4.5 Answers

4.5.1 Solution to 4.2 Non-Consecutive Odd Number Sequence

(i)

p	1	2	3	4	...	m	...	n	...
T_p	1	3	5	7	...	$2m-1$	...	$2n-1$	...

[2 marks]

(ii) Let the two odd numbers be $(2m-1)$ and $(2n-1)$ for some $m, n \in \mathbb{N}$, in which case,

$$\begin{aligned}(2m-1)(2n-1) &= 4mn - 2m - 2n + 1 \\ &= 2(2mn - m - n) + 1\end{aligned}$$

which is one more than a multiple of 2, (an odd number) □

[3 marks]

4.5.2 Solution to 4.3 A Less Obvious Use of Odd and Even

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^3 + 2 &= (2k)^3 + 2 \\ &= 8k^3 + 2\end{aligned}$$

Clearly, 2 more than a multiple of 8 is not a multiple of 8 so

So $n^3 + 2$ does not divide by 8 when n is even.

- Secondly, suppose n is odd in which case $n = 2k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^3 + 2 &= (2k+1)^3 + 2 \\ &= \quad 1 \times (2k)^3 \times 1^0 \\ &\quad + 3 \times (2k)^2 \times 1^1 \\ &\quad + 3 \times (2k)^1 \times 1^2 \\ &\quad + 1 \times (2k)^0 \times 1^3 + 2 \qquad \text{(Using the binomial theorem)} \\ &= 8k^3 + 12k^2 + 6k + 3 \\ &= 2\left(4k^3 + 6k^2 + 3k + 1\right) + 1\end{aligned}$$

This is one more than a multiple of 2, an odd number, and not a multiple of 8.

So $n^3 + 2$ does not divide by 8 when n is odd.

Overall, we deduce that $n^3 + 2$ is not divisible by 8 for any natural number n.

[4 marks]

4.5.3 Solutions to 4.4 Exercise

Answer 1

Let the two odd numbers be $(2m - 1)$ and $(2n - 1)$ for some $m, n \in \mathbb{N}$, in which case,

$$\begin{aligned}(2m - 1) + (2n - 1) &= 2m + 2n - 2 \\ &= 2(m + n - 1)\end{aligned}$$

which is a multiple of 2, in other words an even number. ☐

[2 marks]

Answer 2

Let the two even numbers be $(2m)$ and $(2n)$ for some $m, n \in \mathbb{N}$, in which case,

$$(2m)(2n) = 4mn$$

which is a multiple of 4 and therefore divisible by 4, as we were asked to show. ☐

[2 marks]

Answer 3

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^2 + 5 &= (2k)^2 + 5 \\ &= 4k^2 + 4 + 1 \\ &= 4\left(k^2 + 1\right) + 1\end{aligned}$$

  which is 1 more than a multiple of 4

- Secondly, suppose n is odd in which case $n = 2k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^2 + 5 &= (2k + 1)^2 + 5 \\ &= 4k^2 + 4k + 1 + 4 + 1 \\ &= 4\left(k^2 + k + 1\right) + 2\end{aligned}$$

  which is 2 more than a multiple of 4

Overall, we deduce that $n^2 + 5$ is either 1 or 2 more than a multiple of 4 ☐

[4 marks]

Answer 4

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 + 2 &= (2k)^2 + 2 \\ &= 4k^2 + 2 \end{aligned}$$

  which is 2 more than a multiple of 4 and so not divisible by 4

- Secondly, suppose n is odd in which case $n = 2k+1$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 + 2 &= (2k + 1)^2 + 2 \\ &= 4k^2 + 4k + 1 + 2 \\ &= 4\left(k^2 + k\right) + 3 \end{aligned}$$

  which is 3 more than a multiple of 4 and so not divisible by 4

Overall, we deduce that $n^2 + 2$ is never divisible 4 □

[4 marks]

Answer 5

Either the natural number n is a multiple of 3 or one more than a multiple of 3 or 2 more than a multiple of 3.

