, s, and a general point $(p,\ q,\ 1-(p+q))$,

$$s\big(p,\ q,\ 1-(p+q)\big) = \big(sp,\ sq, s - s(p+q)\big)$$

This is not a point on $x+y+z=1$ because,

$$sp+sq+s-sp-sq \neq 1 \text{ for all } s$$

$\therefore$ The closure under scalar multiplication requirement is not satisfied.

[3 marks]

Answer 4

Showing that any one of the three conditions is not satisfied is sufficient to stop the proposed vector subspace being accepted as such.

Method 1

The zero polynomial, which is the number 0, is not a polynomial of degree three as the definition of a degree three polynomial, a cubic, is that it is of the form, $p(x) = ax^3 + bx^2 + cx + d$ where $a, b, c, d \in \mathbb{R},\ a \neq 0$.

Method 2

Let one polynomial of degree three be, $p(x) = x - x^3$, and another be $q(x) = x^3$, in which case $p(x) + q(x) = x$ which is not a polynomial of degree three. $\therefore$ The closure under addition requirement is not satisfied.

[3 marks]

Answer 5

(i) Symmetric Matrix Part + Antisymmetric Matrix Part

$$= \frac{1}{2}\left(\mathbf{M} + \mathbf{M}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathbf{M} - \mathbf{M}^{\mathrm{T}}\right)$$

$$= \frac{1}{2}\mathbf{M} + \frac{1}{2}\mathbf{M}^{\mathrm{T}} + \frac{1}{2}\mathbf{M} - \frac{1}{2}\mathbf{M}^{\mathrm{T}}$$

$$= \mathbf{M} \qquad \square$$

[2 marks]

(ii)

- A square $n \times n$ matrix, $\mathbf{S}$, is antisymmetric iff $\mathbf{S}^{\mathrm{T}} = -\mathbf{S}$

$$\therefore \begin{pmatrix} s_{11} & s_{21} & \dots & s_{n1} \\ s_{12} & s_{22} & \dots & s_{n2} \\ \dots & \dots & \dots & \dots \\ s_{1n} & s_{2n} & \dots & s_{nn} \end{pmatrix} = \begin{pmatrix} -s_{11} & -s_{12} & \dots & -s_{1n} \\ -s_{21} & -s_{22} & \dots & -s_{2n} \\ \dots & \dots & \dots & \dots \\ -s_{n1} & -s_{n2} & \dots & -s_{nn} \end{pmatrix}$$

In other words, $s_{ij} = -s_{ji}$ for $1 \leqslant i, j \leqslant n$

On the diagonal, let $k = i = j$, and the requirement is that $s_{kk} = -s_{kk}$

Therefore, $s_{kk} = 0$ so an antisymmetric matrix has zeros on its diagonal. $\square$

[2 marks]

(iii) Let a general 3×3 antisymmetric matrix be, $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$

$$\therefore \begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} = 0\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix} - a\begin{vmatrix} -a & c \\ -b & 0 \end{vmatrix} + b\begin{vmatrix} -a & 0 \\ -b & -c \end{vmatrix}$$

$$= 0 - abc + abc$$

$$= 0 \quad \text{and so is singular.} \qquad \square$$

[3 marks]

Answer 6

(i) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \ldots + i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

so a basis for the vector space is,

$$B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

which is 9-dimensional.

[2 marks]

(ii) Working through the three vector subspace conditions;

(1) The zero vector is the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which can be considered antisymmetric and so is in the vector subspace, as required.

(2) Let two general antisymmetric matrices be,

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix}$$

in which case the sum is, $\begin{pmatrix} 0 & a+d & b+e \\ -(a+d) & 0 & c+f \\ -(b+e) & -(c+f) & 0 \end{pmatrix}$

which is an antisymmetric matrix.

As required, there is closure under addition.

(3) Given a general scalar, s, and a general antisymmetric matrix,

$$s\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = \begin{pmatrix} 0 & sa & sb \\ -sa & 0 & sc \\ -sb & -sc & 0 \end{pmatrix}$$

which is an antisymmetric matrix.

As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace of real 3×3 antisymmetric matrices is indeed a vector subspace. □

[3 marks]

(iii)

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

A suitable basis is, for example,

$$B = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

[2 marks]

(iv) The vector subspace is 3-dimensional.

