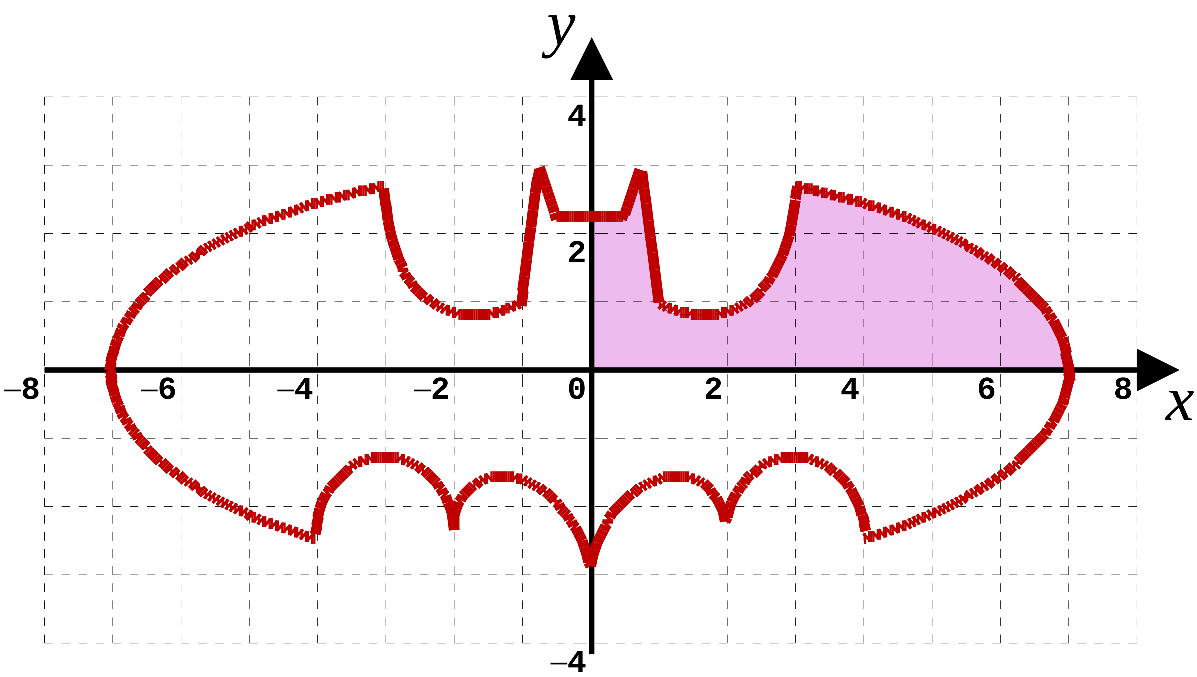
A-Level Pure Mathematics
Year 2

INTEGRATION

III



"The Batman Curve"

$$\textit{Four curves base}\ :\ f(x) = \left|\frac{x}{2}\right| - \frac{3\sqrt{33}-7}{122}x^2 - 3 + \sqrt{1-\left(\left|\,|x|-2\,\right|-1\right)^2}$$

$$\textit{Sides of face}\ :\ g(x) = 9\sqrt{\frac{\left|-(|x|-1)(|x|-0.75)\right|}{-(|x|-1)(|x|-0.75)}} - 8|x|$$

$$\textit{Ears}\ :\ h(x) = 3|x| + 0.75\sqrt{\frac{\left|-(|x|-0.5)(|x|-0.75)\right|}{-(|x|-0.5)(|x|-0.75)}}$$

$$\textit{Forehead}\ :\ p(x) = 2.25\sqrt{\frac{\left|-(x-0.5)(x+0.5)\right|}{-(x-0.5)(x+0.5)}}$$

$$\textit{Shoulders}\ :\ q(x) = \frac{6\sqrt{10}}{7} + (1.5-0.5|x|)\sqrt{\frac{\left|\,|x|-1\,\right|}{|x|-1}} - \frac{6\sqrt{10}}{14}\sqrt{4-(|x|-1)^2}$$

$$\textit{Wings}\ :\ r(x) = 3\sqrt{1-\left(\frac{x}{7}\right)^2}\sqrt{\frac{\left|\,|x|-3\,\right|}{|x|-3}} \qquad s(x) = -3\sqrt{1-\left(\frac{x}{7}\right)^2}\sqrt{\frac{\left|\,|x|-4\,\right|}{|x|-4}}$$

1.1 Integration by Substitution

Example

A-Level Examination Question from January 2006, Paper C4, Q3 (Edexcel)

Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$\int_1^5 \frac{3x}{\sqrt{2x - 1}}\, dx$$

[8 marks]

1.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Evaluate the following integral using the substitution given;

$$\int_{3}^{11} \frac{x}{\sqrt{x - 2}}\, dx \qquad u^2 = x - 2$$

[8 marks]

Question 2

Evaluate the following integral using the substitution given;

$$\int_{6}^{10} \frac{x\,(x-4)}{(x-2)^2}\,dx \qquad u = x - 2$$

[8 marks]

Question 3

Evaluate exactly the following integral using the substitution given;

$$\int_2^6 \frac{x - 1}{\sqrt{2x - 3}}\, dx \qquad u^2 = 2x - 3$$

[8 marks]

Question 4
Evaluate the following integral using the substitution given;

$$\int_{0.5}^{1} \frac{1}{\sqrt{1 - x^2}}\, dx \qquad\qquad x = \sin\theta$$

Warning : You must use **RADIANS**

[8 marks]

Question 5

Evaluate the following integral using the substitution given;

$$\int_{2}^{2.5} \frac{2x + 1}{(3 - x)^6}\, dx \qquad\qquad u = 3 - x$$

[9 marks]

Question 6

Evaluate the following integral using the substitution given;

$$\int_{-0.25}^{0.5} \frac{48x}{(2x+1)^5}\,dx \qquad u = 2x + 1$$

[9 marks]

1.3 Answers

1.3.1 Solution to 1.2 Example

Example

$$u^2 = 2x - 1 \Rightarrow x = \frac{u^2}{2} + \frac{1}{2} \Rightarrow \frac{dx}{du} = u$$

When $x = 5,\ u = 3$ and when $x = 1,\ u = 1$

$$\int_1^5 \frac{3x}{\sqrt{2x - 1}}\,dx = \frac{3}{2}\int_1^3 \frac{u^2 + 1}{\sqrt{u^2}}\,u\,du$$

$$= \frac{3}{2}\int_1^3 u^2 + 1\,du$$

$$= \frac{3}{2}\left[\frac{u^3}{3} + u\right]_1^3$$

$$= \frac{3}{2}\left\{[9 + 3] - \left[\frac{1}{3} + 1\right]\right\}$$

$$= 16$$

[8 marks]

1.3.2 Solutions to 1.3 Exercise

Answer 1

$$u^2 = x - 2 \Rightarrow x = u^2 + 2 \Rightarrow \frac{dx}{du} = 2u$$

When $x = 11,\ u = 3$ and when $x = 3,\ u = 1$

$$\int_3^{11} \frac{x}{\sqrt{x - 2}}\,dx = \int_1^3 \frac{u^2 + 2}{\sqrt{u^2}}\,2u\,du$$

$$= 2\int_1^3 u^2 + 2\,du$$

$$= 2\left[\frac{u^3}{3} + 2u\right]_1^3$$

$$= 2\left\{[9 + 6] - \left[\frac{1}{3} + 2\right]\right\}$$

$$= 25\frac{1}{3}$$

[8 marks]

Answer 2

$$u = x - 2 \Rightarrow x = u + 2 \Rightarrow \frac{dx}{du} = 1$$

When $x = 10,\ u = 8$ and when $x = 6,\ u = 4$

$$\int_6^{10} \frac{x(x-4)}{(x-2)^2}\, dx = \int_4^8 \frac{(u+2)(u-2)}{u^2}\, du$$

$$= \int_4^8 \frac{u^2 - 4}{u^2}\, du$$

$$= \int_4^8 1 - 4u^{-2}\, du$$

$$= \left[u - \frac{4u^{-1}}{(-1)} \right]_4^8$$

$$= \left[u + \frac{4}{u} \right]_4^8$$

$$= \left\{ \left[8 + \frac{1}{2} \right] - \left[4 + 1 \right] \right\}$$

$$= 3\frac{1}{2}$$

[8 marks]

Answer 3

$$u^2 = 2x - 3 \Rightarrow x = \frac{u^2}{2} + \frac{3}{2} \Rightarrow \frac{dx}{du} = u$$

When $x = 6,\ u = 3$ and when $x = 2,\ u = 1$

$$\int_2^6 \frac{x - 1}{\sqrt{2x - 3}}\, dx = \int_1^3 \frac{\left(\frac{u^2}{2} + \frac{3}{2} - 1 \right)}{\sqrt{u^2}}\ u\ du$$

$$= \int_1^3 \frac{u^2}{2} + \frac{1}{2}\, du$$

$$= \left[\frac{u^3}{6} + \frac{u}{2} \right]_1^3$$

$$= \left\{ \left[\frac{9}{2} + \frac{3}{2} \right] - \left[\frac{1}{6} + \frac{1}{2} \right] \right\}$$

$$= \left\{ \left[6 \right] - \left[\frac{2}{3} \right] \right\}$$

$$= 5\frac{1}{3}$$

[8 marks]

Answer 4

$$x = \sin\theta \Rightarrow \theta = \arcsin x \Rightarrow \frac{dx}{d\theta} = \cos\theta$$

When $x = 1,\ \theta = \frac{\pi}{2}$ and when $x = 0.5,\ \theta = \frac{\pi}{6}$

$$\int_{0.5}^{1} \frac{1}{\sqrt{1 - x^2}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \sin^2\theta}} \cos\theta \ \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sqrt{\cos^2\theta}} \cos\theta \ \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\cos\theta} \cos\theta \ \ d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 \ \ d\theta$$

$$= \left[\,\theta\,\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left\{ \left[\frac{\pi}{2}\right] - \left[\frac{\pi}{6}\right] \right\}$$

$$= \frac{\pi}{3}$$

[8 marks]

Answer 5

$$u = 3 - x \Rightarrow x = 3 - u \Rightarrow \frac{dx}{du} = -1$$

When $x = 2.5$, $u = 0.5$ and when $x = 2$, $u = 1$

$$\int_2^{2.5} \frac{2x + 1}{(3 - x)^6}\, dx = \int_1^{0.5} \frac{2(3 - u) + 1}{u^6} \, (-1)\, du$$

$$= \int_1^{0.5} \frac{7 - 2u}{u^6} \, (-1)\, du$$

$$= \int_1^{0.5} \frac{2u - 7}{u^6} \, du$$

$$= \int_1^{0.5} 2u^{-5} - 7u^{-6} \, du$$

$$= \left[\frac{2u^{-4}}{(-4)} - \frac{7u^{-5}}{(-5)} \right]_1^{0.5}$$

$$= \left[-\frac{1}{2u^4} + \frac{7}{5u^5} \right]_1^{0.5}$$

$$= \left\{ \left[-8 + \frac{7 \times 32}{5} \right] - \left[-\frac{1}{2} + \frac{7}{5} \right] \right\}$$

$$= [36.8] - [0.9]$$

$$= 35.9$$

[9 marks]

Answer 6

$$u = 2x + 1 \Rightarrow x = \frac{u}{2} - \frac{1}{2} \Rightarrow \frac{dx}{du} = \frac{1}{2}$$

When $x = 0.5$, $u = 2$ and when $x = -0.25$, $u = 0.5$

$$\int_{-0.25}^{0.5} \frac{48x}{(2x+1)^5}\,dx = \int_{0.5}^{2} \frac{48\left(\frac{u}{2} - \frac{1}{2}\right)}{u^5} \times \frac{1}{2}\,du$$

$$= \int_{0.5}^{2} \frac{12u - 12}{u^5}\,du$$

$$= \int_{0.5}^{2} \frac{12u}{u^5} - \frac{12}{u^5}\,du$$

$$= \int_{0.5}^{2} 12\,u^{-4} - 12\,u^{-5}\,du$$

$$= \left[\frac{12\,u^{-3}}{(-3)} - \frac{12\,u^{-4}}{(-4)}\right]_{0.5}^{2}$$

$$= \left[-\frac{4}{u^3} + \frac{3}{u^4}\right]_{0.5}^{2}$$

$$= \left\{\left[-\frac{1}{2} + \frac{3}{16}\right] - \left[-32 + 48\right]\right\}$$

$$= \left[-\frac{5}{16}\right] - \left[16\right]$$

$$= -16\frac{5}{16}$$

[9 marks]

2.1 Integration by Parts

This is a product rule for integration.
Use it when a product is to be integrated, for example;

$$\int x \ \sin x \ dx$$

Integration by parts expands such integrals into four pieces. It requires some differentiation, ***D***, as well as integration, ***I***, and some leaving alone, ***L***.

The mnemonic ***L I D I*** may help. (*Lie -Die*)

In the examples we'll be integrating and differentiating *sin x*, *cos x*, *ln x* and e^x
Here is a reminder of their derivatives:

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Example N° 1

$$\int x \ \sin x \ dx$$

$$= \ \boldsymbol{L}(x) \quad \boldsymbol{I}(\sin x) - \int \boldsymbol{D}(x) \quad \boldsymbol{I}(\sin x) \ dx$$

$$= \ x \quad (-\cos x) - \int 1 \quad (-\cos x) \ dx$$

$$= -x \ \cos x + \int \cos x \ dx$$

$$= -x \ \cos x + \sin x + c$$

[3 marks]

Example N° 2

$$\int x^3 \; ln\, x \; dx$$

First, swap the order

as we can $\boldsymbol{D}(ln\, x)$ but not (yet!) $\boldsymbol{I}(ln\, x)$

$$= \int ln\, x \;\; x^3 \; dx$$

$$= \boldsymbol{L}(ln\, x) \;\; \boldsymbol{I}\left(x^3\right) - \int \boldsymbol{D}(ln\, x) \;\; \boldsymbol{I}\left(x^3\right) \; dx$$

$$= \quad ln\, x \qquad \frac{x^4}{4} \; - \; \int \; \frac{1}{x} \quad \frac{x^4}{4} \quad dx$$

$$= \frac{x^4 \, ln\, x}{4} - \int \frac{x^3}{4} \; dx$$

$$= \frac{x^4 \, ln\, x}{4} - \frac{x^4}{16} + c$$

[3 marks]

Example N° 3

$$\int ln\, x \; dx$$

Sneaky question because this

does not look like a product

$$= \int ln\, x \;\; \times \; 1 \; dx$$

$$= \boldsymbol{L}(ln\, x) \;\; \boldsymbol{I}(1) - \int \boldsymbol{D}(ln\, x) \;\; \boldsymbol{I}(1) \; dx$$

$$= \quad ln\, x \qquad x \; - \; \int \; \frac{1}{x} \quad x \quad dx$$

$$= x \, ln\, x - \int 1 \; dx$$

$$= x \, ln\, x - x + c$$

A mystery is solved; *ln x* can be integrated as well as differentiated !

