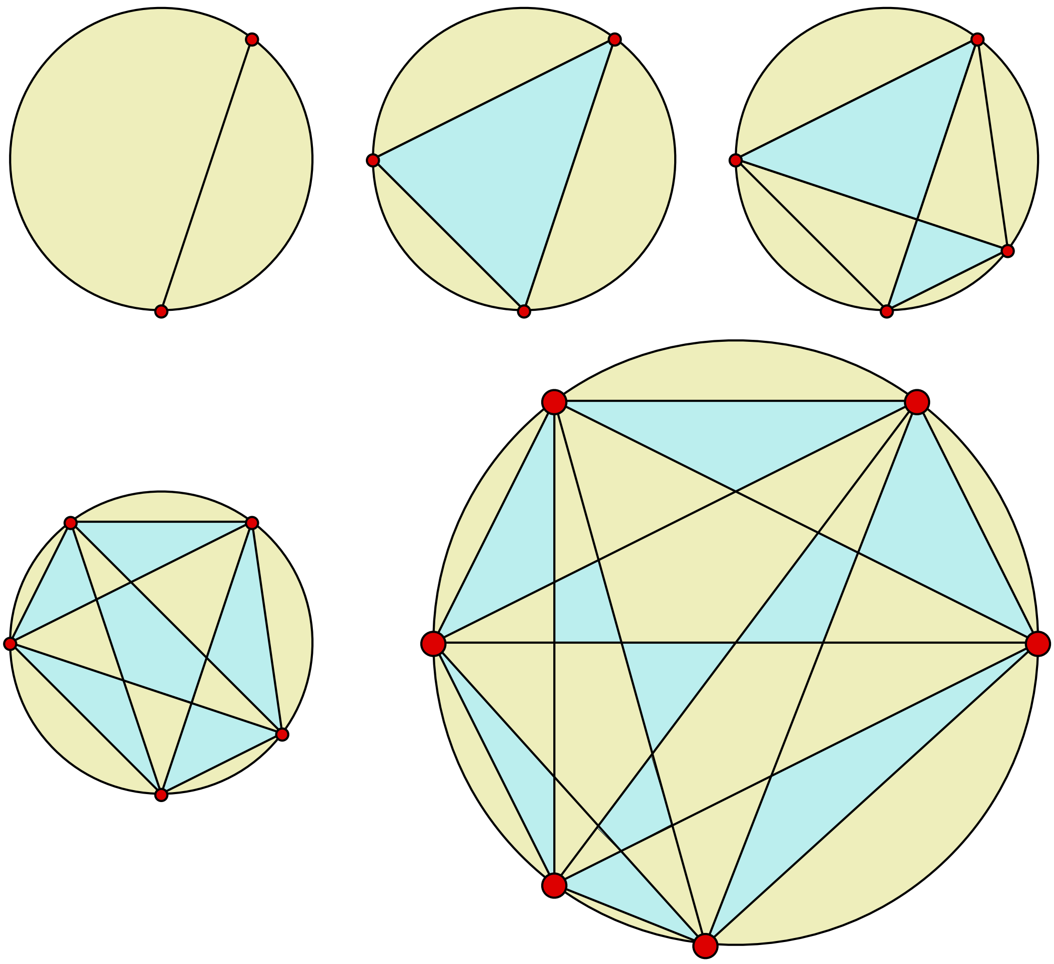
3.5 Answers

3.5.1 Solution to 3.2 Finding a Counterexample

The following colouring of the diagrams makes it easier to count the number of regions.



n	2	3	4	5	6
2^{n-1}	2	4	8	16	32
From Diagram	2	4	8	16	31

An excellent illustration of why proof is needed before a pattern that seems to be there is accepted into mathematics !

[6 marks]

3.5.2 Solution to 3.3 When a Counterexample Can't Be Found

Either prove, or find a counterexample to the conjecture that, “if you add 1 to the product of four consecutive natural numbers the answer is always a perfect square”.

Exploring some numerical cases to see if a counterexample emerges...

$$1 \times 2 \times 3 \times 4 + 1 = 25 = 5^2$$

$$2 \times 3 \times 4 \times 5 + 1 = 121 = 11^2$$

$$3 \times 4 \times 5 \times 6 + 1 = 361 = 19^2$$

$$4 \times 5 \times 6 \times 7 + 1 = 841 = 29^2$$

The claim, expressed in algebra, is that,

$$(n-1)n(n+1)(n+2) + 1 = m^2 \qquad \text{for } n, m \in \mathbb{N}$$

$$\begin{aligned}
\text{LHS} &= (n-1)n(n+1)(n+2) + 1 \\
&= \left(n^2 - 1\right)\left(n^2 + 2n\right) + 1 \\
&= n^4 + 2n^3 - n^2 - 2n + 1 \\
&= \left(n^2 + an \pm 1\right)^2 && \text{for some } a \in \mathbb{N} \\
&= \left(n^2 + n - 1\right)^2 \\
&= m^2 && \text{where } m = n^2 - n + 1 \\
&= \text{RHS} && \square
\end{aligned}$$

