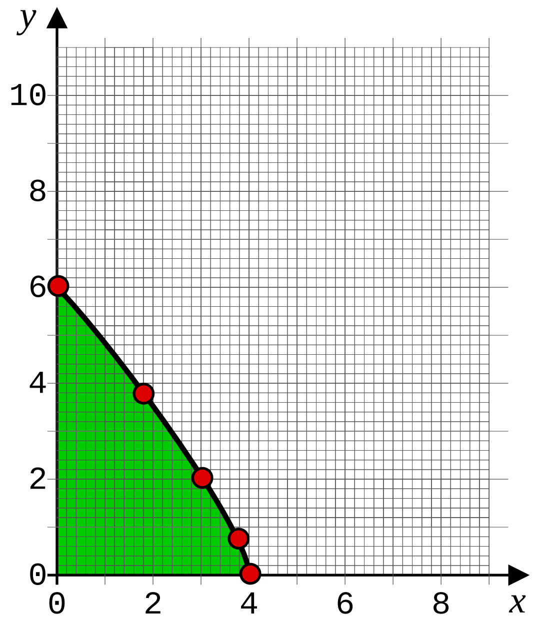
7.3.1 Solution to 7.1 Example

(i) $x = 4 - t^2$ and $y = t\,(t + 1)$ for $t \geqslant 0$

t	0	0.5	1	1.5	2
x	4	3.75	3	1.75	0
y	0	0.75	2	3.75	6

[2 marks]

(ii)



[2 marks]

(iii) First, note that $x = 4 - t^2 \Rightarrow \dfrac{dx}{dt} = -2t$

$$Area = \int y\,dx = \int y\,\frac{dx}{dt}\,dt = \int_{x=0}^{x=4} t\,(t + 1)\,(-2t)\,dt$$

$$= \int_{t=2}^{t=0} -2t^3 - 2t^2\,dt$$

$$= \int_{t=0}^{t=2} 2t^3 + 2t^2\,dt \qquad \text{Limit Swap Rule}$$

$$= \left[\frac{t^4}{2} + \frac{2t^3}{3}\right]_0^2$$

$$= \left(8 + \frac{16}{3}\right) - (0)$$

$$= 13.\dot{3}$$

[6 marks]

7.3.2 Solution to 7.2 Exercise

Answer 1

(a) The curve will cut the x-axis when, $y = 0$

$$y = 0$$

$$t\left(9 - t^2\right) = 0$$

$$\therefore t = \{-3,\ 0,\ 3\}$$

When $t = \pm 3$,

$$x = 5\,(\pm 3)^2 - 4$$

$$= 41 \qquad \therefore B\ (41,\ 0)$$

When $t = 0$,

$$x = 5 \times 0^2 - 4$$

$$= -4 \qquad \therefore C\ (-4,\ 0)$$

[3 marks]

(b)

First, note that $x = 5t^2 - 4 \Rightarrow \dfrac{dx}{dt} = 10\,t$

$$Area = \int y\,dx$$

$$= \int y\,\frac{dx}{dt}\,dt$$

$$= 2\int_{x=-4}^{x=41} t\left(9 - t^2\right)\left(10t\right)dt$$

$$= 2\int_{t=0}^{t=3} 90\,t^2 - 10\,t^4\,dt$$

$$= 2\left[30\,t^3 - 2\,t^5\right]_0^3$$

$$= 2\,(810 - 486 - 0 + 0)$$

$$= 648 \text{ units}^2$$

[6 marks]

Answer 2

(a)

$$x = 3 + 2\sin t$$

$$\sin t = \frac{x-3}{2}$$

$$\text{As } \cos^2 t + \sin^2 t = 1$$

$$\text{and } \cos^2 t - \sin^2 t = \cos 2t \quad \text{SUBTRACT}$$

$$1 - \cos 2t = 2\sin^2 t$$

$$\cos 2t = 1 - 2\sin^2 t$$

$$\cos 2t = 1 - 2\left(\frac{x-3}{2}\right)^2$$

$$y = 4 + 2\cos 2t$$

$$= 4 + 2\left(1 - 2\left(\frac{x-3}{2}\right)^2\right)$$

$$= 4 + 2 - (x-3)^2$$

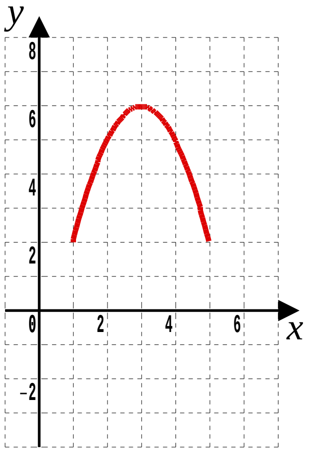
$$= 6 - (x-3)^2 \qquad \square$$

[2 marks]

(b) **(i)** The curve $y = 6 - (x-3)^2$ is in completed square form.