[3 marks]

2.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Find the integral;

$$\int x \; cos\, x \; dx$$

[3 marks]

Question 2

Find the integral;

$$\int x \; sin\, 4x \; dx$$

[4 marks]

Question 3

This is the only question in the exercise were it's necessary to swap the order of the product in order to determine the integral, like Example N° 2

$$\int x^5 \; ln\, x \; dx$$

[3 marks]

Question 4
Find the integral;

$$\int x \; e^{x} \; dx$$

[3 marks]

Question 5
Find the integral;

$$\int x \; e^{-5x} \; dx$$

[4 marks]

Question 6
Find the integral;

$$\int \frac{x}{2\, e^{x}} \; dx$$

[4 marks]

Question 7

(i) By setting up a chain rule backwards, find

$$\int (3x + 1)^6 \, dx$$

[2 marks]

(ii) Use your part (i) answer and integration by parts to show that

$$\int x(3x + 1)^6 \, dx = \frac{x(3x + 1)^7}{21} - \frac{(3x + 1)^8}{21 \times 24} + c$$

[4 marks]

(iii) Simplify your answer by showing that

$$\frac{x(3x + 1)^7}{21} - \frac{(3x + 1)^8}{21 \times 24} + c = \frac{1}{504}(3x + 1)^7(21x - 1) + c$$

[3 marks]

Question 8
Evaluate, giving an exact answer,

$$\int_0^{\frac{\pi}{3}} x \ sin\, 3x \ dx$$

[6 marks]

Question 9
Evaluate giving an exact answer,

$$\int_0^1 (\, 2x \ + \ 1 \,) \ e^x \ dx$$

[6 marks]

Question 10

Do not use integration by parts at any stage in this question !

Instead, use the substitution $u = 3x - 5$ to evaluate,

$$7\int_1^2 x^2\ (3x - 5)^4\ dx$$

[8 marks]

Answer 1

$$\int x \ \cos x \ dx$$

$$= \boldsymbol{L}(x) \quad \boldsymbol{I}(\cos x) - \int \boldsymbol{D}(x) \quad \boldsymbol{I}(\cos x) \ dx$$

$$= x \quad \sin x - \int 1 \quad \sin x \quad dx$$

$$= x \ \sin x - \int \sin x \ dx$$

$$= x \ \sin x - (- \cos x) + c$$

$$= x \sin x + \cos x + c$$

[3 marks]

Answer 2

$$\int x \ \sin 4x \ dx$$

$$= \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= x \quad \left(-\frac{\cos 4x}{4}\right) - \int 1 \quad \left(-\frac{\cos 4x}{4}\right) \quad dx$$

$$= -\frac{x \cos 4x}{4} + \frac{1}{4} \int \cos 4x \ dx$$

$$= -\frac{x \cos 4x}{4} + \frac{1}{4} \ \frac{\sin 4x}{4} + c$$

$$= -\frac{x \cos 4x}{4} + \frac{\sin 4x}{16} + c$$

[4 marks]

Answer 3

$$\int x^5 \ \ln x \ dx$$

$$= \int \ln x \ x^5 \ dx \qquad \text{Swap the order as } D(\ln x) \text{ simplifies}$$

$$= \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= \ln x \quad \frac{x^6}{6} - \int \frac{1}{x} \quad \frac{x^6}{6} \quad dx$$

$$= \frac{x^6 \ln x}{6} - \frac{1}{6} \int x^5 \ dx$$

$$= \frac{x^6 \ln x}{6} - \frac{x^6}{36} + c$$

[3 marks]

Answer 4

$$\int x\ e^{x}\ dx$$

$$= \ \boldsymbol{L} \quad \boldsymbol{I} \qquad \boldsymbol{D} \quad \boldsymbol{I}$$

$$= x\ \ e^{x} - \int 1\ \ e^{x}\ dx$$

$$= x\ e^{x} - \int e^{x}\ dx$$

$$= x\ e^{x} - e^{x} + c$$

[3 marks]

Answer 5

$$\int x\ e^{-5x}\ dx$$

$$= \ \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= x\ \ \frac{e^{-5x}}{(-5)} - \int 1\ \frac{e^{-5x}}{(-5)}\ dx$$

$$= -\frac{x\,e^{-5x}}{5} + \frac{1}{5}\int e^{-5x}\ dx$$

$$= -\frac{x\,e^{-5x}}{5} + \frac{1}{5}\ \frac{e^{-5x}}{(-5)} + c$$

$$= -\frac{x\,e^{-5x}}{5} - \frac{e^{-5x}}{25} + c$$

[4 marks]

Answer 6

$$\int \frac{x}{2\,e^{x}}\ dx = \frac{1}{2}\int x\ e^{-x}\ dx$$

$$= \qquad \boldsymbol{L} \quad \boldsymbol{I} \qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= \frac{1}{2}\left(x\ \frac{e^{-x}}{(-1)} - \int 1\ \frac{e^{-x}}{(-1)}\ dx\right)$$

$$= \frac{1}{2}\left(-x\,e^{-x} + \int e^{-x}\ dx\right)$$

$$= -\frac{x\,e^{-x}}{2} - \frac{1}{2}\,e^{-x} + c$$

[4 marks]

Answer 7

(i)

$$\int (3x+1)^6\,dx = \frac{1}{3}\int 3\,(3x+1)^6\,dx$$

$$= \frac{1}{3}\;\frac{(3x+1)^7}{7} + c$$

$$= \frac{(3x+1)^7}{21} + c$$

[2 marks]

(ii)

$$\int x\,(3x+1)^6\,dx$$

$$= \boldsymbol{L}\qquad \boldsymbol{I}\qquad \boldsymbol{D}\qquad \boldsymbol{I}$$

$$= x\;\frac{(3x+1)^7}{21} - \int 1\;\frac{(3x+1)^7}{21}\;dx$$

$$= \frac{x\,(3x+1)^7}{21} - \frac{1}{21}\int (3x+1)^7\;dx$$

$$= \frac{x\,(3x+1)^7}{21} - \frac{1}{21}\;\frac{1}{3}\int 3\,(3x+1)^7\;dx$$

$$= \frac{x\,(3x+1)^7}{21} - \frac{1}{21\times 3}\;\frac{(3x+1)^8}{8} + c$$

$$= \frac{x\,(3x+1)^7}{21} - \frac{(3x+1)^8}{21\times 24} + c$$

[4 marks]

(iii)

$$= \frac{(3x+1)^7}{21}\left(x - \frac{3x+1}{24}\right) + c$$

$$= \frac{(3x+1)^7}{21}\left(\frac{24x-3x-1}{24}\right) + c$$

$$= \frac{(3x+1)^7}{21}\left(\frac{21x-1}{24}\right) + c$$

$$= \frac{1}{504}\;(3x+1)^7(21x-1) + c$$

[3 marks]

Answer 8

$$\int x \; sin\, 3x \; dx$$

$$= \quad L \qquad I \qquad\qquad D \qquad I$$

$$= x \left(- \frac{cos\, 3x}{3} \right) - \int 1 \left(- \frac{cos\, 3x}{3} \right) dx$$

$$= - \frac{x\, cos\, 3x}{3} + \frac{1}{3} \int cos\, 3x \; dx$$

$$= - \frac{x\, cos\, 3x}{3} + \frac{1}{3} \; \frac{sin\, 3x}{3} + c$$

$$= - \frac{x\, cos\, 3x}{3} + \frac{sin\, 3x}{9} + c$$

$$\therefore \int_0^{\frac{\pi}{3}} x \; sin\, 3x \; dx = \left[- \frac{1}{3} x\, cos\, 3x + \frac{1}{9} sin\, 3x \right]_0^{\frac{\pi}{3}}$$

$$= \left[- \frac{1}{3} \; \frac{\pi}{3} \; cos\; (\pi) + \frac{1}{9} \; sin\; (\pi) \right] - [\, 0 \,]$$

$$= \frac{\pi}{9}$$

[6 marks]

Answer 9

$$\int (\, 2x + 1 \,) \; e^x \; dx$$

$$= \qquad L \qquad I \qquad\qquad D \quad I$$

$$= (\, 2x + 1 \,) \; e^x - \int 2 \; e^x \; dx$$

$$= (\, 2x + 1 \,) \; e^x - 2 \; e^x + c$$

$$\therefore \int_0^1 (\, 2x + 1 \,) \; e^x \; dx = \Big[\, (\, 2x + 1 \,)\, e^x - 2 \; e^x \Big]_0^1$$

$$= \left[3\, e^1 - 2\, e^1 \right] - \left[e^0 - 2\, e^0 \right]$$

$$= e - [-1]$$

$$= e + 1$$

[6 marks]

Answer 10

$$u = 3x - 5 \Rightarrow x = \frac{u}{3} + \frac{5}{3} \Rightarrow \frac{dx}{du} = \frac{1}{3}$$

When $x = 2,\ u = 1$ and when $x = 1,\ u = -2$

$$\begin{aligned}
7\int_1^2 x^2 \,(3x - 5)^4 \,dx &= 7\int_{-2}^{1}\left(\frac{u}{3} + \frac{5}{3}\right)^2 u^4 \,\frac{1}{3}\,du \\
&= \frac{7}{3}\int_{-2}^{1}\frac{1}{3^2}(u + 5)^2\,u^4\,du \\
&= \frac{7}{27}\int_{-2}^{1}(u + 5)^2\,u^4\,du \\
&= \frac{7}{27}\int_{-2}^{1}\left(u^2 + 10u + 25\right)u^4\,du \\
&= \frac{7}{27}\int_{-2}^{1}\left(u^6 + 10\,u^5 + 25\,u^4\right)du \\
&= \frac{7}{27}\left[\frac{u^7}{7} + \frac{10\,u^6}{6} + \frac{25\,u^5}{5}\right]_{-2}^{1} \\
&= \frac{7}{27}\left\{\left[\frac{1}{7} + \frac{5}{3} + 5\right] - \left[-\frac{128}{7} + \frac{320}{3} - 160\right]\right\} \\
&= \frac{7}{27}\left\{\left[\frac{143}{21}\right] - \left[-\frac{1504}{21}\right]\right\} \\
&= \frac{7}{27}\left\{\frac{549}{7}\right\} \\
&= \frac{61}{3} \\
&= 20\,\frac{1}{3}
\end{aligned}$$

[8 marks]

Lesson 3

A-Level Pure Mathematics : Year 2
Integration III

3.1 Tricky Integration by Parts

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$
$\ln x$	$\frac{1}{x}$
e^x	e^x

Using the table of derivatives from right to left we can see that, for example;

$$\int \sin x \; dx = -\cos x + c$$

$$\int \cos x \; dx = \sin x + c$$

However, there are some obvious omissions.
For example;

$$\int \tan x \; dx$$

Some cunning is needed to find this integral.

$$\begin{aligned}
\int \tan x \; dx &= \int \frac{\sin x}{\cos x} \; dx \\
&= \int (\sin x)(\cos x)^{-1} \; dx \\
&= (-1) \int (-\sin x)(\cos x)^{-1} \; dx \\
&= (-1) \ln | \cos x | + c \\
&= \ln | \cos x |^{-1} + c \\
&= \ln \left| \frac{1}{\cos x} \right| + c \\
&= \ln | \sec x | + c
\end{aligned}$$

[4 marks]

3.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

Use the result, just proved, to find show that;

$$\int_0^{\frac{\pi}{6}} tan\,x \;\; dx \;=\; ln\,2 \;-\; \frac{1}{2}\;ln\,3$$

[4 marks]

Question 2

Use integration by parts to find;

$$\int \; x \;\; sec^2 x \;\;\; dx$$

[4 marks]

Question 3

(a) Use the fact that;

$$\cos^2 x + \sin^2 x = 1$$

to prove that;

$$\tan^2 x = \sec^2 x - 1$$

[2 marks]

(b) Hence, or otherwise, find;

$$\int x \tan^2 x \, dx$$

[3 marks]

Question 4

Mirror the “cunning” used to integrate $\tan x$ to find an expression for;

$$\int \cot x \, dx$$

[4 marks]

Question 5
Use your question 4 result to find the ***exact*** value of;

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} cot\, x \; dx$$

[4 marks]

Question 6
Use integration by parts to find;

$$\int x \; csc^2 x \; dx$$

[5 marks]

Question 7
Using a trigonometric formula and integration by parts, or otherwise, find;

$$\int x \ cot^2 x \ \ dx$$

[5 marks]

Question 8
Using a trigonometric formulae first, or otherwise, find;

$$\int cos^2 x \ \ dx$$

[5 marks]

Question 9

Use integration by parts, and your question 8 result, to find;

$$\int x \ cos^2 x \ \ dx$$

[7 marks]

Question 10

Use integration by parts to help find;

$$\int \frac{ln\, x}{x} \ \ dx$$

[7 marks]

Answer 1

$$\int_0^{\frac{\pi}{6}} \tan x \; dx = \left[\ln (\sec x) \right]_0^{\frac{\pi}{6}}$$

$$= \left[\ln \left(\frac{1}{\cos x} \right) \right]_0^{\frac{\pi}{6}}$$

$$= \ln \left(\frac{1}{\cos \left(\frac{\pi}{6} \right)} \right) - \ln \left(\frac{1}{\cos 0} \right)$$

$$= \ln \left(\frac{2}{\sqrt{3}} \right) - \ln \left(\frac{1}{1} \right)$$

$$= \ln (2) - \ln (\sqrt{3}) - \ln (1)$$

$$= \ln (2) - \frac{1}{2} \sqrt{3} \qquad \square$$

[4 marks]

Answer 2

$$\int x \; \sec^2 x \; dx$$

$$= \textbf{L} \quad \textbf{I} \qquad \textbf{D} \quad \textbf{I}$$

$$= x \;\; \tan x - \int 1 \;\; \tan x \;\; dx$$

$$= x \; \tan x - \int \tan x \;\; dx$$

$$= x \; \tan x - \ln \mid \sec x \mid + c$$

[4 marks]

Answer 3

(a)

$$\cos^2 x + \sin^2 x = 1$$

Divide through by $\cos^2 x$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1 \qquad \square$$

[2 marks]

(b)

$$\int x \ \tan^2 x \ dx = \int x \ (\sec^2 x - 1) \ dx$$

$$= \int x \ \sec^2 x \ dx - \int x \ dx$$

$$= x \ \tan x - \ln | \sec x | - \frac{x^2}{2} + c$$

[3 marks]

Answer 4

$$\int \cot x \ dx = \int \frac{\cos x}{\sin x} \ dx$$

$$= \int (\cos x) \ (\sin x)^{-1} \ dx$$

$$= \ln | \ \sin x \ | + c$$

[4 marks]

Answer 5

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cot x \ dx = [\ \ln | \sin x | \ ]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \ln \left| \sin \left(\frac{\pi}{4} \right) \right| - \ln \left| \sin \left(\frac{\pi}{6} \right) \right|$$

$$= \ln \left(\frac{1}{\sqrt{2}} \right) - \ln \left(\frac{1}{2} \right)$$

$$= \ln (2)^{-\frac{1}{2}} - \ln (2)^{-1}$$

$$= -\frac{1}{2} \ln (2) + \ln (2)$$

$$= \frac{1}{2} \ \ln \ 2$$

[4 marks]

Answer 6

$$\int x \ \csc^2 x \ dx$$

$$= \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= x \ (-\cot x) - \int 1 \ (-\cot x) \ dx$$

$$= -x \ \cot x + \int \cot x \ dx$$

$$= -x \ \cot x + \ln | \ \sin x | + c$$

[5 marks]

Answer 7

First the required trigonometric formulae...

$$\cos^2 x + \sin^2 x = 1$$

Divide through by $\sin^2 x$

$$\frac{\cos^2 x}{\sin^2 x} + \frac{\sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$

$$\cot^2 x + 1 = \csc^2 x$$

$$\cot^2 x = \csc^2 x - 1$$

Now to tackle the integration...