[1 mark]

(v) The number of vectors in the basis of an $n \times n$ antisymmetric matrix can be explored for a few low values of n, as in part (iii) and (iv),

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

2×2 — 1-dimensional

3×3 — 3-dimensional

4×4 — 6-dimensional

The dimensionality is clearly given by triangular numbers and a triangle can even be drawn around the linearly independent entries as shown.

$\therefore$ Number of dimensions of the vector subspace is $\dfrac{n(n-1)}{2}$

[2 marks]

Answer 7

(i) This question is about showing that any vector $\boldsymbol{v} \in \mathbb{R}^3$ can be written as a linear combination of the basis vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$

That is, $a\,\boldsymbol{v}_1 + b\,\boldsymbol{v}_2 + c\,\boldsymbol{v}_3 = \boldsymbol{v}$ for some a, b, and c to be determined.

$$a\begin{pmatrix}1\\-1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\-1\end{pmatrix} + c\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \quad \left\{\begin{array}{rcl} a + 2c &=& x \\ -a + b &=& y \\ -b + c &=& z \end{array}\right.$$

These simultaneous equations can be expressed in matrix form as,

$$\begin{pmatrix}1 & 0 & 2\\-1 & 1 & 0\\0 & -1 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

From a calculator,

$$\text{If } \mathbf{M} = \begin{pmatrix}1 & 0 & 2\\-1 & 1 & 0\\0 & -1 & 1\end{pmatrix} \text{ then } \mathbf{M}^{-1} = \frac{1}{3}\begin{pmatrix}1 & -2 & -2\\1 & 1 & -2\\1 & 1 & 1\end{pmatrix}$$

(Classwiz fx911ex: “MENU 4”, “1” to input matrix, “Option 3”, “x^{-1}”, “=”)

$$\therefore \begin{pmatrix}a\\b\\c\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & -2 & -2\\1 & 1 & -2\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\therefore \ a = \frac{x - 2y - 2z}{3}, \ b = \frac{x + y - 2z}{3} \text{ and } c = \frac{x + y + z}{3}$$

This shows that, for any specified x. y and z, suitable values can be found for a, b and c such that a linear combination of the basis vectors yields that specified point.

[5 marks]

(ii)
$$\left(\frac{x - 2y - 2z}{3}\right)\boldsymbol{v}_1 + \left(\frac{x + y - 2z}{3}\right)\boldsymbol{v}_2 + \left(\frac{x + y + z}{3}\right)\boldsymbol{v}_3 = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

[2 marks]

Answer 8

Statement : The set of all orthogonal matrices do not form a vector space.

Proof,

The zero vector is not orthogonal because $\mathbf{O}^{\mathrm{T}}\mathbf{O} = \mathbf{O} \neq \mathbf{I}$

$\therefore$ At least one of the necessary conditions to be a vector subspace is not satisfied. □

Alternatively, it could be shown that under either addition or scalar multiplication the closure property does not hold. For example,

$\mathbf{I}^{\mathrm{T}}\mathbf{I} = \mathbf{I}$ so $\mathbf{I}$ is orthogonal but $2\mathbf{I}$ is not orthogonal as $(2\mathbf{I})^{\mathrm{T}}(2\mathbf{I}) = 4\mathbf{I} \neq \mathbf{I}$

[5 marks]

Answer 9

If $B = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is a suitable basis for a vector space it will be possible to get to any point in that vector space via a linear combination of the basis vectors. As any vector space for which B is suitable will be two dimensional (because there are two vectors in B) let a point in that vector space be (x, y) in which case,

$$a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where a and b and the addition and multiplication are from the field over which the vector space acts.

$$\begin{array}{rrcl} & a + b & = & x \\ \text{ADD} & -a + b & = & y \\ \hline & 2b & = & x + y \end{array}$$

$$b = \frac{x + y}{2} \quad \text{and so} \quad a = \frac{x - y}{2}$$

This shows that,

(i) The proposed basis is not suitable for the vector space $\mathbb{Z}_p^2$ because this vector space is over a field of integers modulo p and integer values of x and y will not necessarily result in integer values of a and b.

(ii) The proposed basis is suitable for the vector space $\mathbb{R}^2$ because this vector space is over a field of real numbers and real values of x and y will result in real values for a and b.

