

5.3 Answers

A-Level Pure Mathematics : Year 2

Integration III

5.3.1 Solutions to 5.2 Exercise

Answer 1

$$x = 2 \cos u \Rightarrow u = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{dx}{du} = -2 \sin u$$

$$\text{When } x = \sqrt{2}, u = \frac{\pi}{4}$$

$$\text{When } x = 1, u = \frac{\pi}{3}$$

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{1}{x^2 \sqrt{4-x^2}} dx &= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4 \cos^2 u \sqrt{4-4 \cos^2 u}} (-2 \sin u) du \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4 \cos^2 u \sqrt{4(1-\cos^2 u)}} (-2 \sin u) du \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4 \cos^2 u \sqrt{4 \sin^2 u}} (-2 \sin u) du \\ &= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4 \cos^2 u \cdot 2 \sin u} (-2 \sin u) du \\ &= -\frac{1}{4} \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{\cos^2 u} du \\ &= -\frac{1}{4} \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sec^2 u du \\ &= -\frac{1}{4} \left[\tan u \right]_{\frac{\pi}{3}}^{\frac{\pi}{4}} \\ &= -\frac{1}{4} \{ 1 - \sqrt{3} \} \\ &= \frac{1}{4} (\sqrt{3} - 1) \end{aligned}$$

[7 marks]

Answer 2

$$1 - 2 \cos x = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \text{ for A and } \frac{5\pi}{3} \text{ for B}$$

[3 marks]

Answer 3**(a)**

$$x = u^2 \Rightarrow u = \sqrt{x} \quad (\text{Positive root taken as told } u > 0)$$

$$\Rightarrow \frac{dx}{du} = 2u$$

$$\int \frac{1}{x(2\sqrt{x} - 1)} dx = \int \frac{1}{u^2(2\sqrt{u^2} - 1)} 2u du$$

$$= \int \frac{2}{u(2u - 1)} du$$

□

[3 marks]**(b)**

Partial fractions !

$$\text{Let } \frac{2}{u(2u - 1)} = \frac{A}{u} + \frac{B}{2u - 1}$$

$$2 = A(2u - 1) + Bu$$

$$u = 0 : 2 = -A \quad \therefore \quad A = -2$$

$$u = 0.5 : 2 = 0.5B \quad \therefore \quad B = 4$$

$$\int_1^9 \frac{1}{x(2\sqrt{x} - 1)} dx = \int_1^3 -\frac{2}{u} + \frac{4}{2u - 1} du$$

$$= \left[-2 \ln u + 2 \ln(2u - 1) \right]_1^3$$

$$= (-2 \ln 3 + 2 \ln 5) - (-2 \ln 1 + 2 \ln 1)$$

$$= 2(\ln 5 - \ln 3)$$

$$= 2 \ln \left[\frac{5}{3} \right]$$

[7 marks]

Answer 4

$$u = 1 + \sin x \Rightarrow \frac{du}{dx} = \cos x$$

$$\begin{aligned}\int \sin x \cos x (1 + \sin x)^5 dx &= \int (u - 1) \cos x u^5 \frac{du}{\cos x} \\&= \int (u - 1) u^5 du \\&= \int u^6 - u^5 du \\&= \frac{u^7}{7} - \frac{u^6}{6} + c \\&= u^6 \left(\frac{u}{7} - \frac{1}{6} \right) + c \\&= u^6 \left(\frac{6u - 7}{42} \right) + c \\&= (1 + \sin x)^6 \left(\frac{6(1 + \sin x) - 7}{42} \right) + c \\&= \frac{1}{42} (1 + \sin x)^6 (6 \sin x - 1) + c \quad \square\end{aligned}$$

[8 marks]