	n^2	n	-1
n^2	n^4	n^3	$-n^2$
n	n^3	n^2	$-n$
-1	$-n^2$	$-n$	1

3.5.3 Solutions to 3.4 Exercise

Answer 1

$$x^2 = y^2$$

$$x^2 - y^2 = 0$$

$$(x - y)(x + y) = 0 \qquad \text{Difference of two squares}$$

Either $x - y = 0$ in which case $x = y$

Or $x + y = 0$ in which case $x = -y$

$\therefore$ Any counterexample that satisfies $x = -y$ will suffice

[2 marks]

Answer 2

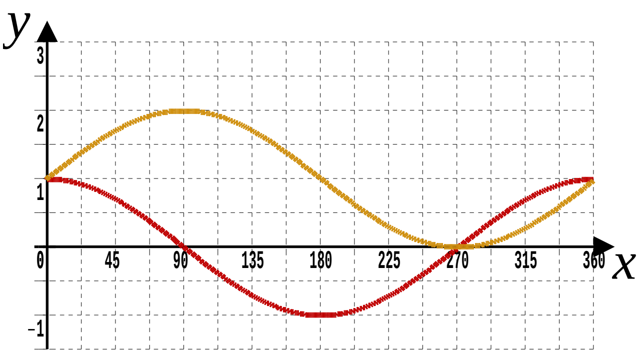
Any counterexample where $|a| > |b|$ with $a < 0$ and $b > 0$

For example,

$$a = -3, \quad b = 2 \text{ gives } a^2 - b^2 = 9 - 4 > 0 \quad \text{with } a - b < 0$$

[2 marks]

Answer 3



In red : $y = \cos x$

In gold : $y = 1 + \sin x$

From the graph it can be seen that any counterexample will have to be between $x = 270°$ and $x = 360°$

[2 marks]

Answer 4

Counterexamples can be found by taking any Pythagorean integer triple, for example, $3^2 + 4^2 = 5^2$ and dividing through by, in this case, the 5^2. This then gives the counterexample $x = \frac{3}{5}$, $y = \frac{4}{5}$ for which $x^2 + y^2 = 1$

[2 marks]

Answer 5

"If I add 3 to a number and square the sum, the result is greater than the square of the original number".

Let the original number be x in which case the above statement claims that,

$$(3 + x)^2 > x^2$$

from which is deduced that, $$9 + 6x + x^2 > x^2$$

$$9 + 6x > 0$$

$$6x > -9$$

$$x > -\frac{3}{2}$$

The statement would be true if x was a natural number.

However, the question does not say this so if x was a negative integer less than negative 1 then the statement would be false, for example if $x = -2$

[2 marks]

Answer 6

One approach is to use a bit of common sense to pick a suitable value for x such as $x = -\frac{5}{2}$ for then $|2x + 1| = \left|2 \times \left(-\frac{5}{2}\right) + 1\right|$

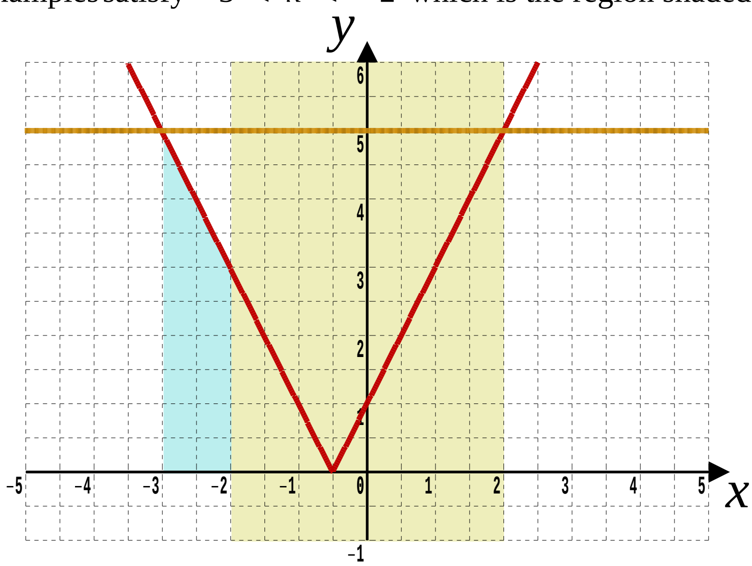
$$= |-4|$$

$$= 4$$

which is less than 5 as required but with $|x|$ greater than 2

The bigger picture is given by the following graph where $y = |2x + 1|$ is in red, $y = 5$ in gold, such that $|2x + 1| \leqslant 5$ is the part of the red graph that's under or on the gold graph. $|x| \leqslant 2$ is the region shaded yellow.

Counterexamples satisfy $-3 \leqslant x < -2$ which is the region shaded blue.