[5 marks]

Answer 10

Clearly, any vector in S can be written as,

$$\begin{pmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{pmatrix} = a\begin{pmatrix} 3 \\ 6 \\ -9 \\ -3 \end{pmatrix} + b\begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} + c\begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix}$$

However, these three vectors are not linearly independent because,

$\begin{pmatrix} 3 \\ 6 \\ -9 \\ -3 \end{pmatrix} = (-3)\begin{pmatrix} -1 \\ -2 \\ 3 \\ 1 \end{pmatrix}$ so a suitable basis is, for example, $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 5 \\ 1 \end{pmatrix} \right\}$

which means the vector subspace spanned by this basis is 2-dimensional.

[4 marks]

3.1 Merging Vector Subspaces

Having established what a vector subspace is, and how to determine whether a proposed vector subspace is or is not a vector subspace, there are various ways of merging two vector subspaces. In this work it will be assumed that the two vector subspaces to be merged are both subspaces of the same vector space.

Suppose that a vector space V has two subspaces U and W.
One type of merge of particular interest is the sum of the subspaces, $U + W$.
This is itself a vector subspace, as will be proven in the exercises.
It is the set of all possible sums $\boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$.

If the basis of U and W are,

$$B_U = \{ \boldsymbol{u}_{b1}, \boldsymbol{u}_{b2}, \dots, \boldsymbol{u}_{bn} \} \text{ and } B_W = \{ \boldsymbol{w}_{b1}, \boldsymbol{w}_{b2}, \dots, \boldsymbol{w}_{bm} \}$$

then the basis of $U + W$ is buried somewhere in the list of vectors,

$$\boldsymbol{u}_{b1}, \boldsymbol{u}_{b2}, \dots, \boldsymbol{u}_{bn}, \boldsymbol{w}_{b1}, \boldsymbol{w}_{b2}, \dots, \boldsymbol{w}_{bm}$$

Definition : Span

Given a list of vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n$, the set of all linear combinations of those vectors is termed their span, and is denoted $span(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n)$

The problem the list of vectors presents is that it may not be linearly independent and so may not be a ready made basis for the vector subspace $U + W$.
It may be that it's easy to spot that one of the vectors in the list is a multiple of another, in which case it can be removed, but with a list of more than two vectors it's possible for none to be a multiple of one other and yet still the system is not linearly independent because one is a linear combination of the others.

Definition: Linearly independent

Given a list of vectors, $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \dots, \boldsymbol{v}_n$ in V over a field $\mathbb{F}$, if the only choice of $a_1, a_2, a_3, \dots, a_n \in \mathbb{F}$ that makes $a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 + \dots + a_n\boldsymbol{v}_n$ equal to zero is $a_1 = a_2 = a_3 = \dots = a_n = 0$ then the list of vectors are said to be linearly independent.

3.2 Example

(i) Show that the following list of vectors is not linearly independent over $\mathbb{R}^3$,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

(ii) Remove a vector show the remaining two are linearly independent.

[4, 2 marks]

Teaching Video: http://www.NumberWonder.co.uk/v9112/3.mp4

<= The video will present a solution to the example

[6 marks]

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Let V be a real vector space over the field $\mathbb{R}^3$

Let U and W be the vector subspaces with basis,

$$B_U = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \; B_W = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(i) Describe geometrically the following vector subspaces of V,

(a) U

[1 mark]

(b) W

[1 mark]

(c) $U \cap W$

[1 mark]

(ii) **(a)** Write down a basis for $U + W$

(Be sure to check that it is linearly independent)

[2 marks]

(b) Describe geometrically the vector subspace $U + W$

[2 marks]

Question 2

Let V be a real vector space over the field $\mathbb{R}^3$

Let U and W be the vector subspaces with basis,

$$B_U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \; B_W = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(i) Describe geometrically the following vector subspaces of V,

(a) U

[2 mark]

(b) W

[2 mark]

(c) $U \cap W$

[2 mark]

(ii) **(a)** Write down a basis for $U + W$

(Be sure to check that it is linearly independent)

[2 marks]

(b) Describe geometrically the vector subspace $U + W$

[2 marks]

Question 3

Let V be a real vector space over the field $\mathbb{R}^3$

Let U and W be the vector subspaces with basis,

$$B_U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \ B_W = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(i) Describe geometrically the following vector subspaces of V,

(a) U

[3 mark]

(b) W

[3 mark]

(c) $U \cap W$

[3 mark]

(ii) **(a)** Write down a basis for $U + W$

(Be sure to check that it is linearly independent)

[4 marks]

(b) Describe geometrically the vector subspace $U + W$

[2 marks]