(Using the result from Question 6)

$$\int x \cot^2 x \, dx = \int x \, (\csc^2 x - 1) \, dx$$

$$= \int x \csc^2 x \, dx - \int x \, dx$$

$$= -x \cot x + \ln | \sin x | - \frac{x^2}{2} + c$$

[5 marks]

Answer 8

First the required trigonometric formulae...

$$\cos^2 x + \sin^2 x = 1$$

$$\text{ADD} \quad \cos^2 x - \sin^2 x = \cos 2x$$

$$2 \cos^2 x = 1 + \cos 2x$$

$$\therefore \quad \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

Thus;

$$\int \cos^2 x \, dx = \int \frac{1}{2} + \frac{1}{2} \cos 2x \, dx$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + c$$

[5 marks]

Answer 9

$$\int x \cos^2 x \ dx$$

$$= \quad \boldsymbol{L} \qquad\qquad \boldsymbol{I} \qquad\qquad\qquad \boldsymbol{D} \qquad\qquad \boldsymbol{I}$$

$$= x \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) - \int 1 \left(\frac{x}{2} + \frac{\sin 2x}{4} \right) dx$$

$$= \frac{x^2}{2} + \frac{x \sin 2x}{4} - \frac{x^2}{4} - \frac{1}{8} (-\cos 2x) + c$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + c$$

[7 marks]

Answer 10

$$\int \frac{\ln x}{x} \ dx = \int \ln x \ \frac{1}{x} \ dx$$

$$= \quad \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \quad \boldsymbol{I}$$

$$= \ln x \ \ \ln x - \int \frac{1}{x} \ \ln x \ dx + c_1$$

$$\therefore \int \frac{\ln x}{x} \ dx = (\ln x)^2 - \int \frac{\ln x}{x} \ dx + c_1$$

$$\therefore 2 \int \frac{\ln x}{x} \ dx = (\ln x)^2 + c_1$$

$$\therefore \int \frac{\ln x}{x} \ dx = \frac{1}{2} (\ln x)^2 + c_2$$

[7 marks]

Lesson 4

A-Level Pure Mathematics : Year 2

Integration III

4.1 Examination Questions on Integration by Parts

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$
$\ln x$	$\frac{1}{x}$
$\ln\lvert \sec x \rvert$	$\tan x$
$\ln\lvert \sin x \rvert$	$\cot x$
e^x	e^x

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 47

Question 1

A-Level Examination Question from January 2011, Paper C4, Q1 (Edexcel)

Use integration by parts to find the exact value of

$$\int_0^{\frac{\pi}{2}} x \sin 2x \, dx$$

[6 marks]

Question 2

A-Level Examination Question from January 2012, Paper C4, Q2 (Edexcel)

(a) Use integration by parts to find

$$\int x \sin 3x \, dx$$

[3 marks]

(b) Using your answer to part (a), find

$$\int x^2 \cos 3x \, dx$$

[3 marks]

Question 3

A-Level Examination Question from June 2007, Paper C4, Q3 (Edexcel)

(a) Find

$$\int x \cos 2x \, dx$$

[4 marks]

(b) Hence, using the identity

$$\cos 2x = 2\cos^2 x - 1$$

deduce

$$\int x \cos^2 x \, dx$$

[3 marks]

Question 4

A-Level Examination Question from January 2008, Paper C4, Q4 (Edexcel)

(i) Find

$$\int ln\left(\frac{x}{2}\right) dx$$

[4 marks]

(ii) Find the exact value of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sin^2 x \; dx$$

[5 marks]

Question 5

A-Level Examination Question from June 2008, Paper C4, Q2 (Edexcel)

(a) Use integration by parts to find

$$\int x\, e^{x}\ dx$$

[3 marks]

(b) Hence find

$$\int x^{2}\, e^{x}\ dx$$

[3 marks]

Question 6

A-Level Examination Question from January 2009, Paper C4, Q6 (Edexcel)

(a) Find

$$\int tan^2 x \ dx$$

[2 marks]

(b) Use integration by parts to find

$$\int \frac{1}{x^3} \ln x \ dx$$

[4 marks]

(c) Use the substitution $u = 1 + e^x$ to show that,

$$\int \frac{e^{3x}}{1 + e^x}\, dx = \frac{1}{2} e^{2x} - e^x + ln\,(1 + e^x) + k$$

where k is a constant.

[7 marks]

4.3 Answers

4.3.1 Solutions to 4.2 Exercise

Answer 1

$$\int x \ \sin 2x \ dx$$

$$= \quad \boldsymbol{L} \qquad \boldsymbol{I} \qquad\qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= x \left(-\frac{\cos 2x}{2} \right) - \int 1 \left(-\frac{\cos 2x}{2} \right) dx$$

$$= -\frac{x \cos 2x}{2} + \frac{1}{2} \int \cos 2x \ dx$$

$$= -\frac{x \cos 2x}{2} + \frac{1}{2} \ \frac{\sin 2x}{2} + c$$

$$= -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} + c$$

$$\therefore \int_0^{\frac{\pi}{2}} x \ \sin 2x \ dx = \left[-\frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{2}}$$

$$= \left[-\frac{1}{2} \ \frac{\pi}{2} \ \cos (\pi) + \frac{1}{4} \sin (\pi) \right] - [\, 0 \,]$$

$$= \frac{\pi}{4}$$

[6 marks]

Answer 2

(a)

$$\int x \ \sin 3x \ dx$$

$$= \boldsymbol{L} \qquad \boldsymbol{I} \qquad\qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= x \left(-\frac{\cos 3x}{3}\right) - \int 1 \left(-\frac{\cos 3x}{3}\right) dx$$

$$= -\frac{x\cos 3x}{3} + \frac{1}{3}\int \cos 3x \ dx$$

$$= -\frac{x\cos 3x}{3} + \frac{1}{3} \ \frac{\sin 3x}{3} + c$$

$$= -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + c$$

[3 marks]

(b)

$$\int x^2 \cos 3x \ dx$$

$$= \boldsymbol{L} \qquad \boldsymbol{I} \qquad\qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= x^2 \left(\frac{\sin 3x}{3}\right) - \int 2x \left(\frac{\sin 3x}{3}\right) dx$$

$$= \frac{x^2 \sin 3x}{3} - \frac{2}{3}\int x \ \sin 3x \ dx$$

$$= \frac{x^2 \sin 3x}{3} - \frac{2}{3}\left[-\frac{x\cos 3x}{3} + \frac{\sin 3x}{9}\right] + c$$

$$= \frac{x^2 \sin 3x}{3} + \frac{2x\cos 3x}{9} - \frac{2\sin 3x}{27} + c$$

[3 marks]

Answer 3

(a)

$$\int x \ \cos 2x \ dx$$

$$= \ \boldsymbol{L} \qquad \boldsymbol{I} \qquad\qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= x \left(\frac{\sin 2x}{2} \right) - \int 1 \left(\frac{\sin 2x}{2} \right) dx$$

$$= \frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x \ dx$$

$$= \frac{x \sin 2x}{2} - \frac{1}{2} \left(- \frac{\cos 2x}{2} \right) + c$$

$$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + c$$

[4 marks]

(b)

$$\int x \cos^2 x \ dx = \int x \left(\frac{\cos 2x + 1}{2} \right) dx$$

$$= \frac{1}{2} \int x \cos 2x \ dx + \frac{1}{2} \int x \ dx$$

$$= \frac{1}{2} \left[\frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right] + \frac{1}{2} \ \frac{x^2}{2} + c$$

$$= \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + \frac{x^2}{4} + c$$

[3 marks]

Answer 4
(i)

$$\int ln\left(\frac{x}{2}\right) dx = \int ln\,x\,dx - \int ln\,2\;\,dx$$
$$= \int ln\,x \quad \times\ 1\;\,dx - x\,ln\,2$$
$$= \quad \boldsymbol{L} \quad \boldsymbol{I} \qquad \boldsymbol{D} \quad \boldsymbol{I}$$
$$= ln\,x\;\;x - \int \frac{1}{x}\;\;x\;\,dx - x\,ln\,2$$
$$= x\,ln\,x - \int 1\;\,dx - x\,ln\,2$$
$$= x\,ln\,x - x - x\,ln\,2 + c$$
$$= x\;\,ln\left(\frac{x}{2}\right) - x + c$$

[4 marks]

(ii)

First the required trigonometric formulae...

$$cos^2 x + sin^2 x = 1$$
$$\text{SUBTRACT} \qquad cos^2 x - sin^2 x = cos\,2x$$
$$2\,sin^2 x = 1 - cos\,2x$$
$$\therefore \quad sin^2 x = \frac{1}{2} - \frac{1}{2}\,cos\,2x$$

Thus;

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} sin^2 x \;\; dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 - \;\; cos\,2x \quad dx$$
$$= \frac{1}{2}\left[\, x - \frac{sin\,2x}{2}\,\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= \frac{1}{2}\left\{\left[\,\frac{\pi}{2} - \frac{sin\,\pi}{2}\,\right] - \left[\,\frac{\pi}{4} - \frac{sin\,\left(\frac{\pi}{2}\right)}{2}\,\right]\right\}$$
$$= \frac{1}{2}\left\{\,\frac{\pi}{2} - 0 - \frac{\pi}{4} + \frac{1}{2}\,\right\}$$
$$= \frac{\pi}{8} + \frac{1}{4}$$

[3 marks]

Answer 5

(a) $\int x \ e^x \ dx$

$= \ \boldsymbol{L} \quad \boldsymbol{I} \qquad \boldsymbol{D} \quad \boldsymbol{I}$

$= \ x \ \ e^x \ - \int 1 \ \ e^x \ \ dx$

$= \ x \ e^x \ - \int e^x \ dx$

$= \ x \ e^x \ - \ e^x \ + \ c$

[3 marks]

(b) $\int x^2 \ e^x \ dx$

$= \ \boldsymbol{L} \quad \boldsymbol{I} \qquad \boldsymbol{D} \quad \boldsymbol{I}$

$= \ x^2 \ \ e^x \ - \int 2x \ \ e^x \ \ dx$

$= \ x^2 \ e^x \ - \ 2 \int x \, e^x \ \ dx$

$= \ x^2 \ e^x \ - \ 2 \left(x \ e^x \ - \ e^x \right) \ + \ c$

[3 marks]

Answer 6

(a) $cos^2 x \ + \ sin^2 x \ = \ 1$

Divide through by $cos^2 x$

$1 \ + \ tan^2 x \ = \ sec^2 x$

Thus, $\int tan^2 x \ dx \ = \int sec^2 x \ - \ 1 \ \ dx$

$= \ tan \, x \ - \ x \ + \ c$

[2 marks]

(b) $\int \dfrac{1}{x^3} \, ln \, x \ \ dx \ = \int \ ln \, x \ \ x^{-3} \ \ dx$

$= \ \boldsymbol{L} \qquad \boldsymbol{I} \qquad \boldsymbol{D} \quad \boldsymbol{I}$

$= \ ln \, x \quad \dfrac{x^{-2}}{(-2)} \ - \int \dfrac{1}{x} \ \dfrac{x^{-2}}{(-2)} \ \ dx$

$= \ - \ \dfrac{ln \, x}{2 \, x^2} \ + \ \dfrac{1}{2} \int x^{-3} \, dx$

$= \ - \ \dfrac{ln \, x}{2 \, x^2} \ - \ \dfrac{x^{-2}}{4} \ + \ c$

$= \ - \ \dfrac{ln \, x}{2 \, x^2} \ - \ \dfrac{1}{4 \, x^2} \ + \ c$

[4 marks]

(c)

$$u = 1 + e^x \Rightarrow x = ln\,(u - 1) \Rightarrow \frac{du}{dx} = e^x$$

$$\int \frac{e^{3x}}{1 + e^x}\,dx = \int \frac{e^{2x}\,e^x}{u}\;\frac{1}{e^x}\,du$$

$$= \int \frac{e^{2\,ln\,(u-1)}}{u}\,du$$

$$= \int \frac{e^{\,ln\,(u-1)^2}}{u}\,du$$

$$= \int \frac{(u - 1)^2}{u}\,du$$

$$= \int \frac{u^2}{u} - \frac{2u}{u} + \frac{1}{u}\,du$$

$$= \int u - 2 + \frac{1}{u}\;du$$

$$= \frac{u^2}{2} - 2u + ln\,u + c$$

$$= \frac{(1 + e^x)^2}{2} - 2\left(1 + e^x\right) + ln\left(1 + e^x\right) + c_1$$

$$= \frac{1}{2} + e^x + \frac{e^{2x}}{2} - 2 - 2\,e^x + ln\left(1 + e^x\right) + c_1$$

$$= \frac{e^{2x}}{2} - e^x + ln\left(1 + e^x\right) + k \qquad \square$$

[7 marks]

5.1 Year 2 Integration : Examination Questions

The formulae book provided in the examination gives the derivative of many functions. These are identified with a * in the table below, which also highlights several key results that are NOT provided. Used backward, the table gives integrals.