[2 marks]

Answer 7

For $n = 1$: $n^2 - n + 3 = 3$ prime

For $n = 2$: $n^2 - n + 3 = 5$ prime

For $n = 3$: $n^2 - n + 3 = 9$ not prime

$\therefore$ $n = 3$ is the smallest counterexample (there are infinitely more...)

[2 marks]

Answer 8

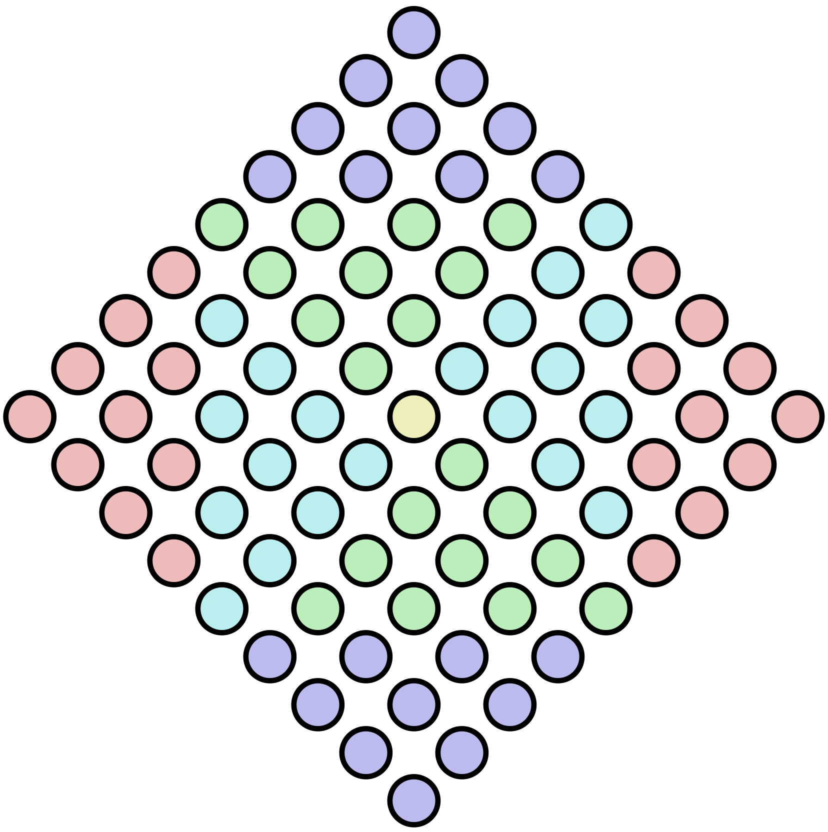
"For all $n \in \mathbb{N}$, $8T_n + 1$ is always a square number"

Algebraic Proof

$$\begin{aligned} 8T_n + 1 &= 8 \times \frac{1}{2} n(n+1) + 1 \\ &= 4n(n+1) + 1 \\ &= 4n^2 + 4n + 1 \\ &= (2n+1)^2 \\ &= m^2 \qquad \text{where } m = 2n+1 \end{aligned}$$

which is a square number, as required. $\square$

Geometric Proof (without words)



[5 marks]

Answer 9

"The difference between the squares of two consecutive triangular numbers is always a cube"

Proof

$$\left(\frac{1}{2} n (n + 1) \right)^2 - \left(\frac{1}{2} (n - 1) n \right)^2 = \frac{1}{4} n^2 \Big((n + 1)^2 - (n + 1)^2 \Big)$$

$$= \frac{1}{4} n^2 \left(n^2 + 2n + 1 - n^2 - 2n - 1 \right)$$

$$= \frac{1}{4} n^2 \left(4n \right)$$

$$= n^3$$

which is a cube number, as required. ☐

[5 marks]

Answer 10

(i) As m^2 is divisible by 4, $m^2 = 4n$ for some $m \in \mathbb{N}$
Taking positive square roots, $m = 2\sqrt{n}$ and so m will only be divisible by 4 if n contains a factor of 4.
Thus, although sometimes true, in general this conjecture is false.

[3 marks]

(ii) For $p, q \in \mathbb{N}$, the cube of any odd integer p plus the square of any even integer q is always odd.

Proof

Let $p = 2m + 1$ be an odd integer for some $m \in \mathbb{Z}$

Let $q = 2n$ be an even integer for some $n \in \mathbb{Z}$

$$p^3 + q^2 = (2m + 1)^3 + (2n)^2$$

$$= 8m^3 + 12m^2 + 6m + 1 + 4n^2$$

$$= \left(8m^3 + 12m^2 + 6m + 4n^2 \right) + 1$$

$$= 2\left(4m^3 + 6m^2 + 3m + 2n^2 \right) + 1$$

$$= 2w + 1 \quad \text{for } w = 4m^3 + 6m^2 + 3m + 2n^2$$

which is an odd number, as required. ☐

[3 marks]