Question 4

Consider a real vector space V and two vector subspaces U and W of V

(a) Prove that the intersection $U \cap W$ is a vector subspace of V

[6 marks]

(b) Prove that $U + W$ is a vector subspace of V

($U + W$ is the set of all sums $\boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$)

[6 marks]

(c) The dimensions of the above vector spaces are related by,

$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$

Verify this formula for the specific example where $V = \mathbb{R}^3$, and where

- U is spanned by $\boldsymbol{u}_1 = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{u}_2 = \boldsymbol{k}$
- W is spanned by $\boldsymbol{w}_1 = \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{w}_2 = -\boldsymbol{i} + 2\boldsymbol{j}$

[6 marks]

3.4.1 Solution to 3.2 Example (Teaching Video)

(i) From studying the list of vectors,

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

it is clear that there is no non-zero value of k such that,

$$\boldsymbol{v}_1 = k\boldsymbol{v}_2 \ \text{ or } \ \boldsymbol{v}_1 = k\boldsymbol{v}_3 \ \text{ or } \ \boldsymbol{v}_2 = k\boldsymbol{v}_3$$

However, we are told this list is not linearly independent and so there must be some combination of two that gives the other. To force the issue solve,

$$a\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} a - b = -1 \\ 2a + b = 4 \\ -a = -1 \end{cases}$$

From the lower equation $a = 1$ and then from the upper $b = 2$

Crucially, check that those two solutions satisfy the middle equation;

$$\begin{aligned} \text{LHS} &= 2a + b \\ &= 2 \times 1 + 2 \\ &= 4 \\ &= \text{RHS} \qquad \square \end{aligned}$$

Thus $\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and so list $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is not linearly independent.

[4 marks]

(ii) Now $\boldsymbol{v}_3$ can be removed from the list of vectors.

With a list of only two vectors, to show they are linearly independent simply observe that there is no non-zero value of k such that $\boldsymbol{v}_1 = k\boldsymbol{v}_2$.

Alternatively, show that the only value of a and b that makes $a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = \mathbf{0}$ is $a = b = 0$. That is,

$$a\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a - b = 0 \\ 2a + b = 0 \\ -a = 0 \end{cases}$$

From the lower equation, $a = 0$, and then the upper equation gives $b = 0$

$\therefore$ $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. $\square$

[2 marks]

3.4.2 Solutions to 3.3 Exercise

Answer 1

(i) **(a)** The y-axis

(b) The line $y = x$ across the x-y plane (when $z = 0$)

As a vector straight line equation, $\boldsymbol{r} = \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(c) The point (0, 0, 0)

[3 marks]

(ii) **(a)** $B = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

To verify the two vectors used in the basis are linearly independent simply note that there is no non-zero value for k such that,

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

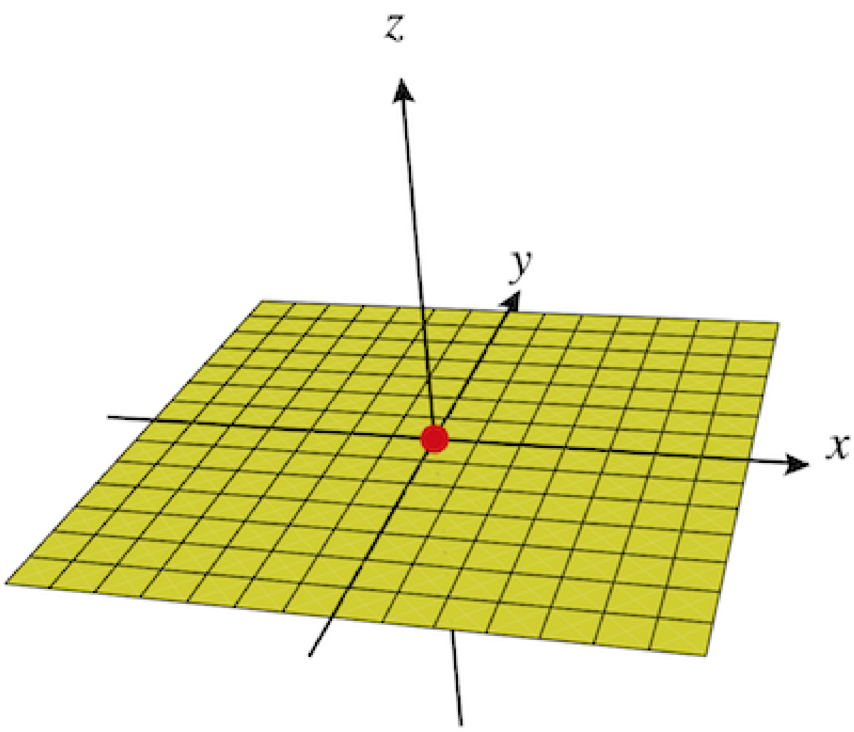
[2 marks]