$f(x)$	$f'(x)$	Given ?
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	*
$\sec x$	$\sec x \tan x$	*
$\csc x$	$-\csc x \cot x$	*
$\cot x$	$-\csc^2 x$	*
$\ln x$	$\frac{1}{x}$	
$\ln\lvert \sec x \rvert$	$\tan x$	*
$\ln\lvert \sin x \rvert$	$\cot x$	*
$\ln\lvert \sec x + \tan x \rvert$	$\sec x$	*
$\ln\left\lvert \tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right) \right\rvert$	$\sec x$	*
$-\ln\lvert \csc x + \cot x \rvert$	$\csc x$	*
$\ln\left\lvert \tan\left(\frac{1}{2}x\right) \right\rvert$	$\csc x$	*
e^x	e^x	

As has been seen, many questions require the application of a trigonometric identity, but the useful identities are not given explicitly.

The ***three key*** identities should be memorised;

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$2\sin\theta\cos\theta = \sin 2\theta$$

From the ***three key***, the ***following four*** are easily obtained;

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

$2\cos^2\theta = 1 + \cos 2\theta$ Essential to find $\int \cos^2\theta \, d\theta$

$2\sin^2\theta = 1 - \cos 2\theta$ Essential to find $\int \sin^2\theta \, d\theta$

Either memorise the ***following four***, or learn how to obtain them from the ***three key***.

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 28

When trigonometry and calculus mix, **RADIANS MUST BE USED !**

Question 1

A-Level Examination Question from January 2010, Paper C4, Q8 (a) (Edexcel)

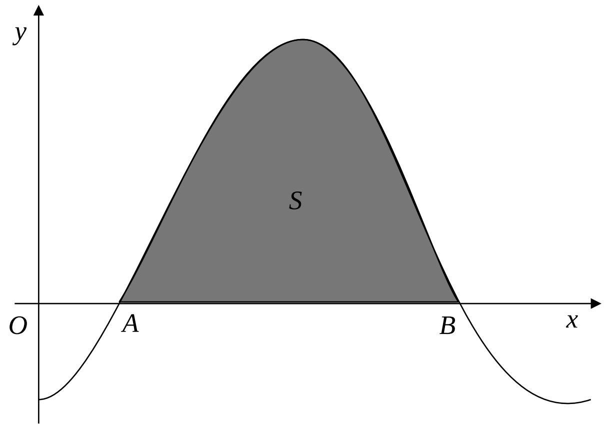
Using the substitution $x = 2\cos u$, or otherwise, find the exact value of

$$\int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{4 - x^2}}\,dx$$

[7 marks]

Question 2

A-Level Examination Question from January 2013, Paper C4, Q6 (a) (Edexcel)



Shown is a sketch of the curve with equation $y = 1 - 2\cos x$, where x is measured in radians. The curve crosses the x-axis at the point A and the point B.
Find, in terms of π, the x coordinate of the point A and the x coordinate of the point B.

[3 marks]

Question 3

A-Level Examination Question from June 2013, Paper C4, Q5 (Edexcel)

(a) Use the substitution $x = u^2$, $u > 0$, to show that,

$$\int \frac{1}{x\,(\,2\sqrt{x} - 1\,)}\,dx = \int \frac{2}{u\,(\,2u - 1\,)}\,du$$

[3 marks]

(b) Hence show that

$$\int_1^9 \frac{1}{x\,(\,2\sqrt{x} - 1\,)}\,dx = 2\,ln\left(\frac{a}{b}\right)$$

where a and b are integers to be determined.

[7 marks]

Question 4

A-Level Examination Question from June 2004, Paper P3, Q4 (Edexcel)

Use the substitution $u = 1 + \sin x$ and integration to show that

$$\int \sin x \cos x \, (1 + \sin x)^5 \, dx = \frac{1}{42}(1 + \sin x)^6 \, (6 \sin x - 1) + constant$$

[8 marks]

5.3 Answers

5.3.1 Solutions to 5.2 Exercise

Answer 1

$$x = 2\cos u \Rightarrow u = \arccos\left(\frac{x}{2}\right) \Rightarrow \frac{dx}{du} = -2\sin u$$

$$\text{When } x = \sqrt{2},\ u = \frac{\pi}{4}$$

$$\text{When } x = 1,\ u = \frac{\pi}{3}$$

$$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4 - x^2}}\, dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4\cos^2 u\sqrt{4 - 4\cos^2 u}}(-2\sin u)\ du$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4\cos^2 u\sqrt{4(1 - \cos^2 u)}}(-2\sin u)\ du$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4\cos^2 u\sqrt{4\sin^2 u}}(-2\sin u)\ du$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4\cos^2 u\ \ 2\sin u}(-2\sin u)\ du$$

$$= -\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{\cos^2 u}\ du$$

$$= -\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sec^2 u\ du$$

$$= -\frac{1}{4}\left[\tan u\right]_{\frac{\pi}{3}}^{\frac{\pi}{4}}$$

$$= -\frac{1}{4}\left\{1 - \sqrt{3}\right\}$$

$$= \frac{1}{4}\left(\sqrt{3} - 1\right)$$

[7 marks]

Answer 2

$$1 - 2\cos x = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3} \text{ for } A \text{ and } \frac{5\pi}{3} \text{ for } B$$

[3 marks]

Answer 3

(a)

$$x = u^2 \Rightarrow u = \sqrt{x} \qquad \text{(Positive root taken as told } u > 0\text{)}$$

$$\Rightarrow \frac{dx}{du} = 2u$$

$$\int \frac{1}{x\left(2\sqrt{x} - 1\right)}\, dx = \int \frac{1}{u^2\left(2\sqrt{u^2} - 1\right)} 2u \; du$$

$$= \int \frac{2}{u\,(2u - 1)}\, du \qquad \square$$

[3 marks]

(b)

Partial fractions !

$$Let \; \frac{2}{u\,(2u - 1)} = \frac{A}{u} + \frac{B}{2u - 1}$$

$$2 = A\,(2u - 1) + B\,u$$

$$u = 0 \; : \; 2 = -A \qquad \therefore \qquad A = -2$$

$$u = 0.5 \; : \; 2 = 0.5\,B \qquad \therefore \qquad B = 4$$

$$\int_1^9 \frac{1}{x\,(2\sqrt{x} - 1)}\, dx = \int_1^3 -\frac{2}{u} + \frac{4}{2u - 1}\, du$$

$$= \left[-2\,ln\,u + 2\,ln\,(2u - 1)\right]_1^3$$

$$= (-2\,ln\,3 + 2\,ln\,5) - (-2\,ln\,1 + 2\,ln\,1)$$

$$= 2\,(ln\,5 - ln\,3)$$

$$= 2\,ln\left[\frac{5}{3}\right]$$

[7 marks]

Answer 4

$$u = 1 + \sin x \Rightarrow \frac{du}{dx} = \cos x$$

$$\begin{aligned}\int \sin x \cos x \, (1 + \sin x)^5 \, dx &= \int (u - 1) \cos x \; u^5 \, \frac{du}{\cos x} \\ &= \int (u - 1) \, u^5 \, du \\ &= \int u^6 - u^5 \, du \\ &= \frac{u^7}{7} - \frac{u^6}{6} + c \\ &= u^6 \left(\frac{u}{7} - \frac{1}{6} \right) + c \\ &= u^6 \left(\frac{6u - 7}{42} \right) + c \\ &= (1 + \sin x)^6 \left(\frac{6 (1 + \sin x) - 7}{42} \right) + c \\ &= \frac{1}{42} (1 + \sin x)^6 (6 \sin x - 1) + c \qquad \square\end{aligned}$$

[8 marks]

6.1 Parametric Integration

Example

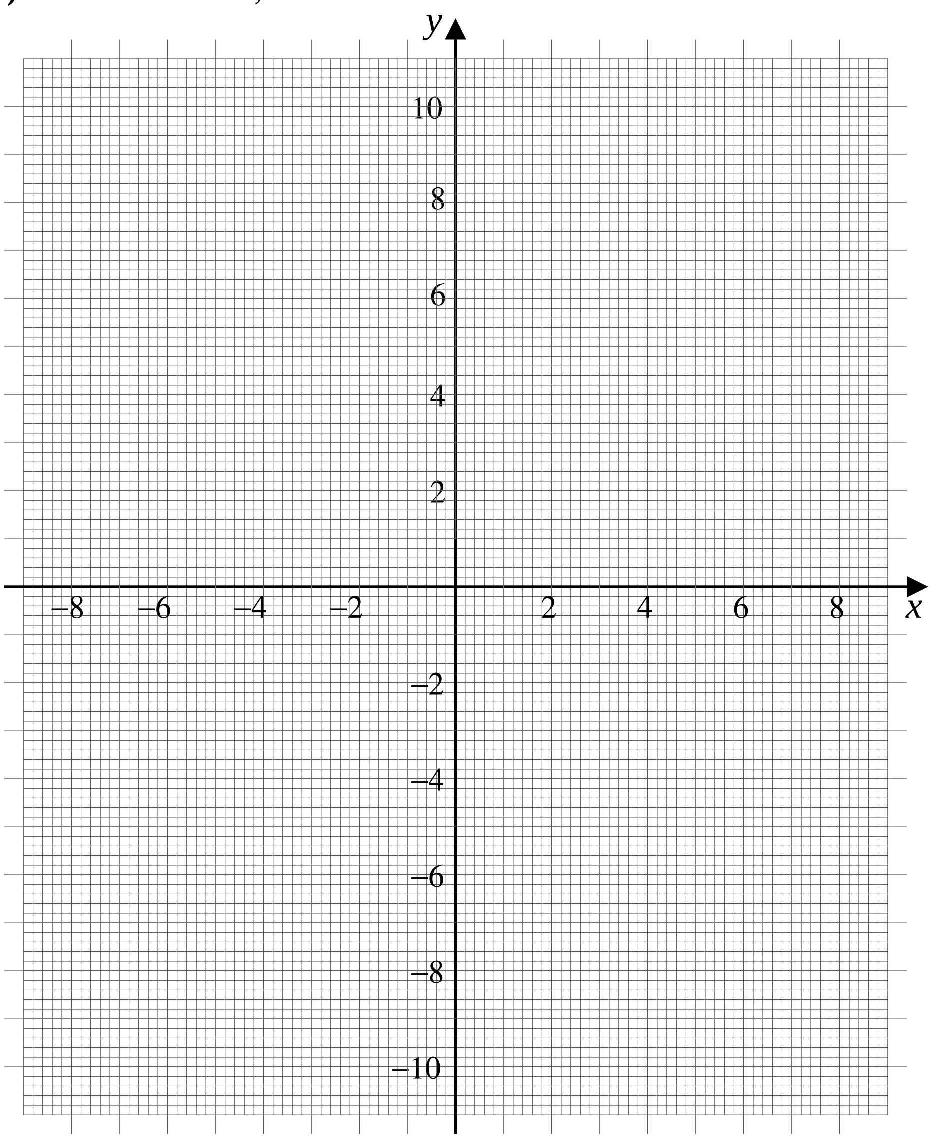
(i) Complete the following table to determine some points on the curve with parametric equations,

$$x = t^2 - 9 \qquad y = 2t$$

t	-4	-3	-2	-1	0	1	2	3	4
x									
y									

[3 marks]

(ii) Plot the curve;



[3 marks]

(iii) Shade in the region bounded by the y-axis and the curve.
This is the area to be found by parametric integration.

[1 mark]

(iv) Find the area of the region bounded by the y-axis and the curve with parametric equations,

$$x = t^2 - 9 \qquad\qquad y = 2t$$

[4 marks]

(v) Eliminate t to obtain the Cartesian equation of the curve.

[2 marks]

(vi) Verify your part (iv) answer by integrating the Cartesian equation of the curve between appropriate limits.

[3 marks]

6.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 20

Question 1

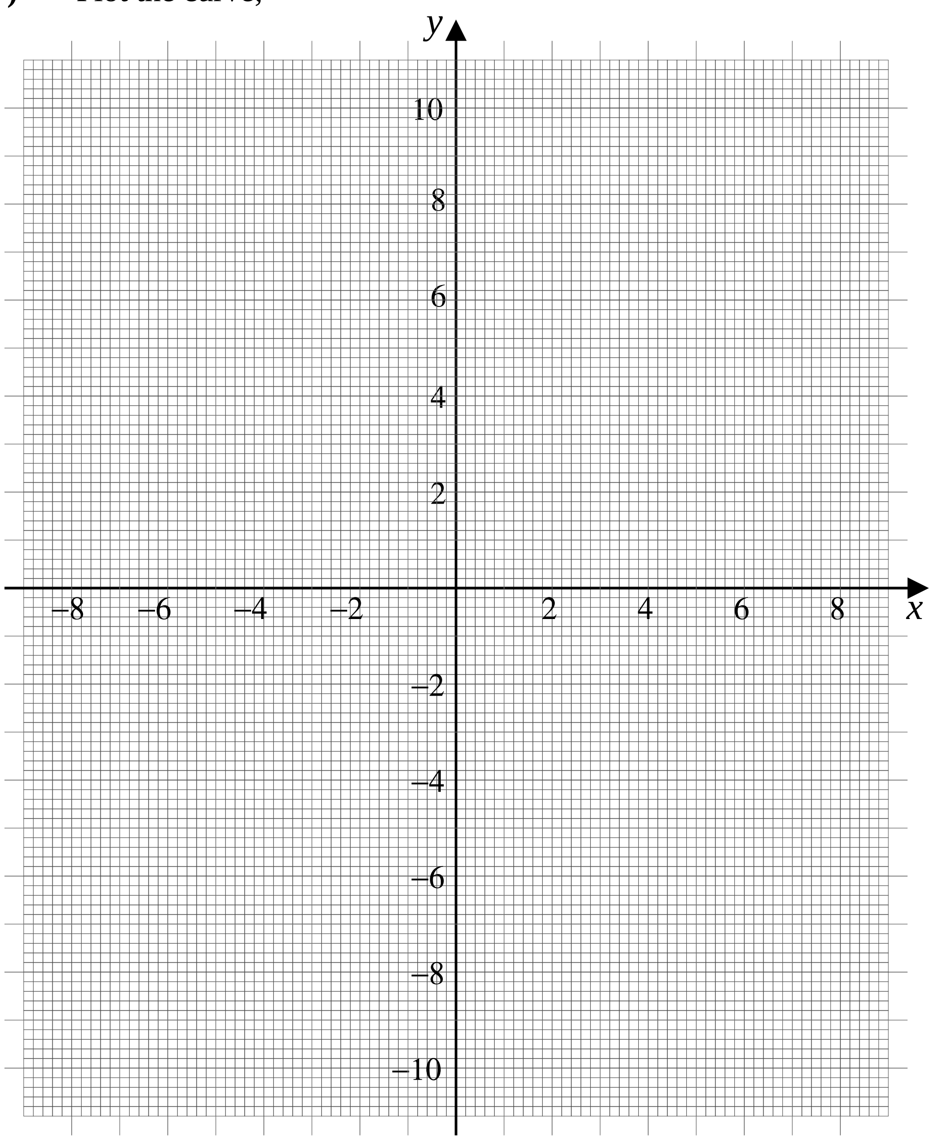
(i) Complete the following table to determine some points on the curve with parametric equations

$$x = 4 - t^2 \qquad\qquad y = t\left(t^2 - 9\right)$$

t	− 3.4	− 3	− 2	− 1	0	1	2	3	3.4
x	−7.6								− 7.6
y	− 8.7								8.7

[3 marks]

(ii) Plot the curve;



[3 marks]

(iii) Shade in the region bounded by the loop of the curve.
This is the area to be found by parametric integration.