$\therefore$ There is a turning point at (3, 6).

This is a maximum because, when the brackets are expanded the dominant term is $-x^2$. A rewrite as $y = -x^2 + 12x - 3$ also shows that $(0, -3)$ is on the curve.



(ii) As $-1 \leqslant \sin t \leqslant 1$ it follows that $1 \leqslant x \leqslant 5$

As $-1 \leqslant \cos 2t \leqslant 1$ it follows that $2 \leqslant y \leqslant 6$

[3 marks]

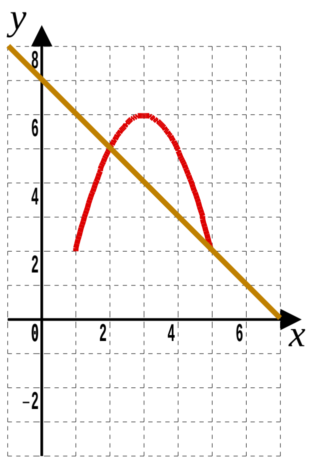
(c)

$$x + y = k$$

$$y = -x + k$$

This can be thought of as a line with a 45° downward slope that can be translated vertically.

One limit (the easier one) will be when the line passes through one of the curves endpoints at (5, 2) corresponding to $k = 7$



The other limit is more tricky to obtain.
The curve and the line intersect when,

$$-x + k = 6 - (x - 3)^2$$

$$-x + k = 6 - \left(x^2 - 6x + 9\right)$$

$$-x + k = 6 - x^2 + 6x - 9$$

$$x^2 - 7x + 3 + k = 0$$

For this to have two distinct roots, the discriminant must be positive,

$$D > 0$$

$$b^2 - 4ac > 0$$

$$(-7)^2 - 4 \times 1 \times (3 + k) > 0$$

$$49 - 12 - 4k > 0$$

$$37 - 4k > 0$$

$$-4k > -37$$

$$4k < 37$$

$$k < 9.25$$

Thus,

$$7 \leqslant k < 9.25, \quad k \in \mathbb{R}$$

[5 marks]

Answer 3

(a) $$x = \ln(t+2) \Leftrightarrow t = e^x - 2$$

$$\frac{dx}{dt} = \frac{1}{t+2}$$

$$\begin{aligned} Area &= \int_{\ln 2}^{\ln 4} y\, dx \\ &= \int_{x=\ln 2}^{x=\ln 4} y \frac{dx}{dt}\, dt \\ &= \int_{t=0}^{t=2} \left(\frac{1}{t+1}\right)\left(\frac{1}{t+2}\right) dt \\ &= \int_0^2 \frac{1}{(t+1)(t+2)}\, dt \qquad \square \end{aligned}$$

[4 marks]

(b) Use partial fractions,

$$\frac{1}{[\,(t+1)(t+2)\,]} = \frac{A}{t+1} + \frac{B}{t+2}$$

Multiply through by the red bracket

$$\frac{1 \times [\,\dots\dots\,]}{[\,(t+1)(t+2)\,]} = \frac{A \times [\,\dots\dots\,]}{t+1} + \frac{B \times [\,\dots\dots\,]}{t+2}$$

$$1 = A(t+2) + B(t+1)$$

Let $t = -1 \;:\; 1 = A \quad \therefore A = 1$

Let $t = -2 \;:\; 1 = -B \quad \therefore B = -1$

Thus,

$$\begin{aligned} \int_0^2 \frac{1}{(t+1)(t+2)}\, dt &= \int_0^2 \frac{1}{t+1} - \frac{1}{t+2}\, dt \\ &= \Big[\ln|t+1| - \ln|t+2|\Big]_0^2 \\ &= \ln 3 - \ln 4 - \ln 1 + \ln 2 \\ &= \ln 3 - \ln 2^2 - 0 + \ln 2 \\ &= \ln 3 - 2\ln 2 + \ln 2 \\ &= \ln 3 - \ln 2 \\ &= \ln\left(\frac{3}{2}\right) \end{aligned}$$

[6 marks]

(c)

$$x = ln\ (t + 2)$$
$$\therefore t = e^x - 2 \quad \text{as before}$$

$$y = \frac{1}{t + 1}$$
$$= \frac{1}{e^x - 2 + 1}$$
$$= \frac{1}{e^x - 1} \quad \text{is in Cartesian form, as asked for.}$$

[4 marks]

(d) Clearly, x cannot equal zero for then a division by zero would result, from

$$y = \frac{1}{e^x - 1}$$
$$= \frac{1}{e^0 - 1}$$
$$= \frac{1}{1 - 1}$$
$$= \frac{1}{0} \quad \text{Division by zero is undefined}$$

More than that, however, we have that,

$$x = ln\ (t + 2)$$

and as the log of a negative number (and zero)is not defined, $t > - 2$
and consequently the domain is $x > 0$

You could also have guessed this answer from the sketch graph provided by the question.

[1 mark]