(b) As there are two vectors in the basis for $U + W$ it must be a 2-dimensional vector subspace, which will be a plane.

$$a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} \quad\;\; b = x \\ a + b = y \\ \quad\;\; 0 = z \end{cases}$$

As $z = 0$, that is the Cartesian equation of the plane,

[2 marks]

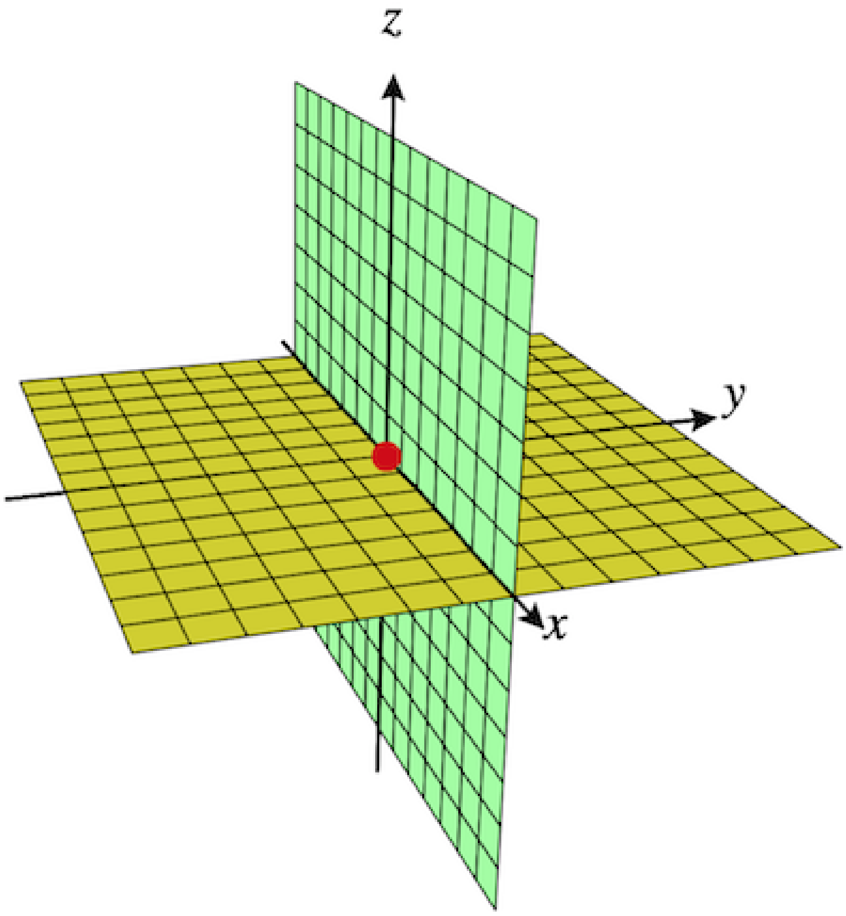


Answer 2

(i) **(a)** The plane $z = 0$

(b) The plane $y = 0$

(c) The x-axis



[3 marks]

(ii) **(a)** In merging together the basis for U with the basis for W we clearly do not need two copies of the vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

The basis for $U + W$ is thus,

$$B_{U+W} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$$

which is immediately recognizable as the standard basis for $\mathbb{R}^3$ and so is linearly independent without any further checking.

[2 marks]

(b) Geometrically $U + W$ is all of the three dimensional vector space $\mathbb{R}^3$ which is also all of V

[2 marks]

Answer 3

(i) **(a)** $a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} a \quad = x \\ \quad b = y \\ a \quad = z \end{cases}$