[1 mark]

(iv) Find the area of the region bounded by the loop of the curve with parametric equations,

$$x = 4 - t^2 \qquad\qquad y = t\left(t^2 - 9\right)$$

[4 marks]

(v) Eliminate *t* to show that the Cartesian equation of the curve is

$$y = \pm\,(4 - x)^{\frac{1}{2}}\,(x + 5)$$

[4 marks]

(vi) Verify your part **(iv)** answer by integrating the Cartesian equation of the curve between appropriate limits.
Use the substitution $u = 4 - x$

[5 marks]

6.3 Answers

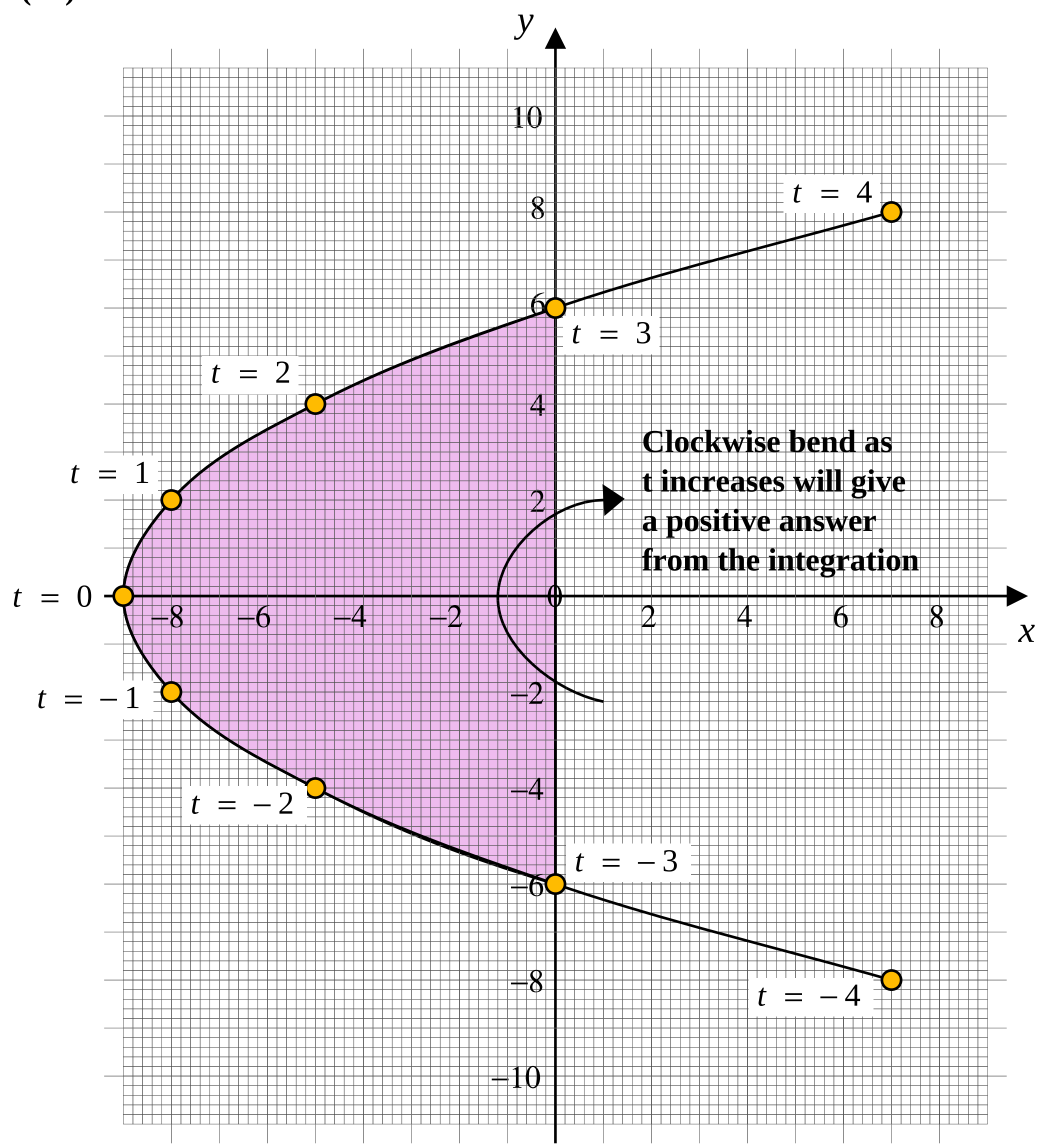
6.3.1 Solution to Introduction Question

(i)

t	-4	-3	-2	-1	0	1	2	3	4
x	7	0	-5	-8	-9	-8	-5	0	7
y	-8	-6	-4	-2	0	2	4	6	8

[3 marks]

(ii)



[3 marks]

(iii) Area shaded in as shown above.

[1 mark]

(iv)

$$x = t^2 - 9 \Rightarrow \frac{dx}{dt} = 2t$$

$$\begin{aligned} Area &= \int y\,dx \\ &= \int_{t=-3}^{t=3} 2t \;\; 2t \;\; dt \\ &= 4\int_{-3}^{3} t^2\,dt \\ &= 4\left[\frac{t^3}{3}\right]_{-3}^{3} \\ &= 4\left\{ [\,9\,] - [-9\,] \right\} \\ &= 72 \end{aligned}$$

[4 marks]

(v)

$$y = 2t \Rightarrow y^2 = 4t^2 \Rightarrow t^2 = \frac{y^2}{4}$$

$$x = t^2 - 9 \Rightarrow x = \frac{y^2}{4} - 9 \Rightarrow y^2 = 4\,(x + 9)$$

$$y = \pm 2\,(x + 9)^{\frac{1}{2}}$$

[2 marks]

(vi)

$$\begin{aligned} \int y\,dx &= 2 \times \int_{x=-9}^{x=0} 2\,(x + 9)^{\frac{1}{2}}\,dx \\ &= 4\int_{-9}^{0} (x + 9)^{\frac{1}{2}}\,dx \\ &= 4\left[\frac{2\,(x + 9)^{\frac{3}{2}}}{3}\right]_{-9}^{0} \\ &= \frac{8}{3}\left[\,(x + 9)^{\frac{3}{2}}\,\right]_{-9}^{0} \\ &= \frac{8}{3}\left\{ 9^{\frac{3}{2}} - 0 \right\} \\ &= 72 \qquad \text{as before} \end{aligned}$$

[3 marks]

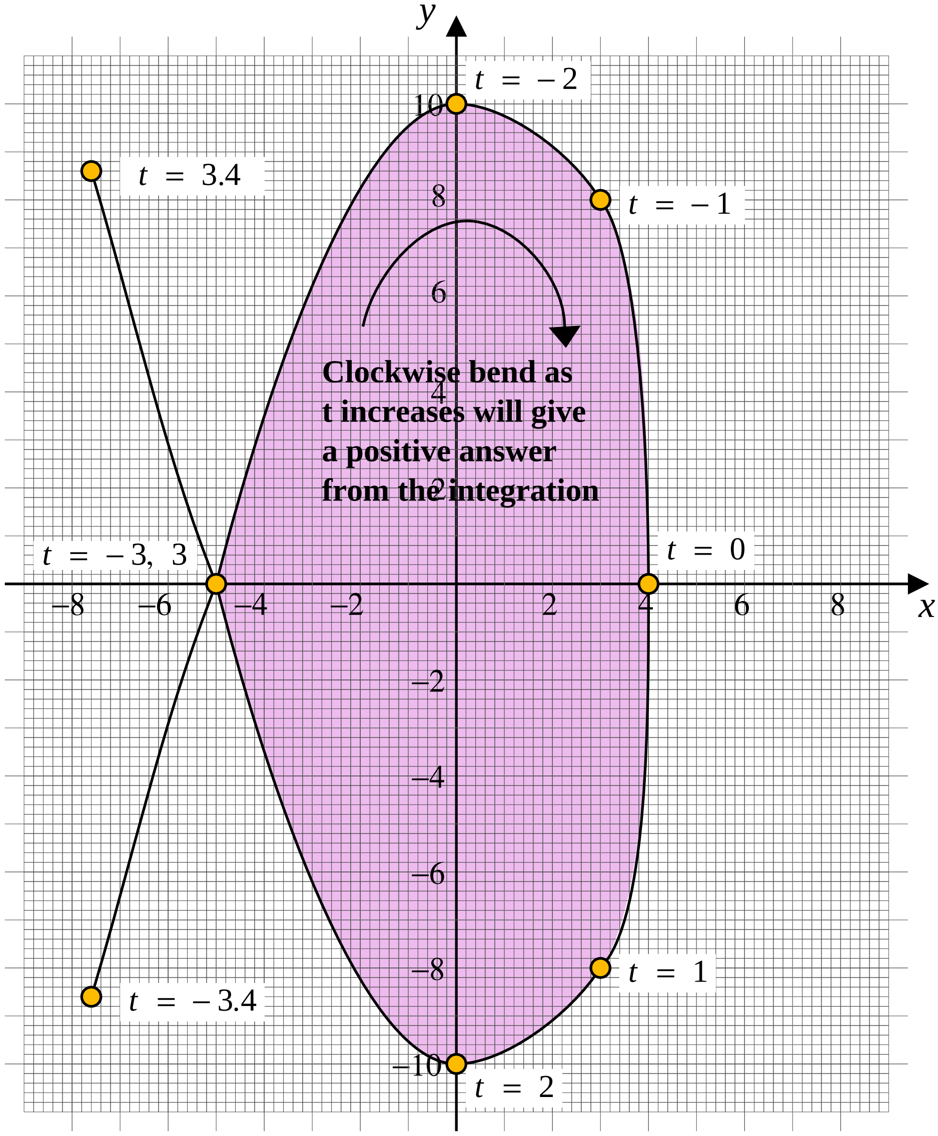
6.3.2 Solution to Exercise 6.2

(i)

t	− 3.4	− 3	− 2	− 1	0	1	2	3	3.4
x	− 7.6	− 5	0	3	4	3	0	− 5	− 7.6
y	− 8.7	0	10	8	0	− 8	− 10	0	8.7

[3 marks]

(ii)



[3 marks]

(iii) Area shaded in as shown above.

[1 mark]

(iv) $x = 4 - t^2 \Rightarrow \dfrac{dx}{dt} = -2t$

$$Area = \int y\,dx$$

$$= \int_{t=-3}^{t=3} t\left(t^2 - 9\right)(-2t)\ dt$$

$$= -2\int_{-3}^{3} t^4 - 9t^2\,dt$$

$$= -2\left[\frac{t^5}{5} - 3t^3\right]_{-3}^{3}$$

$$= -2\left\{\left[48.6 - 81\right] - \left[-48.6 - (-81)\right]\right\}$$

$$= 129.6$$

[4 marks]

(v) $x = 4 - t^2 \Rightarrow t^2 = 4 - x$

$$y = t\left(t^2 - 9\right) \Rightarrow y = \pm(4 - x)^{\frac{1}{2}}(-1)\ (x + 5)$$

$$\therefore\quad y = \pm(4 - x)^{\frac{1}{2}}\ (x + 5)$$

[4 marks]

(vi) $u = 4 - x \Rightarrow x = 4 - u \Rightarrow \dfrac{dx}{du} = -1$

When $x = -5,\ u = 9$

When $x = 4,\ u = 0$

$$Area = 2\int_{-5}^{4}(4 - x)^{\frac{1}{2}}\ (x + 5)\ dx$$

$$= 2\int_{9}^{0} u^{\frac{1}{2}}\ (9 - u)\ (-1)\ du$$

$$= 2\int_{9}^{0} u^{\frac{3}{2}} - 9u^{\frac{1}{2}}\ du$$

$$= 2\left[\frac{2u^{\frac{5}{2}}}{5} - 6u^{\frac{3}{2}}\right]_{0}^{9}$$

$$= 2 \times \left\{0 - (-64.8)\right\}$$

$= 129.6$ as before

[5 marks]

7.1 Sweating the Exam

Example

This question is about finding the area in the quadrant between the positive x-axis and the positive y-axis and the curve, C, with parametric equations,

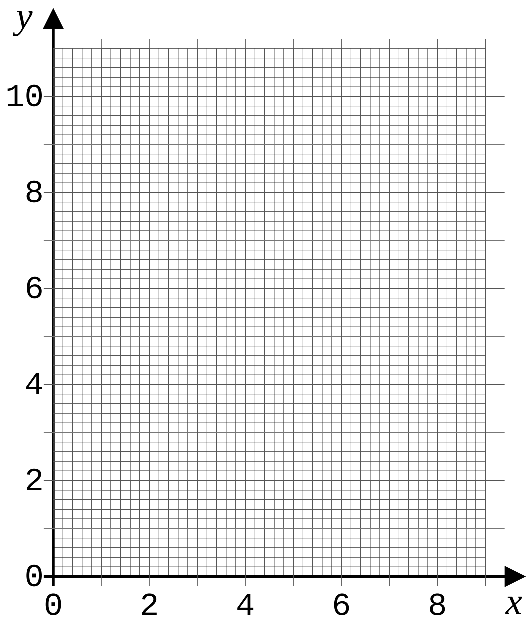
$$x = 4 - t^2 \text{ and } \quad y = t\,(t+1) \quad \text{for} \quad t \geqslant 0$$

(i) Complete the following table,

t	0	0.5	1	1.5	2
x					
y					

[2 marks]

(ii) Use the table to sketch the curve, C, and shade the area to be found.



