1.7 Answers

1.7.1 Solutions to Teaching Examples

#1.2 Three consecutive natural numbers have a sum of 84.
What are the three consecutive natural numbers ?

Method 1 : Form and Solve an Equation

$$(n-1)+(n)+(n+1) = 84$$
$$3n = 84$$
$$n = 28$$

$\therefore$ The three consecutive numbers are 27, 28, 29

[2 marks]

Method 2 : Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x+x+x = 84$$
$$3x = 84$$
$$x = 28$$

The numbers are around 28. Experimentation gives 27, 28, 29

[2 marks]

#1.3 Three consecutive natural numbers have a product of 85140.
What are the three consecutive natural numbers ?

Method 1 : Form and Solve an Equation

$$(n-1)(n)(n+1) = 85140$$
$$n\left(n^2-1\right) = 85140 \qquad \text{difference of two squares}$$
$$n^3+0n^2-n-85140 = 0 \qquad \text{this method in trouble !}$$
$$n = 44 \qquad \text{using equation solver (cheating)}$$

so the numbers are 43, 44, 45

[2 marks]

Method 2 : Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x \times x \times x = 85140$$
$$x^3 = 85140$$
$$x = \sqrt[3]{85140}$$
$$= 43.99$$

The numbers are around 44. Experimentation gives 43, 44, 45

[2 marks]

#1.4

p	1	2	3	4	...	n	$n + 1$	$n + 2$	...
T_p	1	3	5	7	...	$2n - 1$	$2n + 1$	$2n + 3$	...

[4 marks]

#1.5

(a) **(i)** $1 \times 3 = 3$

(ii) $3 \times 5 = 15$

(iii) $5 \times 7 = 35$

(iv) $7 \times 9 = 63$

(v) $9 \times 11 = 99$

[2 marks]

(b) Let the two consecutive odd numbers be ($2n - 1$) and ($2n + 1$)

in which case $(2n - 1)(2n + 1) = (2n)^2 - 1^2$

$= m^2 - 1$ where $m = 2n$

which is one less than a square number. ☐

[4 marks]

1.7.2 Solutions to 1.6 Exercise

Answer 1

Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x + x + x = 204$$
$$3x = 204$$
$$x = 68$$

The numbers are around 68. Experimentation gives 67, 68, 69

[2 marks]

Answer 2

Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x \times x \times x = 778596$$
$$x^3 = 778596$$
$$x = \sqrt[3]{778596}$$
$$= 91.99$$

The numbers are around 92. Experimentation gives 91, 92, 93

[2 marks]

Answer 3

Make a Simplifying Assumption

As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x + x + x + x = 78$$
$$4x = 78$$
$$x = 19.5$$

The numbers are around 19.5. Experimentation gives 18, 19, 20, 21

[2 marks]

Answer 4

Make a Simplifying Assumption

As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x \times x \times x \times x \times x = 95040$$
$$x^5 = 95040$$
$$x = \sqrt[5]{95040}$$
$$= 9.90$$

The numbers are around 9.9. Experimentation gives 8, 9, 10, 11, 12

[3 marks]

Answer 5

(a) **(i)** $3^2 - 1^2 = 8$

(ii) $5^2 - 3^2 = 16$

(iii) $7^2 - 5^2 = 24$

(iv) $9^2 - 7^2 = 32$

(v) $11^2 - 9^2 = 40$

[3 marks]

(b) Let the two consecutive odd numbers be $(2n - 1)$ and $(2n + 1)$ in which case,

$$(2n + 1)^2 - (2n - 1)^2 = \left(4n^2 + 4n + 1\right) - \left(4n^2 - 4n + 1\right)$$
$$= 4n^2 + 4n + 1 - 4n^2 + 4n - 1$$
$$= 8n$$

which is divisible by 8 for all $n \in \mathbb{N}$ $\square$

[5 marks]

Answer 6

Let the four consecutive odd numbers be,

$$(2n - 3), (2n - 1), (2n + 1), (2n + 3)$$

in which case,

$$(2n - 3) + (2n - 1) + (2n + 1) + (2n + 3) = 8n$$

which is divisible by 8 for all $n \in \mathbb{N}$ □

[4 marks]

Answer 7

p	1	2	3	4	...	$n - 1$	n	$n + 1$	...
T_p	2	4	6	8	...	$2n - 2$	$2n$	$2n + 2$	...

[5 marks]

Answer 8

Let the three consecutive even numbers be,

$$(2n - 2), (2n), (2n + 2)$$

in which case,

$$(2n - 2) + (2n) + (2n + 2) = 72$$

$$6n = 72$$

$$n = 12$$

The three consecutive even numbers are, 22, 24, 26

[4 marks]

Answer 9

Let the three consecutive even numbers be,

$$(2n - 2), (2n), (2n + 2)$$

in which case,

$$(2n - 2)^2 + (2n)^2 + (2n + 2)^2$$

$$= 4n^2 - 8n + 4 + 4n^2 + 4n^2 + 8n + 4$$

$$= 12n^2 + 8$$

$$= 4\left(3n^2 + 2\right)$$

$= 4m$ for some natural number m

which is divisible by 4 for all $m \in \mathbb{N}$ □

[5 marks]

Answer 10

(a) Let the odd number n be expressed in the form $2m + 1$ for $m \in \mathbb{N} \cup \{0\}$.
in which case,

$$\begin{aligned} n^3 - n &= n\left[n^2 - 1 \right] \\ &= (2m+1)\left[(2m+1)^2 - 1 \right] \\ &= (2m+1)\left[4m^2 + 4m + 1 - 1 \right] \\ &= (2m+1)\left[4m^2 + 4m \right] \\ &= 4(2m+1)\left[m^2 + m \right] \end{aligned}$$

which is divisible by 4 ☐

[4 marks]

(b) In part (a) the result has been proven true for all odd numbers.
So, in trying to show it's not always true, the counterexample has to be amongst the even numbers.
Let $n = 2$ in which case,

$$\begin{aligned} n^3 - n &= 2^3 - 2 \\ &= 8 - 2 \\ &= 6 \text{ which is not divisible by 4} \end{aligned}$$

$\therefore\ n^3 - n$ is not always divisible by 4 ☐

[1 mark]

Note : Counterexamples are of the form $4m + 2$ for $m \in \mathbb{N} \cup \{0\}$