

1.3 Answers

A-Level Pure Mathematics : Year 2

Integration III

1.3.1 Solution to 1.2 Example

Example

$$u^2 = 2x - 1 \Rightarrow x = \frac{u^2}{2} + \frac{1}{2} \Rightarrow \frac{dx}{du} = u$$

When $x = 5$, $u = 3$ and when $x = 1$, $u = 1$

$$\begin{aligned} \int_1^5 \frac{3x}{\sqrt{2x-1}} dx &= \frac{3}{2} \int_1^3 \frac{u^2 + 1}{\sqrt{u^2}} u du \\ &= \frac{3}{2} \int_1^3 u^2 + 1 du \\ &= \frac{3}{2} \left[\frac{u^3}{3} + u \right]_1^3 \\ &= \frac{3}{2} \left\{ [9 + 3] - \left[\frac{1}{3} + 1 \right] \right\} \\ &= 16 \end{aligned}$$

[8 marks]

1.3.2 Solutions to 1.3 Exercise

Answer 1

$$u^2 = x - 2 \Rightarrow x = u^2 + 2 \Rightarrow \frac{dx}{du} = 2u$$

When $x = 11$, $u = 3$ and when $x = 3$, $u = 1$

$$\begin{aligned} \int_3^{11} \frac{x}{\sqrt{x-2}} dx &= \int_1^3 \frac{u^2 + 2}{\sqrt{u^2}} 2u du \\ &= 2 \int_1^3 u^2 + 2 du \\ &= 2 \left[\frac{u^3}{3} + 2u \right]_1^3 \\ &= 2 \left\{ [9 + 6] - \left[\frac{1}{3} + 2 \right] \right\} \\ &= 25\frac{1}{3} \end{aligned}$$

[8 marks]

Answer 2

$$u = x - 2 \Rightarrow x = u + 2 \Rightarrow \frac{dx}{du} = 1$$

When $x = 10$, $u = 8$ and when $x = 6$, $u = 4$

$$\begin{aligned} \int_6^{10} \frac{x(x-4)}{(x-2)^2} dx &= \int_4^8 \frac{(u+2)(u-2)}{u^2} du \\ &= \int_4^8 \frac{u^2 - 4}{u^2} du \\ &= \int_4^8 1 - 4u^{-2} du \\ &= \left[u - \frac{4u^{-1}}{(-1)} \right]_4^8 \\ &= \left[u + \frac{4}{u} \right]_4^8 \\ &= \left\{ \left[8 + \frac{1}{2} \right] - [4 + 1] \right\} \\ &= 3\frac{1}{2} \end{aligned}$$

[8 marks]

Answer 3

$$u^2 = 2x - 3 \Rightarrow x = \frac{u^2}{2} + \frac{3}{2} \Rightarrow \frac{dx}{du} = u$$

When $x = 6$, $u = 3$ and when $x = 2$, $u = 1$

$$\begin{aligned} \int_2^6 \frac{x-1}{\sqrt{2x-3}} dx &= \int_1^3 \frac{\left(\frac{u^2}{2} + \frac{3}{2} - 1\right)}{\sqrt{u^2}} u du \\ &= \int_1^3 \frac{u^2}{2} + \frac{1}{2} du \\ &= \left[\frac{u^3}{6} + \frac{u}{2} \right]_1^3 \\ &= \left\{ \left[\frac{9}{2} + \frac{3}{2} \right] - \left[\frac{1}{6} + \frac{1}{2} \right] \right\} \\ &= \left\{ [6] - \left[\frac{2}{3} \right] \right\} \\ &= 5\frac{1}{3} \end{aligned}$$

[8 marks]

Answer 4

$$x = \sin \theta \Rightarrow \theta = \arcsin x \Rightarrow \frac{dx}{d\theta} = \cos \theta$$

When $x = 1$, $\theta = \frac{\pi}{2}$ and when $x = 0.5$, $\theta = \frac{\pi}{6}$

$$\begin{aligned} \int_{0.5}^1 \frac{1}{\sqrt{1-x^2}} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2 \theta}} \cos \theta \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos^2 \theta}} \cos \theta \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\cos \theta} \cos \theta \, d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 \, d\theta \\ &= \left[\theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left\{ \left[\frac{\pi}{2} \right] - \left[\frac{\pi}{6} \right] \right\} \\ &= \frac{\pi}{3} \end{aligned}$$

[8 marks]

Answer 5

$$u = 3 - x \Rightarrow x = 3 - u \Rightarrow \frac{dx}{du} = -1$$

When $x = 2.5$, $u = 0.5$ and when $x = 2$, $u = 1$

$$\begin{aligned}\int_2^{2.5} \frac{2x + 1}{(3 - x)^6} dx &= \int_1^{0.5} \frac{2(3 - u) + 1}{u^6} (-1) du \\&= \int_1^{0.5} \frac{7 - 2u}{u^6} (-1) du \\&= \int_1^{0.5} \frac{2u - 7}{u^6} du \\&= \int_1^{0.5} 2u^{-5} - 7u^{-6} du \\&= \left[\frac{2u^{-4}}{(-4)} - \frac{7u^{-5}}{(-5)} \right]_1^{0.5} \\&= \left[-\frac{1}{2u^4} + \frac{7}{5u^5} \right]_1^{0.5} \\&= \left\{ \left[-8 + \frac{7 \times 32}{5} \right] - \left[-\frac{1}{2} + \frac{7}{5} \right] \right\} \\&= [36.8] - [0.9] \\&= 35.9\end{aligned}$$

[9 marks]

Answer 6

$$u = 2x + 1 \Rightarrow x = \frac{u}{2} - \frac{1}{2} \Rightarrow \frac{dx}{du} = \frac{1}{2}$$

When $x = 0.5$, $u = 2$ and when $x = -0.25$, $u = 0.5$

$$\begin{aligned} \int_{-0.25}^{0.5} \frac{48x}{(2x+1)^5} dx &= \int_{0.5}^2 \frac{48 \left(\frac{u}{2} - \frac{1}{2} \right)}{u^5} \times \frac{1}{2} du \\ &= \int_{0.5}^2 \frac{12u - 12}{u^5} du \\ &= \int_{0.5}^2 \frac{12u}{u^5} - \frac{12}{u^5} du \\ &= \int_{0.5}^2 12u^{-4} - 12u^{-5} du \\ &= \left[\frac{12u^{-3}}{(-3)} - \frac{12u^{-4}}{(-4)} \right]_{0.5}^2 \\ &= \left[-\frac{4}{u^3} + \frac{3}{u^4} \right]_{0.5}^2 \\ &= \left\{ \left[-\frac{1}{2} + \frac{3}{16} \right] - \left[-32 + 48 \right] \right\} \\ &= \left[-\frac{5}{16} \right] - [16] \\ &= -16\frac{5}{16} \end{aligned}$$

[9 marks]