9.2 Answers

Answer 1

(a) Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int (2x - 1)^{\frac{3}{2}}\, dx = \frac{1}{2} \int 2(2x - 1)^{\frac{3}{2}}\, dx$$

$$= \frac{1}{2} \times \frac{2(2x - 1)^{\frac{5}{2}}}{5} + c$$

$$= \frac{(2x - 1)^{\frac{5}{2}}}{5} + c$$

[3 marks]

(b) Equate curve with line, then solve.

$$(2x - 1)^{\frac{3}{2}} = 8$$

cube root both sides

$$(2x - 1)^{\frac{1}{2}} = 2$$

square both sides

$$2x - 1 = 4$$

$$2x = 5$$

$$x = \frac{5}{2} \qquad \therefore k = \frac{5}{2}$$

[2 marks]

(c) Shaded area, S, is area of rectangle less area under the curve.

$$S = 8 \times 2.5 - \int_{\frac{1}{2}}^{\frac{5}{2}} (2x - 1)^{\frac{3}{2}}\, dx$$

$$= 20 - \left[\frac{(2x - 1)^{\frac{5}{2}}}{5} \right]_{\frac{1}{2}}^{\frac{5}{2}}$$

$$= 20 - \left[\frac{32}{5} - 0 \right]$$

$$= 13.6$$

[4 marks]

Answer 2

(a) $u = 1 + \sqrt{x} \Rightarrow \dfrac{du}{dx} = \dfrac{1}{2} x^{-\frac{1}{2}} \quad \therefore \dfrac{dx}{du} = 2\sqrt{x}$

Also $\sqrt{x} = u - 1$

When $x = 16,\ u = 5$ and when $x = 0,\ u = 1$

$$\int_0^{16} \frac{x}{1 + \sqrt{x}}\, dx = \int_1^5 \frac{x}{u}\, 2\sqrt{x}\, du$$

$$= \int_1^5 \frac{2x^{\frac{3}{2}}}{u}\, du$$

$$= \int_1^5 \frac{2(\sqrt{x})^3}{u}\, du$$

$$= \int_1^5 \frac{2(u - 1)^3}{u}\, du$$

Which is of the form required with $p = 1$ and $q = 5$

[3 marks]

(b) $$\int_0^{16} \frac{x}{1 + \sqrt{x}}\, dx = \int_1^5 \frac{2(u - 1)^3}{u}\, du$$

$$= 2\int_1^5 \frac{u^3 - 3u^2 + 3u - 1}{u}\, du$$

$$= 2\int_1^5 u^2 - 3u + 3 - \frac{1}{u}\, du$$

$$= 2\left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u \right]_1^5$$

$$= 2\left[\left(\frac{125}{3} - \frac{75}{2} + 15 - \ln 5 \right) - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1 \right) \right]$$

$$= 2\left[\frac{124}{3} - \frac{72}{2} + 12 - \ln 5 \right]$$

$$= 2\left[\frac{52}{3} - \ln 5 \right]$$

$$= \frac{104}{3} - 2\ln 5$$

Which is of the form required with $A = \dfrac{104}{3},\ B = 2$ and $A = 5$

[4 marks]

Answer 3

(a) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\{ (y_0 + y_4) + 2(y_1 + y_2 + y_3) \}$$

$$R \approx \frac{1}{2} \times \frac{1}{2}\{(0.4805 + 1.9218) + 2(0.8396 + 1.2069 + 1.5694)\}$$

$$\approx 2.41$$

[3 marks]

(b) Integration by parts with a sneaky 1 inserted...

$$R = \int_2^4 (\ln x)^2 \times 1 \; dx$$

Without worrying about the limits for now...

(Or the constant of algebraic integration)

$$= \quad \boldsymbol{L} \qquad \boldsymbol{I} \qquad\qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= (\ln x)^2 x \quad - \int 2\,(\ln x)\,\frac{1}{x}\,x \quad dx$$

$$= (\ln x)^2 x \quad - 2\int (\ln x) \times 1 \; dx$$

$$= (\ln x)^2 x \quad - 2\left[(\ln x)\,x - \int \frac{1}{x}\,x\,dx \right]$$

$$= (\ln x)^2 x \quad - 2x\ln x + \int 2\,dx$$

Now to put in the limits...