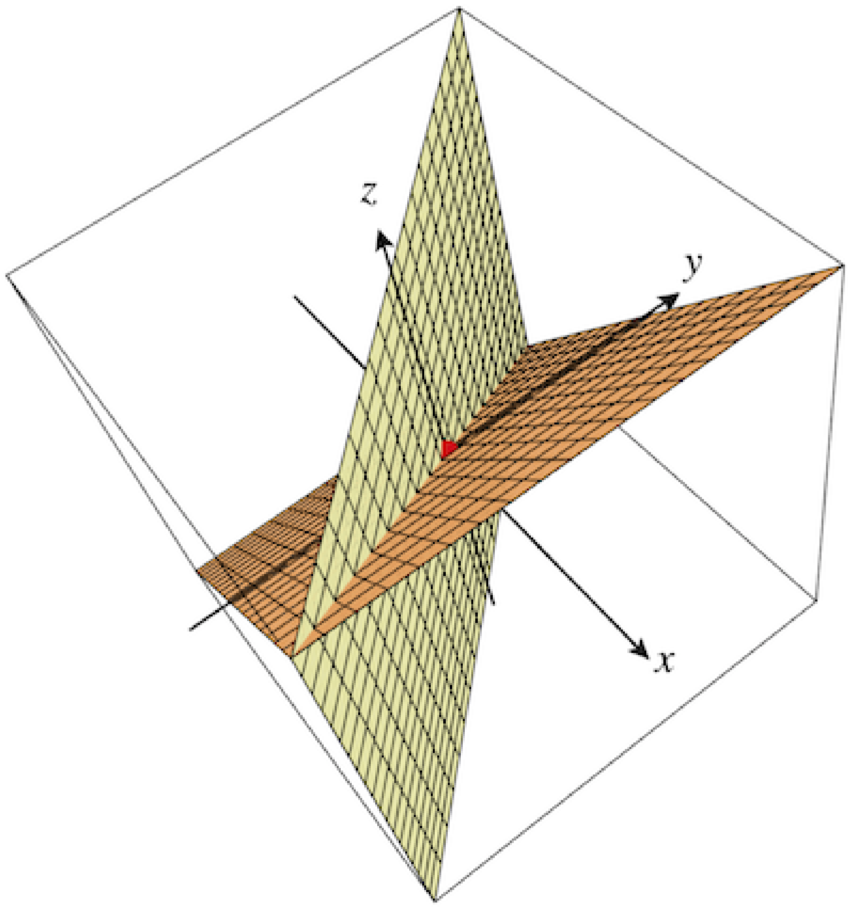
From the upper and lower equations, the equation of the 2-dimensional plane is $x - z = 0$

[3 marks]

(b) $a\begin{pmatrix}1\\-1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} a \quad = x \\ -a \quad = y \\ \quad b = z \end{cases}$

From the two uppermost equations, the equation of the 2-dimensional plane is $x + y = 0$

[3 marks]



(c) The two planes will intersect in the line with equation

$x = -y = z$ (in Cartesian form)

$\boldsymbol{r} = \mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$ (in vector form)

[3 marks]

(ii) **(a)** The list of vectors from the merge of the two basis is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These cannot be linearly independent because the vector subspace has to have a dimension not greater than that of its parent.
A basis of four vectors is 4-dimensional, the parent is 3-dimensional.

That is, $dim(U + W) \leqslant dim(\mathbb{R}^3)$

$$dim(U + W) \leqslant 3$$

Notice that $\boldsymbol{v}_1 - \boldsymbol{v}_2 - \boldsymbol{v}_3 = \boldsymbol{v}_4$, so $\boldsymbol{v}_4$ can be dropped from the list.

If the remaining three vectors are linearly independent than the only solution to the following equation will be that $a = b = c = 0$.

$$a\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a \quad + c = 0 \\ \quad b - c = 0 \\ a \quad\quad = 0 \end{cases}$$

The lower equation gives $a = 0$, which when substituted into the middle equation gives $b = 0$, and which when substituted into the upper equation gives $c = 0$.

$\therefore$ These three vectors are a suitable basis for $U + W$.

$$B_{U+W} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

(In fact any three chosen from the four would be suitable)

[4 marks]

(b) As the basis has three vectors it is three dimensional and so must be all of V. In other words $U + W$ is $\mathbb{R}^3$.

[2 marks]

Answer 4

(a) Working through the three vector subspace conditions;

(1) By definition, as we are told that U and W are vector subspaces, each must contain the zero vector.
Hence the zero vector is in the intersection of U and W.
$\therefore$ The requirement the zero be in the subspace $U \cap W$ is satisfied.