[2 marks]

(iii) Use parametric integration to find the area.
The following teaching video demonstrates a method of solution, and may be used to help write out your solution, if required.
http://www.NumberWonder.co.uk/Video/v9045(11a).mp4

Notice that, as the bend is anticlockwise as *t* increases, strictly speaking the integration gives the area sought but with a negative sign. The video has a convincing dodge to argue about which is the lower and which is the upper limit. The following rule is worth keeping in mind;

The Limit Swap Rule

$$\int_A^B -f(x)\ dx = \int_B^A f(x)\ dx$$

The video begins having already done the following step,

$$Area = \int y\,dx = \int y\,\frac{dx}{dt}\,dt$$

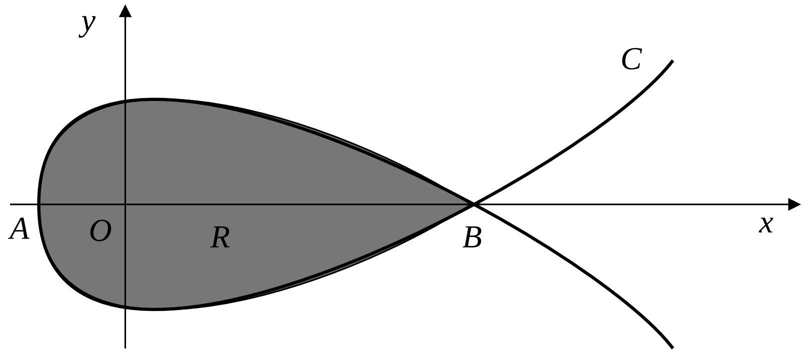
[6 marks]

7.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 34

Question 1

A-Level Examination Question from January 2010, Paper C4, Q7 (Edexcel)



The diagram shows a sketch of the curve C with parametric equations

$$x = 5t^2 - 4 \qquad y = t\left(9 - t^2\right)$$

The curve C cuts the x-axis at the points A and B

(a) Find the x-coordinate at the point A and the x-coordinate at the point B

[3 marks]

The region R, shown shaded, is enclosed by the loop of the curve

(b) Use integration to find the area of R

[6 marks]

Help for Q1 : http://www.NumberWonder.co.uk/Video/v9045(11b).mp4

Question 2

A-Level Examination Question from June 2018, Q14 (Edexcel)

A curve C has parametric equations

$$x = 3 + 2\sin t \qquad y = 4 + 2\cos 2t \qquad 0 \leqslant t < 2\pi$$

(a) Show that all points on C satisfy

$$y = 6 - (x - 3)^2$$

[2 marks]

(b) **(i)** Sketch the curve C

(ii) Explain briefly why C does not include all points of

$$y = 6 - (x - 3)^2 \qquad x \in \mathbb{R}$$

[3 marks]

The line with equation $x + y = k$ where k is a constant, intersects C at two distinct points

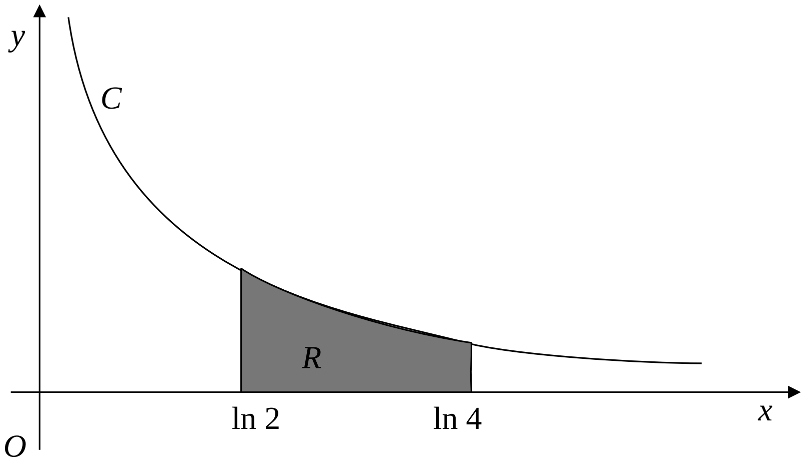
(c) State the range of values of k writing your answer in set notation

[5 marks]

Help for Q2 : http://www.NumberWonder.co.uk/Video/v9045(11c).mp4

Question 3

A-Level Examination Question from January 2008, Paper C4 (Edexcel)



The curve *C* has parametric equations

$$x = \ln(t+2) \qquad y = \frac{1}{(t+1)}$$

The finite region *R* between the curve *C* and the *x*-axis, bounded by the lines with equations $x = \ln 2$ and $x = \ln 4$, is shown shaded.

(a) Show that the area of *R* is given by the integral,

$$\int_0^2 \frac{1}{(t+1)(t+2)}\, dt$$

[4 marks]

(b) Hence find an exact value for this area.

[6 marks]

(c) Find a Cartesian equation of the curve C in the form $y = f(x)$

[4 marks]

(d) State the domain of values for x for this curve.

[1 mark]

Help for Q3 : http://www.NumberWonder.co.uk/Video/v9045(11d).mp4

7.3 Answers

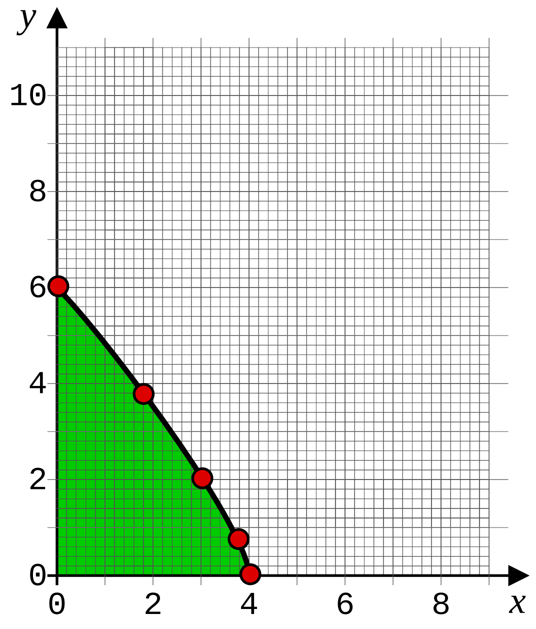
7.3.1 Solution to 7.1 Example

(i) $x = 4 - t^2$ and $y = t\,(t + 1)$ for $t \geqslant 0$

t	0	0.5	1	1.5	2
x	4	3.75	3	1.75	0
y	0	0.75	2	3.75	6

[2 marks]

(ii)



[2 marks]

(iii) First, note that $x = 4 - t^2 \Rightarrow \dfrac{dx}{dt} = -2t$

$$Area = \int y\,dx = \int y\,\frac{dx}{dt}\,dt = \int_{x=0}^{x=4} t\,(t+1)\,(-2t)\,dt$$

$$= \int_{t=2}^{t=0} -2t^3 - 2t^2\,dt$$

$$= \int_{t=0}^{t=2} 2t^3 + 2t^2\,dt \qquad \text{Limit Swap Rule}$$

$$= \left[\frac{t^4}{2} + \frac{2t^3}{3}\right]_0^2$$

$$= \left(8 + \frac{16}{3}\right) - (0)$$

$$= 13.\dot{3}$$

[6 marks]

7.3.2 Solution to 7.2 Exercise

Answer 1

(a) The curve will cut the x-axis when, $y = 0$

$$y = 0$$

$$t\left(9 - t^2\right) = 0$$

$$\therefore t = \{-3,\ 0,\ 3\}$$

When $t = \pm 3$,

$$x = 5\,(\pm 3)^2 - 4$$

$$= 41 \qquad \therefore B\,(41,\ 0)$$

When $t = 0$,

$$x = 5 \times 0^2 - 4$$

$$= -4 \qquad \therefore C\,(-4,\ 0)$$

[3 marks]

(b)

First, note that $x = 5t^2 - 4 \Rightarrow \dfrac{dx}{dt} = 10\,t$

$$Area = \int y\,dx$$

$$= \int y\,\frac{dx}{dt}\,dt$$

$$= 2\int_{x=-4}^{x=41} t\left(9 - t^2\right)\left(10t\right)dt$$

$$= 2\int_{t=0}^{t=3} 90\,t^2 - 10\,t^4\,dt$$

$$= 2\left[30\,t^3 - 2\,t^5\right]_0^3$$

$$= 2\,(810 - 486 - 0 + 0)$$

$$= 648 \text{ units}^2$$

[6 marks]

Answer 2

(a)

$$x = 3 + 2\sin t$$

$$\sin t = \frac{x-3}{2}$$

As $\cos^2 t + \sin^2 t = 1$

and $\cos^2 t - \sin^2 t = \cos 2t$ SUBTRACT

$$1 - \cos 2t = 2\sin^2 t$$

$$\cos 2t = 1 - 2\sin^2 t$$

$$\cos 2t = 1 - 2\left(\frac{x-3}{2}\right)^2$$

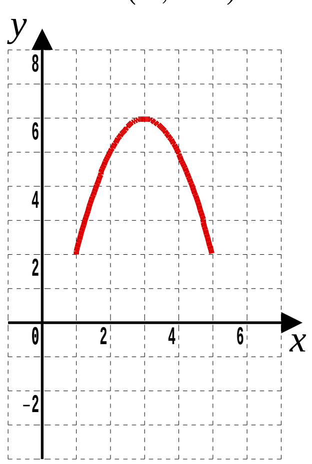
$$\begin{aligned} y &= 4 + 2\cos 2t \\ &= 4 + 2\left(1 - 2\left(\frac{x-3}{2}\right)^2\right) \\ &= 4 + 2 - (x-3)^2 \\ &= 6 - (x-3)^2 \qquad \square \end{aligned}$$

[2 marks]

(b) **(i)** The curve $y = 6 - (x-3)^2$ is in completed square form.

$\therefore$ There is a turning point at (3, 6).

This is a maximum because, when the brackets are expanded the dominant term is $-x^2$. A rewrite as $y = -x^2 + 12x - 3$ also shows that $(0, -3)$ is on the curve.



(ii) As $-1 \leqslant \sin t \leqslant 1$ it follows that $1 \leqslant x \leqslant 5$

As $-1 \leqslant \cos 2t \leqslant 1$ it follows that $2 \leqslant y \leqslant 6$

[3 marks]

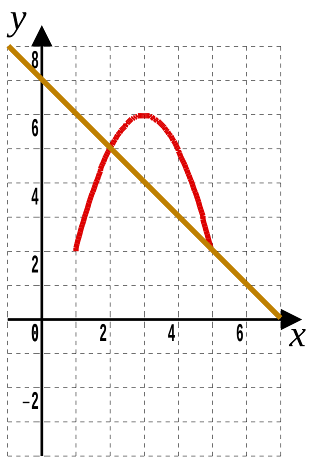
(c)

$$x + y = k$$

$$y = -x + k$$

This can be thought of as a line with a 45° downward slope that can be translated vertically.

One limit (the easier one) will be when the line passes through one of the curves endpoints at (5, 2) corresponding to $k = 7$



The other limit is more tricky to obtain.
The curve and the line intersect when,

$$-x + k = 6 - (x - 3)^2$$

$$-x + k = 6 - \left(x^2 - 6x + 9\right)$$

$$-x + k = 6 - x^2 + 6x - 9$$

$$x^2 - 7x + 3 + k = 0$$

For this to have two distinct roots, the discriminant must be positive,

$$D > 0$$

$$b^2 - 4ac > 0$$

$$(-7)^2 - 4 \times 1 \times (3 + k) > 0$$

$$49 - 12 - 4k > 0$$

$$37 - 4k > 0$$

$$-4k > -37$$

$$4k < 37$$

$$k < 9.25$$

Thus,

$$7 \leqslant k < 9.25, \quad k \in \mathbb{R}$$

[5 marks]

Answer 3

(a) $$x = \ln(t+2) \Leftrightarrow t = e^x - 2$$

$$\frac{dx}{dt} = \frac{1}{t+2}$$

$$\begin{aligned} Area &= \int_{\ln 2}^{\ln 4} y\,dx \\ &= \int_{x=\ln 2}^{x=\ln 4} y\frac{dx}{dt}\,dt \\ &= \int_{t=0}^{t=2}\left(\frac{1}{t+1}\right)\left(\frac{1}{t+2}\right)dt \\ &= \int_0^2 \frac{1}{(t+1)(t+2)}\,dt \qquad \square \end{aligned}$$

[4 marks]

(b) Use partial fractions,

$$\frac{1}{[(t+1)(t+2)]} = \frac{A}{t+1} + \frac{B}{t+2}$$

Multiply through by the red bracket

$$\frac{1 \times [\ldots\ldots]}{[(t+1)(t+2)]} = \frac{A \times [\ldots\ldots]}{t+1} + \frac{B \times [\ldots\ldots]}{t+2}$$

$$1 = A(t+2) + B(t+1)$$

Let $t = -1$: $1 = A \quad \therefore A = 1$

Let $t = -2$: $1 = -B \quad \therefore B = -1$

Thus,

$$\begin{aligned} \int_0^2 \frac{1}{(t+1)(t+2)}\,dt &= \int_0^2 \frac{1}{t+1} - \frac{1}{t+2}\,dt \\ &= \Big[\ln|t+1| - \ln|t+2|\Big]_0^2 \\ &= \ln 3 - \ln 4 - \ln 1 + \ln 2 \\ &= \ln 3 - \ln 2^2 - 0 + \ln 2 \\ &= \ln 3 - 2\ln 2 + \ln 2 \\ &= \ln 3 - \ln 2 \\ &= \ln\left(\frac{3}{2}\right) \end{aligned}$$

[6 marks]

(c)

$$x = ln\ (t + 2)$$
$$\therefore t = e^x - 2 \quad \text{as before}$$

$$y = \frac{1}{t + 1}$$
$$= \frac{1}{e^x - 2 + 1}$$
$$= \frac{1}{e^x - 1} \quad \text{is in Cartesian form, as asked for.}$$

[4 marks]

(d) Clearly, x cannot equal zero for then a division by zero would result, from

$$y = \frac{1}{e^x - 1}$$
$$= \frac{1}{e^0 - 1}$$
$$= \frac{1}{1 - 1}$$
$$= \frac{1}{0} \quad \text{Division by zero is undefined}$$

More than that, however, we have that,

$$x = ln\ (t + 2)$$

and as the log of a negative number (and zero)is not defined, $t > -2$
and consequently the domain is $x > 0$

You could also have guessed this answer from the sketch graph provided by the question.

[1 mark]

8.1 The Trapezium Rule

Integrating algebraically is desirable but not all curves have equations that can be integrated in this way. For such cases, numerical methods can be used. These are less informative because a solution method that uses numbers rather than algebra gives far less information about the bigger picture in which the solution is embedded. One such numerical approximation method is called the trapezium rule.

8.2 The Formulae Book

The trapezium rule is given in the examination formulae booklet;

$$\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}, \text{ where } h = \frac{b-a}{n}$$

The h is termed the strip width. The greater the value of n the more accurate the numerical approximation is to the true answer. However, as n is increased, the numerical calculation takes longer.

