3.3 Answers (3.2 Exercise)

Answer 1

$$\int_0^{\frac{\pi}{6}} \tan x \; dx = \Big[\ln (\sec x) \Big]_0^{\frac{\pi}{6}}$$

$$= \left[\ln \left(\frac{1}{\cos x} \right) \right]_0^{\frac{\pi}{6}}$$

$$= \ln \left(\frac{1}{\cos \left(\frac{\pi}{6} \right)} \right) - \ln \left(\frac{1}{\cos 0} \right)$$

$$= \ln \left(\frac{2}{\sqrt{3}} \right) - \ln \left(\frac{1}{1} \right)$$

$$= \ln (2) - \ln (\sqrt{3}) - \ln (1)$$

$$= \ln (2) - \frac{1}{2} \sqrt{3} \qquad \square$$

[4 marks]

Answer 2

$$\int x \sec^2 x \; dx$$

$$= \textbf{\textit{L}} \quad \textbf{\textit{I}} \qquad \textbf{\textit{D}} \quad \textbf{\textit{I}}$$

$$= x \; \tan x - \int 1 \; \tan x \; dx$$

$$= x \tan x - \int \tan x \; dx$$

$$= x \tan x - \ln | \sec x | + c$$

[4 marks]

Answer 3

(a)

$$\cos^2 x + \sin^2 x = 1$$

Divide through by $\cos^2 x$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1 \qquad \square$$

[2 marks]

(b)

$$\int x\ tan^2 x\ dx = \int x\ (sec^2 x - 1)\ dx$$
$$= \int x\ sec^2 x\ dx - \int x\ dx$$
$$= x\ tan\,x - ln\ |\ sec\,x\ | - \frac{x^2}{2} + c$$

[3 marks]

Answer 4

$$\int cot\,x\ dx = \int \frac{cos\,x}{sin\,x}\ dx$$
$$= \int (\,cos\,x\,)\ (\,sin\,x\,)^{-1}\ dx$$
$$= ln\ |\ sin\,x\ | + c$$

[4 marks]

Answer 5

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cot\,x\ dx = [\,ln\ |\ sin\,x\ |\ ]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$
$$= ln\left|sin\left(\frac{\pi}{4}\right)\right| - ln\left|sin\left(\frac{\pi}{6}\right)\right|$$
$$= ln\left(\frac{1}{\sqrt{2}}\right) - ln\left(\frac{1}{2}\right)$$
$$= ln\ (\,2\,)^{-\frac{1}{2}} - ln\ (\,2\,)^{-1}$$
$$= -\frac{1}{2}\,ln\,(\,2\,) + ln\,(\,2\,)$$
$$= \frac{1}{2}\ ln\ 2$$

[4 marks]

Answer 6

$$\int x\ csc^2 x\ dx$$
$$= \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \qquad \boldsymbol{I}$$
$$= x\ (-\,cot\,x\,) - \int 1\ (-\,cot\,x\,)\ dx$$
$$= -x\ cot\,x + \int cot\,x\ dx$$
$$= -x\ cot\,x + ln\ |\ sin\,x\ | + c$$

[5 marks]

Answer 7
First the required trigonometric formulae...

$$cos^2 x + sin^2 x = 1$$

Divide through by $sin^2 x$

$$\frac{cos^2 x}{sin^2 x} + \frac{sin^2 x}{sin^2 x} = \frac{1}{sin^2 x}$$

$$cot^2 x + 1 = csc^2 x$$

$$cot^2 x = csc^2 x - 1$$

Now to tackle the integration...
(Using the result from Question 6)

$$\int x \ cot^2 x \ dx = \int x \ (csc^2 x - 1) \ dx$$

$$= \int x \ csc^2 x \ dx - \int x \ dx$$

$$= -x \ cot\, x + ln \mid sin\, x \mid - \frac{x^2}{2} + c$$

[5 marks]

Answer 8
First the required trigonometric formulae...

$$cos^2 x + sin^2 x = 1$$

$$\text{ADD} \qquad cos^2 x - sin^2 x = cos\, 2x$$

$$2\, cos^2 x = 1 + cos\, 2x$$

$$\therefore \quad cos^2 x = \frac{1}{2} + \frac{1}{2} cos\, 2x$$

Thus;

$$\int cos^2 x \ \ dx = \int \frac{1}{2} + \frac{1}{2} cos\, 2x \quad dx$$

$$= \frac{x}{2} + \frac{sin\, 2x}{4} + c$$

[5 marks]

Answer 9

$$\int x \; cos^2 x \;\; dx$$

$$= \;\; \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= \; x \;\; \left(\frac{x}{2} + \frac{sin\,2x}{4} \right) - \int 1 \;\; \left(\frac{x}{2} + \frac{sin\,2x}{4} \right) \; dx$$

$$= \; \frac{x^2}{2} + \frac{x\,sin\,2x}{4} - \frac{x^2}{4} - \frac{1}{8} \; (-\,cos\,2x\,) \; + \; c$$

$$= \; \frac{x^2}{4} + \frac{x\,sin\,2x}{4} + \frac{cos\,2x}{8} \;\; + \; c$$

[7 marks]

Answer 10

$$\int \frac{ln\,x}{x} \;\; dx \; = \; \int \; ln\,x \;\; \frac{1}{x} \;\; dx$$

$$= \;\; \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \quad \boldsymbol{I}$$

$$= \; ln\,x \quad ln\,x \; - \int \frac{1}{x} \;\; ln\,x \;\; dx \; + \;\; c_1$$

$$\therefore \;\; \int \frac{ln\,x}{x} \;\; dx \; = \; (\,ln\,x\,)^2 \; - \int \frac{ln\,x}{x} \;\; dx \; + \;\; c_1$$

$$\therefore \;\; 2 \int \frac{ln\,x}{x} \;\; dx \; = \; (\,ln\,x\,)^2 \; + \;\; c_1$$

$$\therefore \int \frac{ln\,x}{x} \;\; dx \; = \; \frac{1}{2}\,(\,ln\,x\,)^2 \; + \;\; c_2$$

[7 marks]