$$= \Big[x\,(\ln x)^2 - 2x\ln x + 2x \Big]_2^4$$

$$= 4\,(\ln 4)^2 - 8\ln 4 + 8 - 2\,(\ln 2)^2 + 4\ln 2 - 4$$

$$= 4\,(2\ln 2)^2 - 16\ln 2 + 8 - 2\,(\ln 2)^2 + 4\ln 2 - 4$$

$$= 14\,(\ln 2)^2 - 12\ln 2 + 4$$

Which is of the form required with $a = 14$, $b = -12$ and $c = 4$

[5 marks]

Answer 4

CLEVER METHOD using a difference of two squares.

$$\begin{aligned} y &= \frac{x - 4}{2 + \sqrt{x}} \\ &= \frac{(\sqrt{x})^2 - 2^2}{\sqrt{x} + 2} \\ &= \frac{(\sqrt{x} + 2)(\sqrt{x} - 2)}{(\sqrt{x} + 2)} \\ &= x^{\frac{1}{2}} - 2 \\ \frac{dy}{dx} &= \frac{1}{2} x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

which is of the required form with $A = 2$

[4 marks]

STANDARD METHOD using the quotient rule.

$$\begin{aligned} y &= \frac{x - 4}{2 + \sqrt{x}} \\ \frac{dy}{dx} &= \frac{(2 + \sqrt{x}) \times 1 - \frac{1}{2} x^{-\frac{1}{2}} (x - 4)}{(2 + \sqrt{x})^2} \times \frac{2x^{\frac{1}{2}}}{2x^{\frac{1}{2}}} \\ &= \frac{4x^{\frac{1}{2}} + 2x - x + 4}{2\sqrt{x}(4 + 4\sqrt{x} + x)} \\ &= \frac{(4 + 4\sqrt{x} + x)}{2\sqrt{x}\,(4 + 4\sqrt{x} + x)} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

which is of the required form with $A = 2$

[4 marks]

Answer 5

(a)

$$x = cos^3\,\theta \;\Rightarrow\; \frac{dx}{d\theta} = 3\,cos^2\,\theta\,(-\,sin\,\theta\,)$$

$$\text{Also, } \theta = arccos\left(\sqrt[3]{x}\,\right)$$

$$\begin{aligned}
Area\;Shaded &= \int y\,dx \\
&= \int_{x=0}^{x=1} 12\,sin\,\theta \times 3\,cos^2\,\theta\,(-\,sin\,\theta\,)\,d\theta \\
&= \int_{\theta=\frac{\pi}{2}}^{\theta=0} -\,36\,sin^2\,\theta\,cos^2\,\theta\,d\theta \\
&= 36\int_0^{\frac{\pi}{2}} sin^2\,\theta\,cos^2\,\theta\,d\theta
\end{aligned}$$

[4 marks]

(b) First note the two relations that...

$$\begin{array}{lrcl}
 & cos^2\,2\theta + sin^2\,2\theta & = & 1 \\
\text{ADD} & cos^2\,2\theta - sin^2\,2\theta & = & cos\,4\theta \\
\hline
 & 2\,cos^2\,2\theta & = & 1 + cos\,4\theta
\end{array}$$

$$\begin{aligned}
\text{LHS} &= cos^2\,\theta\,sin^2\,\theta \\
&= \frac{1}{4}\,(\,2\,sin\,\theta\,cos\,\theta\,)^2 \\
&= \frac{1}{4}\,sin^2\,2\theta \\
&= \frac{1}{4}\left(1 - cos^2\,2\theta\right) \\
&= \frac{1}{8}\left(2 - 2\,cos^2\,2\theta\right) \\
&= \frac{1}{8}\left(2 - 1 - cos\,4\theta\right) \\
&= \frac{1}{8}\left(1 - cos\,4\theta\right) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(c)

$$Total\ Area = 4 \times 36 \int_0^{\frac{\pi}{2}} sin^2\,\theta\, cos^2\,\theta\ d\theta$$

$$= 4 \times 36 \times \frac{1}{8} \int_0^{\frac{\pi}{2}} 1 - cos\,4\theta\ d\theta$$

$$= 18\left[\ \theta - \frac{sin\,4\theta}{4}\ \right]_0^{\frac{\pi}{2}}$$

$$= 18\left[\ \left(\ \frac{\pi}{2} - \frac{sin\,2\pi}{4}\ \right) - \left(\ 0 - \frac{sin\,0}{4}\ \right)\ \right]$$

$$= 18 \times \frac{\pi}{2}$$

$$= 9\pi$$

[4 marks]