8.3 Example

The following is a table of values for $y = \sqrt{\sin x}$ where x is in radians.

x	0	$\frac{\pi}{8}$	$\frac{2\pi}{8}$	$\frac{3\pi}{8}$	$\frac{4\pi}{8}$
y	0	0.619	p	0.961	q

(i) Find the value of p and the value of q

[2 marks]

(ii) Use the trapezium rule and all the values of y in the completed table to obtain an estimate of A, where

$$A = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$$

[3 marks]

(iii) Hence determine an estimate of the related quantity, B, where

$$B = \int_0^{\frac{\pi}{2}} \left(\sqrt{\sin x} + 3\right) dx$$

[2 marks]

8.4 Exercise

Show sufficient working to make your methods clear.

Marks Available : 40

Question 1

The following is a table of values for $y = \sqrt{1 + \sin x}$ where x is in radians.

x	0	0.5	1	1.5	2
y	1	1.216	p	1.413	q

(i) Find the value of p and the value of q

[2 marks]

(ii) Use the trapezium rule and all the values of y in the completed table to obtain an estimate of A, where,

$$A = \int_0^2 \sqrt{1 + \sin x}\, dx$$

[3 marks]

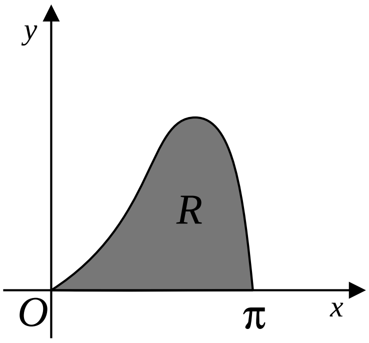
(iii) Hence determine an estimate of the related quantity, B, where,

$$B = \int_0^2 x + \sqrt{1 + \sin x}\ \ dx$$

[2 marks]

Question 2

A-Level Examination Question from January 2008, Paper C4, Q1 (Edexcel)



The curve shown has equation, $y = e^x\sqrt{\sin x}$ $\quad 0 \leqslant x \leqslant \pi$

The finite region R bounded by the curve and the x-axis is shown shaded.

(a) Complete the table with the values of y, correct to 5 decimal places.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0			8.87207	

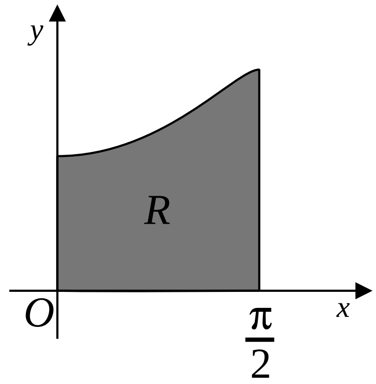
[2 marks]

(b) Use the trapezium rule, with all the values in the completed table, to obtain an estimate for the area of the region R.
Give your answer to 4 decimal places.

[4 marks]

Question 3

A-Level Examination Question from June 2013, Paper C4, Q3 (edit) (Edexcel)



The curve shows the finite region R bounded by the x-axis, the y-axis, the line $x = \frac{\pi}{2}$ and the curve with equation, $y = sec\left(\frac{1}{2}x\right)$ $\quad 0 \leqslant x \leqslant \frac{\pi}{2}$

(a) Complete the table, giving the missing values of y to 6 decimal places.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	1	1.035276		1.414214

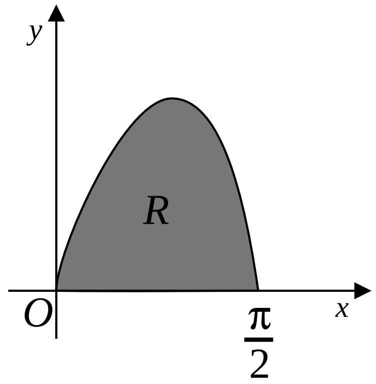
[1 mark]

(b) Using the trapezium rule, with all the values of y from the completed table, find an approximation for the area of R.
Give your answer to 4 decimal places.

[3 marks]

Question 4

A-Level Examination Question from January 2012, Paper C4, Q6 (edit) (Edexcel)



The sketch is of the curve with equation, $y = \dfrac{2\sin 2x}{(1 + \cos x)} \qquad 0 \leqslant x \leqslant \dfrac{\pi}{2}$

(a) Complete the table, giving the missing values of y to 5 decimal places.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0		1.17157	1.02280	0

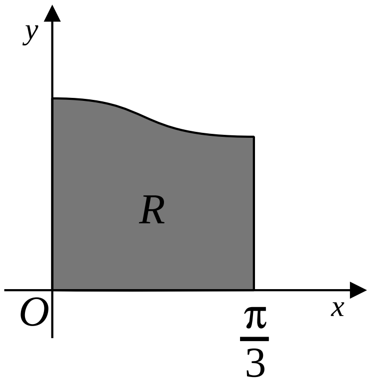
[1 mark]

(b) Use the trapezium rule, with all the values of y from the completed table, to obtain an estimate for the area of R.
Give your answer to 4 decimal places.

[3 marks]

Question 5

A-Level Examination Question from June 2010, Paper C4, Q1 (edit) (Edexcel)



The sketch is of the curve with equation, $y = \sqrt{0.75 + \cos^2 x}$

The finite region R, shown shaded, is bounded by the curve, the y-axis, the x-axis and the line with equation $x = \frac{\pi}{3}$

(a) Complete the table with values of y corresponding to $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1.3229	1.2973			1

[2 marks]

(b) Use the trapezium rule,

(i) with the values of y at $x = 0$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ to find an estimate of the area of R. Give your answer to 3 decimal places.

[3 marks]

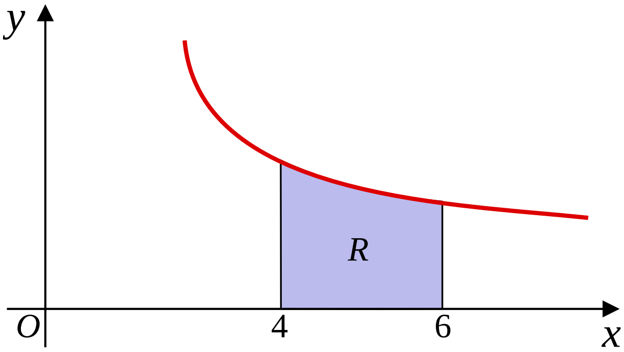
(ii) with the values of y at $x = 0$, $x = \dfrac{\pi}{12}$, $x = \dfrac{\pi}{6}$, $x = \dfrac{\pi}{4}$ and $x = \dfrac{\pi}{3}$ to find a further estimate of the area of R.

Give your answer to 3 decimal places.

[3 marks]

Question 6

A-Level Examination Question from January 2019, Paper C34, Q7 (Edexcel)



The diagram shows a sketch of part of the curve with equation,

$$y = \frac{x+7}{\sqrt{2x-3}}, \qquad x > \frac{3}{2}$$

The region R, shown shaded, is bounded by the curve, the line with equation $x = 4$, the x-axis and the line with equation $x = 6$

(a) Use the trapezium rule with 4 strips of equal width to find an estimate for the area of R, giving your answer to 2 decimal places.

[4 marks]

(b) Using the substitution $u = 2x - 3$, or otherwise, use calculus to find the exact area of R, giving your answer in the form $a + b\sqrt{5}$, where a and b are constants to be found.

[7 marks]

8.5 Answers

8.5.1 Solution to 8.3 Example

$$y = \sqrt{\sin x}$$

(i)

x	0	$\frac{\pi}{8}$	$\frac{2\pi}{8}$	$\frac{3\pi}{8}$	$\frac{4\pi}{8}$
y	0	0.619	p	0.961	q

$$p = \sqrt{\sin\left(\frac{2\pi}{8}\right)} = \sqrt{\frac{\sqrt{2}}{2}} = 0.841$$

$$q = \sqrt{\sin\left(\frac{4\pi}{8}\right)} = \sqrt{1} = 1.000$$

[2 marks]

(ii) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\left\{ \left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right) \right\}$$

$$A = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx \approx \frac{1}{2} \times \frac{\pi}{8}\{(0 + 1) + 2\,(0.619 + 0.841 + 0.961)\}$$

$$\approx \frac{\pi}{16}\{1 + 2 \times 2.421\}$$

$$\approx 1.147$$

[3 marks]

(iii)

$$B = \int_0^{\frac{\pi}{2}} \left(\sqrt{\sin x} + 3\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx + \int_0^{\frac{\pi}{2}} 3\,dx$$

$$\approx 1.147 + \left[\,3x\,\right]_0^{\frac{\pi}{2}}$$

$$\approx 1.147 + 3 \times \frac{\pi}{2}$$

$$\approx 5.859$$

[2 marks]

8.5.2 Solutions to 8.4 Exercise

Answer 1

$$\sqrt{1 + \sin x}$$

(i)

x	0	0.5	1	1.5	2
y	1	1.216	p	1.413	q

$$p = \sqrt{1 + \sin(1)} = 1.357$$

$$q = \sqrt{1 + \sin(2)} = 1.382$$

[2 marks]

(ii) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0 + y_4) + 2(y_1 + y_2 + y_3)\}$$

$$A = \int_0^2 \sqrt{1 + \sin x}\; dx \approx \frac{0.5}{2}\{(1 + 1.382) + 2(1.216 + 1.357 + 1.413)\}$$

$$\approx 0.25\{2.382 + 2 \times 3.986\}$$

$$\approx 2.589$$

[3 marks]

(iii)

$$B = \int_0^2 x + \sqrt{1 + \sin x}\; dx$$

$$= \int_0^2 x\,dx + \int_0^2 \sqrt{1 + \sin x}\; dx$$

$$\approx \left[\frac{x^2}{2}\right]_0^2 + 2.589$$

$$\approx 2 + 2.589$$

$$\approx 4.589$$

[2 marks]

Answer 2

$$y = e^x \sqrt{\sin x}$$

(a)

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	1.84432	4.81047	8.87207	0

[2 marks]

(b) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\{\,(y_0 + y_4) + 2(y_1 + y_2 + y_3)\,\}$$

$$R = \int_0^{\pi} e^x \sqrt{\sin x}\,dx$$

$$\approx \frac{1}{2} \times \frac{\pi}{4}\{(0 + 0) + 2(1.84432 + 4.81047 + 8.87207)\}$$

$$\approx \frac{\pi}{8}\{0 + 2 \times 15.52686\}$$

$$\approx 12.1948$$

[4 marks]

Answer 3

$$y = sec\left(\frac{1}{2}x\right)$$

(a)

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	1	1.035276	1.154701	1.414214

[1 mark]

(b) With three strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\{\,(y_0 + y_3) + 2(y_1 + y_2)\,\}$$

$$\approx \frac{1}{2} \times \frac{\pi}{6}\{(1 + 1.414214) + 2(1.035276 + 1.154701)\}$$

$$\approx \frac{\pi}{12}\{2.414214 + 2 \times 2.189977\}$$

$$\approx 1.7787$$

[3 marks]

Answer 4

$$y = \frac{2 \sin 2x}{(1 + \cos x)}$$

(a)

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y	0	0.73508	1.17157	1.02280	0

[1 mark]

(b) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\{ (y_0 + y_4) + 2(y_1 + y_2 + y_3) \}$$

$$\approx \frac{1}{2} \times \frac{\pi}{8}\{(0 + 0) + 2(0.73508 + 1.17157 + 1.02280)\}$$

$$\approx \frac{\pi}{16}\{ 0 + 2 \times 2.92945 \}$$

$$\approx 1.1504$$

[3 marks]

Answer 5

$$y = \sqrt{0.75 + cos^2 x}$$

(a)

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1.3229	1.2973	1.2247	1.1180	1

[2 marks]

(b) **(i)** With two strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\left\{ \left(y_0 + y_2 \right) + 2\left(y_1 \right) \right\}$$

$$\approx \frac{1}{2} \times \frac{\pi}{6}\{(1.3229 + 1) + 2(1.2247)\}$$

$$\approx \frac{\pi}{12}\{2.3229 + 2 \times 1,2247\}$$

$$\approx 1.249$$

[3 marks]

(ii) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\left\{ \left(y_0 + y_4 \right) + 2\left(y_1 + y_2 + y_3 \right) \right\}$$

$$\approx \frac{1}{2} \times \frac{\pi}{12}\{(1.3229 + 1) + 2(1.2973 + 1.2247 + 1.1180)\}$$

$$\approx \frac{\pi}{24}\{2.3229 + 2 \times 3.6400\}$$

$$\approx 1.257$$

[3 marks]

Answer 6

(a) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\left\{ \left(y_0 + y_4 \right) + 2\left(y_1 + y_2 + y_3 \right) \right\}$$

$$R \approx \frac{1}{2} \times \frac{1}{2}\left\{\left(\frac{11\sqrt{5}}{5} + \frac{13}{3}\right) + 2\left(\frac{23\sqrt{6}}{12} + \frac{12\sqrt{7}}{7} + \frac{25\sqrt{2}}{8}\right)\right\}$$

$$\approx \frac{1}{2} \times \frac{1}{2}\{(4.91 + 4.33 + 2(4.69 + 4.53 + 4.41)\}$$

$$\approx 9.14$$

[4 marks]

(b)

$$\int_4^6 \frac{x + 7}{\sqrt{2x - 3}}\,dx = \int_5^9 \frac{u + 3 + 14}{2\sqrt{u}}\ \frac{du}{2}$$

$$= \int_5^9 \frac{u + 17}{4\sqrt{u}}\,du$$

$$= \frac{1}{4}\int_5^9 u^{\frac{1}{2}} + 17u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\left[\frac{2u^{\frac{3}{2}}}{3} + 34u^{\frac{1}{2}}\right]_5^9$$

$$= 30 - \frac{28}{3}\sqrt{5}$$

[7 marks]

Lesson 9

A-Level Pure Mathematics : Year 2

Integration III

9.1 Revision

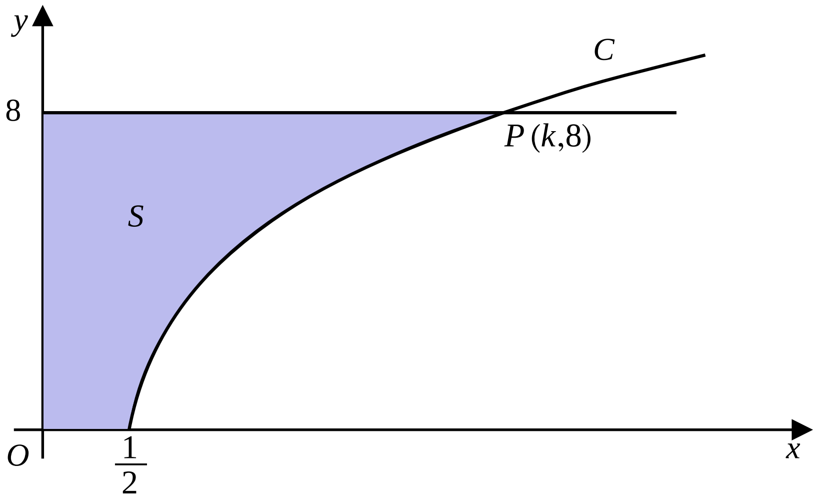
Show sufficient working to make your methods clear.