(2) Let two general elements in the proposed vector subspace $U \cap W$ be $\boldsymbol{x}$ and $\boldsymbol{y}$. Then $\boldsymbol{x}$ is in U and $\boldsymbol{y}$ is in U and so, because we are told that U is a vector subspace, $\boldsymbol{x} + \boldsymbol{y}$ must be in U.
By the same argument, $\boldsymbol{x}$ is in W and $\boldsymbol{y}$ is in W and so $\boldsymbol{x} + \boldsymbol{y}$ is in W.
Therefore, $\boldsymbol{x} + \boldsymbol{y}$ is in $U \cap W$.
As required, there is closure under addition.

(3) Given a general scalar, s, and a general element $\boldsymbol{x}$ in $U \cap W$, then $\boldsymbol{x}$ is in U and so $s\boldsymbol{x}$ is in U as we are told U is a vector subspace.
By the same argument, $\boldsymbol{x}$ is in W and so $s\boldsymbol{x}$ is in W.
Therefore, $s\boldsymbol{x}$ is in $U \cap W$.
As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace $U \cap W$ is indeed a vector subspace. $\square$

[6 marks]

(b) Working through the three vector subspace conditions;

(1) By definition, as we are told that U and W are vector subspaces, each must contain the zero vector. Now, $\mathbf{0} + \mathbf{0} = \mathbf{0}$
Hence the zero vector is in the sum of U and W.
$\therefore$ The requirement the zero be in the subspace $U + W$ is satisfied.

(2) Suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are two general elements (vectors) in the proposed vector subspace $U + W$.
Each can be decomposed into a sum from which it was formed.
In other words,

$\boldsymbol{x} = \boldsymbol{u}_1 + \boldsymbol{w}_1$ and $\boldsymbol{y} = \boldsymbol{u}_2 + \boldsymbol{w}_2 \qquad \boldsymbol{u}_1, \boldsymbol{u}_2 \in U$ and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$

Consider now, the vector $\boldsymbol{x} + \boldsymbol{y}$,

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &= \boldsymbol{u}_1 + \boldsymbol{w}_1 + \boldsymbol{u}_2 + \boldsymbol{w}_2 \\ &= \left(\boldsymbol{u}_1 + \boldsymbol{u}_2 \right) + \left(\boldsymbol{w}_1 + \boldsymbol{w}_2 \right) \end{aligned}$$

(As a vector space over a field has commutativity and associativity)

Note that $\boldsymbol{u}_1 + \boldsymbol{u}_2$ is in U because $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U$ and U is a vector subspace in which, by definition, vector addition holds.

Similarly, $\boldsymbol{w}_1 + \boldsymbol{w}_2$ is in W.

Thus $\left(\boldsymbol{u}_1 + \boldsymbol{u}_2 \right) + \left(\boldsymbol{w}_1 + \boldsymbol{w}_2 \right) \in U + W$

$\therefore \; \boldsymbol{x} + \boldsymbol{y} \in U + W$

As required, there is closure under addition.

(3) Suppose that $\boldsymbol{x}$ is a general element in $U + W$ that, as before, can be decomposed into a sum from which it was formed,

$\boldsymbol{x} = \boldsymbol{u} + \boldsymbol{w} \qquad \boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$

Let s be a general scalar from the field under the vector space in which case $\; s\boldsymbol{x} = s\left(\boldsymbol{u} + \boldsymbol{w} \right) = s\boldsymbol{u} + s\boldsymbol{w}$

(As a vector space over a field is distributive)
Note that $s\boldsymbol{u}$ is in U because $\boldsymbol{u} \in U$ and U is a vector subspace in which, by definition, scalar multiplication holds.
Similarly, $s\boldsymbol{w}$ is in W.
Thus $s\boldsymbol{u} + s\boldsymbol{w} \in U + W$
As required, there is closure under scalar multiplication.

With all three conditions satisfied the proposed vector subspace $U + W$ is indeed a vector subspace. $\square$

[6 marks]

(c) Although not asked for, we can get a feel for this question by working out the equation of each of the two two-dimensional planes that are the geometric interpretation of the vector subspaces U and W.