Marks Available : 40

Question 1

(a) Find $\int (2x - 1)^{\frac{3}{2}}\, dx$ giving your answer in its simplest form.

[3 marks]



The sketch shows part of the curve C with equation $y = (2x - 1)^{\frac{3}{2}}$, $x \geqslant \frac{1}{2}$

which cuts the line $y = 8$ at point P with coordinates $(k, 8)$, where k is a constant.

(b) Find the value of k

[2 marks]

(c) Find the shaded area, S, bounded by the coordinate axes, $y = 8$ and C.

[4 marks]

Question 2

A-Level Examination Question from October 2021, Paper 2, Q12 (Edexcel)

(a) Use the substitution $u = 1 + \sqrt{x}$ to show that,

$$\int_0^{16} \frac{x}{1 + \sqrt{x}}\, dx = \int_p^q \frac{2(u - 1)^3}{u}\, du$$

where p and q are constants to be found.

[3 marks]

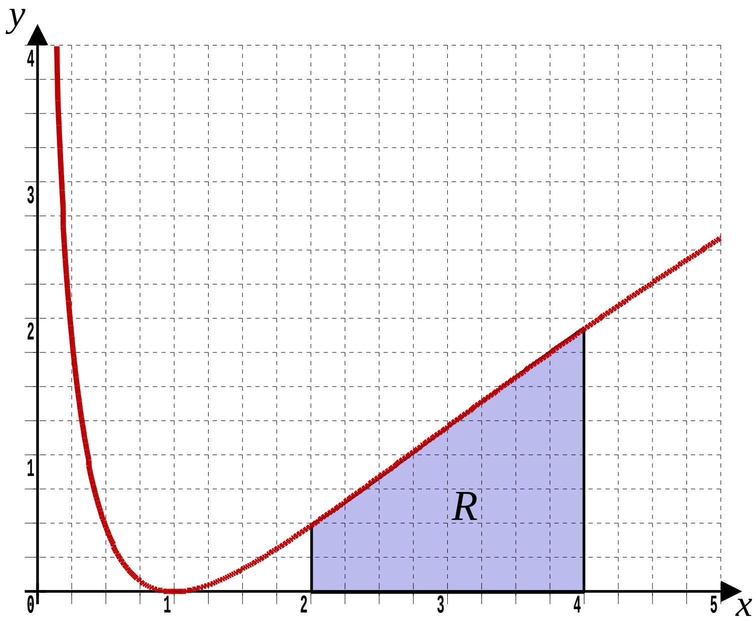
(b) Hence show that, $\int_0^{16} \frac{x}{1 + \sqrt{x}}\, dx = A - B \ln 5$

where A and B are constants to be found.

[4 marks]

Question 3

A-Level Examination Question from October 2021, Paper 1, Q11 (Edexcel)



The graph shows part of the curve with equation, $y = (\ln x)^2, \quad x > 0$. The finite region R, shown shaded, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 4$. The table below shows corresponding values of x and y, with the values of y given to 4 decimal places.

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R. giving your answer to 3 significant figures.

[3 marks]

(b) Use algebraic integration to find the exact area of *R*, giving your answer in the form,

$$y = a(\ln 2)^2 + b\ln 2 + c$$

where *a*, *b* and *c* are integers to be found.

[5 marks]

Question 4

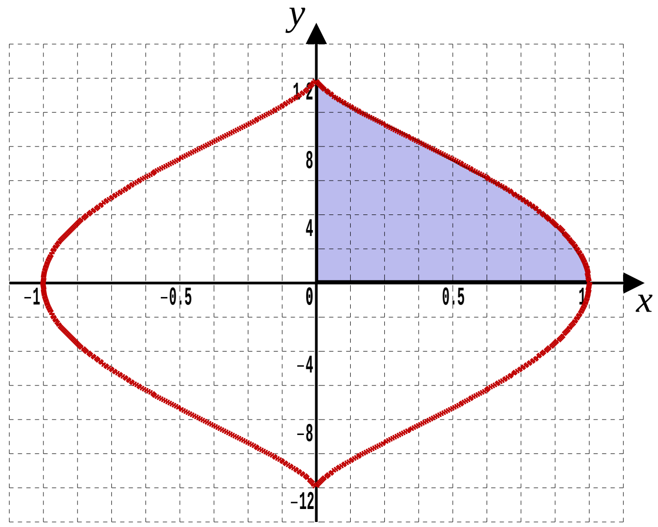
A-Level Examination Question from October 2021, Paper 1, Q14 (Edexcel)

Given that $y = \dfrac{x - 4}{2 + \sqrt{x}}$, $x > 0$, show that $\dfrac{dy}{dx} = \dfrac{1}{A\sqrt{x}}$, $x > 0$

where *A* is a constant to be found.

[4 marks]

Question 5



The graph is of the curve C with parametric equations,

$$x = \cos^3\theta, \quad y = 12\sin\theta, \quad 0 \leqslant \theta < 2\pi$$

The finite region in the first quadrant, bounded by C and the coordinate axes, is shown shaded. The curve is symmetrical in both the x and the y axis.

(a) Show that the area of the shaded region is given by the integral,

$$36\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta$$

[4 marks]

(b) Use trigonometric identities to show that,

$$cos^2\,\theta\; sin^2\,\theta \;=\; \frac{1}{8}\left(1 \;-\; cos\, 4\theta\right)$$

[4 marks]

(c) Hence find, in terms of π, the total area enclosed by C.

[4 marks]

Answer 1

(a) Add in a fiddle factor of $\frac{1}{2} \times 2$

$$\int (2x - 1)^{\frac{3}{2}}\, dx = \frac{1}{2} \int 2(2x - 1)^{\frac{3}{2}}\, dx$$

$$= \frac{1}{2} \times \frac{2(2x - 1)^{\frac{5}{2}}}{5} + c$$

$$= \frac{(2x - 1)^{\frac{5}{2}}}{5} + c$$

[3 marks]

(b) Equate curve with line, then solve.

$$(2x - 1)^{\frac{3}{2}} = 8$$

cube root both sides

$$(2x - 1)^{\frac{1}{2}} = 2$$

square both sides

$$2x - 1 = 4$$

$$2x = 5$$

$$x = \frac{5}{2} \qquad \therefore k = \frac{5}{2}$$

[2 marks]

(c) Shaded area, S, is area of rectangle less area under the curve.

$$S = 8 \times 2.5 - \int_{\frac{1}{2}}^{\frac{5}{2}} (2x - 1)^{\frac{3}{2}}\, dx$$

$$= 20 - \left[\frac{(2x - 1)^{\frac{5}{2}}}{5} \right]_{\frac{1}{2}}^{\frac{5}{2}}$$

$$= 20 - \left[\frac{32}{5} - 0 \right]$$

$$= 13.6$$

[4 marks]

Answer 2

(a) $u = 1 + \sqrt{x} \Rightarrow \dfrac{du}{dx} = \dfrac{1}{2} x^{-\frac{1}{2}} \quad \therefore \dfrac{dx}{du} = 2\sqrt{x}$

Also $\sqrt{x} = u - 1$

When $x = 16,\ u = 5$ and when $x = 0,\ u = 1$

$$\int_0^{16} \frac{x}{1 + \sqrt{x}}\, dx = \int_1^5 \frac{x}{u}\, 2\sqrt{x}\, du$$

$$= \int_1^5 \frac{2x^{\frac{3}{2}}}{u}\, du$$

$$= \int_1^5 \frac{2(\sqrt{x})^3}{u}\, du$$

$$= \int_1^5 \frac{2(u - 1)^3}{u}\, du$$

Which is of the form required with $p = 1$ and $q = 5$

[3 marks]

(b) $$\int_0^{16} \frac{x}{1 + \sqrt{x}}\, dx = \int_1^5 \frac{2(u - 1)^3}{u}\, du$$

$$= 2\int_1^5 \frac{u^3 - 3u^2 + 3u - 1}{u}\, du$$

$$= 2\int_1^5 u^2 - 3u + 3 - \frac{1}{u}\, du$$

$$= 2\left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u \right]_1^5$$

$$= 2\left[\left(\frac{125}{3} - \frac{75}{2} + 15 - \ln 5 \right) - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1 \right) \right]$$

$$= 2\left[\frac{124}{3} - \frac{72}{2} + 12 - \ln 5 \right]$$

$$= 2\left[\frac{52}{3} - \ln 5 \right]$$

$$= \frac{104}{3} - 2\ln 5$$

Which is of the form required with $A = \dfrac{104}{3},\ B = 2$ and $A = 5$

[4 marks]

Answer 3

(a) With four strips the trapezium rule becomes,

$$\int_a^b y\,dx \approx \frac{1}{2}h\{\,(y_0 + y_4) + 2(y_1 + y_2 + y_3)\,\}$$

$$R \approx \frac{1}{2} \times \frac{1}{2}\{(0.4805 + 1.9218) + 2(0.8396 + 1.2069 + 1.5694)\}$$

$$\approx 2.41$$

[3 marks]

(b) Integration by parts with a sneaky 1 inserted...

$$R = \int_2^4 (\,ln\,x\,)^2 \times 1\ dx$$

Without worrying about the limits for now...

(Or the constant of algebraic integration)

$$= \quad \boldsymbol{L} \qquad \boldsymbol{I} \qquad\qquad \boldsymbol{D} \qquad \boldsymbol{I}$$

$$= (\,ln\,x\,)^2\ x \quad - \int 2\ (\,ln\,x\,)\ \frac{1}{x}\ x \quad dx$$

$$= (\,ln\,x\,)^2\ x \quad - \ 2\int (\,ln\,x\,) \times 1\ \ dx$$

$$= (\,ln\,x\,)^2\ x \quad - \ 2\left[\ (\,ln\,x\,)\ x - \int \frac{1}{x}\ x\,dx\ \right]$$

$$= (\,ln\,x\,)^2\ x \quad - \ 2x\,ln\,x + \int 2\,dx$$

Now to put in the limits...

$$= \Big[\ x\ (\,ln\,x\,)^2 \ - 2x\,ln\,x + 2x\,\Big]_2^4$$

$$= 4\ (\,ln\,4\,)^2 - 8\,ln\,4 + 8 - 2\ (\,ln\,2\,)^2 + 4\,ln\,2 - 4$$

$$= 4\ (\,2\,ln\,2\,)^2 - 16\,ln\,2 + 8 - 2\ (\,ln\,2\,)^2 + 4\,ln\,2 - 4$$

$$= 14\ (\,ln\,2\,)^2 - 12\,ln\,2 + 4$$

Which is of the form required with $a = 14$, $b = -12$ and $c = 4$

[5 marks]

Answer 4

CLEVER METHOD using a difference of two squares.

$$y = \frac{x - 4}{2 + \sqrt{x}}$$

$$= \frac{(\sqrt{x})^2 - 2^2}{\sqrt{x} + 2}$$

$$= \frac{(\sqrt{x} + 2)(\sqrt{x} - 2)}{(\sqrt{x} + 2)}$$

$$= x^{\frac{1}{2}} - 2$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

which is of the required form with $A = 2$

[4 marks]

STANDARD METHOD using the quotient rule.

$$y = \frac{x - 4}{2 + \sqrt{x}}$$

$$\frac{dy}{dx} = \frac{(2 + \sqrt{x}) \times 1 - \frac{1}{2} x^{-\frac{1}{2}} (x - 4)}{(2 + \sqrt{x})^2} \times \frac{2x^{\frac{1}{2}}}{2x^{\frac{1}{2}}}$$

$$= \frac{4x^{\frac{1}{2}} + 2x - x + 4}{2\sqrt{x}(4 + 4\sqrt{x} + x)}$$

$$= \frac{(4 + 4\sqrt{x} + x)}{2\sqrt{x}\,(4 + 4\sqrt{x} + x)}$$

$$= \frac{1}{2\sqrt{x}}$$

which is of the required form with $A = 2$

[4 marks]

Answer 5

(a)

$$x = \cos^3\theta \Rightarrow \frac{dx}{d\theta} = 3\cos^2\theta\,(-\sin\theta)$$

$$\text{Also, } \theta = \arccos\left(\sqrt[3]{x}\right)$$

$$\begin{aligned}
Area\ Shaded &= \int y\,dx \\
&= \int_{x=0}^{x=1} 12\sin\theta \times 3\cos^2\theta\,(-\sin\theta)\,d\theta \\
&= \int_{\theta=\frac{\pi}{2}}^{\theta=0} -36\sin^2\theta\cos^2\theta\,d\theta \\
&= 36\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta
\end{aligned}$$

[4 marks]

(b) First note the two relations that...

$$\begin{aligned}
&\cos^2 2\theta + \sin^2 2\theta = 1 \\
\text{ADD}\quad &\cos^2 2\theta - \sin^2 2\theta = \cos 4\theta \\
\hline
&2\cos^2 2\theta = 1 + \cos 4\theta
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \cos^2\theta\sin^2\theta \\
&= \frac{1}{4}(2\sin\theta\cos\theta)^2 \\
&= \frac{1}{4}\sin^2 2\theta \\
&= \frac{1}{4}\left(1 - \cos^2 2\theta\right) \\
&= \frac{1}{8}\left(2 - 2\cos^2 2\theta\right) \\
&= \frac{1}{8}\left(2 - 1 - \cos 4\theta\right) \\
&= \frac{1}{8}\left(1 - \cos 4\theta\right) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[4 marks]

(c)

$$\begin{aligned} Total\ Area &= 4 \times 36 \int_0^{\frac{\pi}{2}} sin^2\,\theta\, cos^2\,\theta\; d\theta \\ &= 4 \times 36 \times \frac{1}{8} \int_0^{\frac{\pi}{2}} 1 - cos\,4\theta\; d\theta \\ &= 18\left[\ \theta - \frac{sin\,4\theta}{4}\ \right]_0^{\frac{\pi}{2}} \\ &= 18\left[\ \left(\frac{\pi}{2} - \frac{sin\,2\pi}{4}\right) - \left(0 - \frac{sin\,0}{4}\right)\ \right] \\ &= 18 \times \frac{\pi}{2} \\ &= 9\pi \end{aligned}$$

[4 marks]