For U,

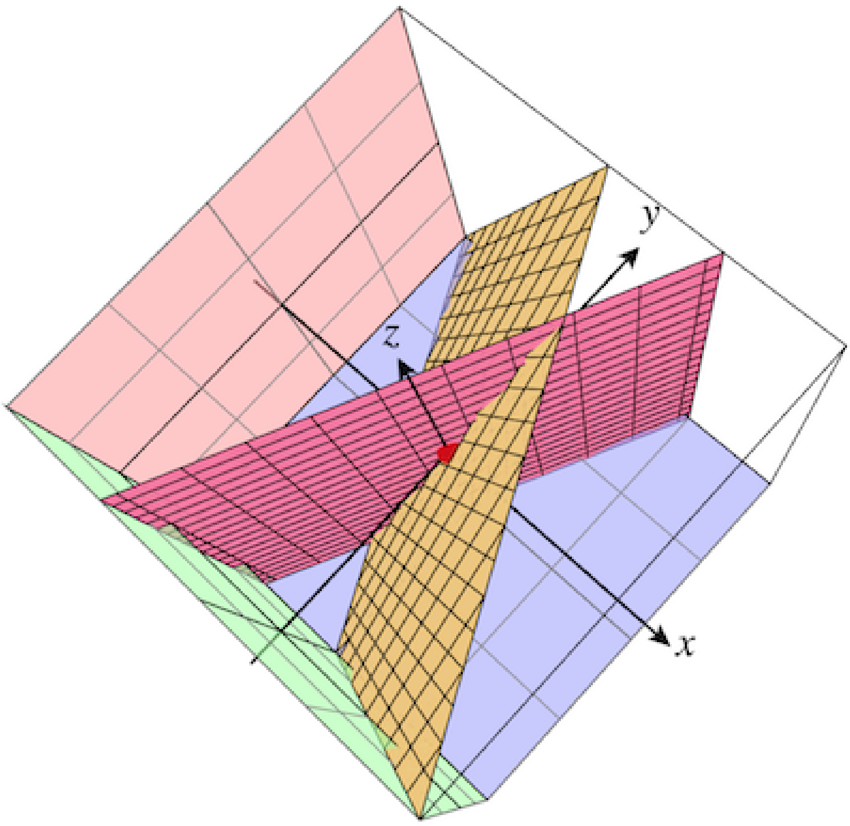
$$a\begin{pmatrix}1\\2\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} a = x \\ 2a = y \\ b = z \end{cases}$$

This subspace is the plane with Cartesian equation $y = 2x$

For W,

$$a\begin{pmatrix}0\\1\\1\end{pmatrix} + b\begin{pmatrix}-1\\2\\0\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \Rightarrow \begin{cases} - b = x \\ a + 2b = y \\ a = z \end{cases}$$

This subspace is the plane with Cartesian equation $2x + y - z = 0$



The intersection of two planes that are not parallel is a line (one-dimensional)
Its equation is,

$$2x = y = \frac{z}{2} \text{ (in Cartesian form)}$$

$$\boldsymbol{r} = \mu\begin{pmatrix}1\\2\\4\end{pmatrix} \text{ (in vector form)}$$

So far, it has been established that

$$dim(U) = 2,\ dim(W) = 2 \text{ and } dim(U \cap W) = 1$$

The remaining issue is to determine $dim(U + W)$

The list of vectors from the merge of the two basis is

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\ \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

These cannot be linearly independent because the vector subspace has to have a dimension not greater than that of its parent.
A basis of four vectors is 4-dimensional, the parent is 3-dimensional.

That is, $dim\left(U + W\right) \leqslant dim\left(\mathbb{R}^3\right)$

$$dim\left(U + W\right) \leqslant 3$$

Notice that $-\boldsymbol{v}_1 - 4\boldsymbol{v}_2 + 4\boldsymbol{v}_3 = \boldsymbol{v}_4$, so $\boldsymbol{v}_4$ can be dropped from the list.
If the remaining three vectors are linearly independent than the only solution to the following equation will be that $a = b = c = 0$.

$$a\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a & & = 0 \\ 2a & + c & = 0 \\ & b + c & = 0 \end{cases}$$

The upper equation gives $a = 0$, which when substituted into the middle equation gives $c = 0$. Substituted $c = 0$ into the lower equation gives $b = 0$.
$\therefore$ These three vectors are a suitable basis for $U + W$.

$$B_{U+W} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

(In fact any three chosen from the four would be suitable)

$$\therefore dim\ (U + W) = 3$$

Verification: The claim is that, for this example,

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

$$\begin{aligned} \text{RHS} &= dim(U) + dim(W) - dim(U \cap W) \\ &= 2 + 2 - 1 \\ &= 3 \\ &= \text{LHS} \end{aligned}$$

$\therefore$ The proposed formula holds in this case. □

[6 marks]