- Firstly, suppose n is a multiple of 3 in which case $n = 3k$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 + n + 2 &= (3k)^2 + (3k) + 2 \\ &= 9k^2 + 3k + 2 \\ &= 3\left(3k^2 + k\right) + 2 \end{aligned}$$

  which is 2 more than a multiple of 3 and so not divisible by 3

- Secondly, suppose n is one more than a multiple of 3 in which case $n = 3k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 + n + 2 &= (3k + 1)^2 + (3k + 1) + 2 \\ &= 9k^2 + 6k + 1 + 3k + 1 + 2 \\ &= 3\left(3k^2 + 3k + 1\right) + 1 \end{aligned}$$

  which is 1 more than a multiple of 3 and so not divisible by 3

- Thirdly, suppose n is two more than a multiple of 3 in which case $n = 3k + 2$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 + n + 2 &= (3k + 2)^2 + (3k + 2) + 2 \\ &= 9k^2 + 12k + 4 + 3k + 2 + 2 \\ &= 3\left(3k^2 + 5k + 2\right) + 2 \end{aligned}$$

  which is 2 more than a multiple of 3 and so not divisible by 3

Overall, we deduce that $n^2 + n + 2$ is never divisible 3 □

[4 marks]

Answer 6

Either the natural number n is a multiple of 3 or one more than a multiple of 3 or 2 more than a multiple of 3.

- Firstly, suppose n is a multiple of 3 in which case $n = 3k$ for some $k \in \mathbb{N}$.

$$n^2 = (3k)^2$$
$$= 9k^2$$

which is a multiple of 3

- Secondly, suppose n is one more than a multiple of 3 in which case $n = 3k + 1$ for some $k \in \mathbb{N}$.

$$n^2 = (3k + 1)^2$$
$$= 9k^2 + 6k + 1$$
$$= 3\left(3k^2 + 2k\right) + 1$$

which is one more than a multiple of 3

- Thirdly, suppose n is two more than a multiple of 3 in which case $n = 3k + 2$ for some $k \in \mathbb{N}$.

$$n^2 = (3k + 2)^2$$
$$= 9k^2 + 12k + 4$$
$$= 3\left(3k^2 + 4k + 1\right) + 1$$

which is one more than a multiple of 3

Overall, it has been shown that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3, as required. □

[4 marks]

Answer 7

To show $3n^2 - 1$ is a not perfect square show that it cannot be written in the form $3k$ or $3k + 1$, using the result from question 6.

In other words, show $3n^2 - 1$ is of the form $3k + 2$

Let $n = m + 1$, for some integer m, in which case,

$$3n^2 - 1 = 3(m + 1)^2 - 1$$
$$= 3m^2 + 6m + 3 - 1$$
$$= 3\left(m^2 + 2m\right) + 2$$
$$= 3k + 2$$

Thus, $3n^2 - 1$ cannot ever be a perfect square for any $n \in \mathbb{N}$. □

[4 marks]

Answer 8

(i) From being told that we have a perfect square that is even we can write that,

$$n^2 = 2k \qquad \text{for some integer } k$$

From this we have that,

$$n = \sqrt{2k} \qquad \text{ignoring the negative square root an } n \in \mathbb{N}$$

The $\sqrt{2k}$ must be a natural number and so k must be of the form

$2\,m^2$, for some $m \in \mathbb{N}$.

Thus, $\quad n^2 = 2 \times 2\,m^2$

$\qquad\qquad = 4\,m^2$

showing that n^2 is divisible by 4, as requested. $\qquad \square$

[3 marks]

(ii) Let the two odd numbers be $x = (2m - 1)$ and $y = (2n - 1)$ for some $m, n \in \mathbb{N}$, in which case,

$$\begin{aligned} x^2 + y^2 &= (2m - 1)^2 + (2n - 1)^2 \\ &= 4\,m^2 - 4m + 1 + 4\,n^2 - 4n + 1 \\ &= 4\left(m^2 - m + n^2 - n\right) + 2 \end{aligned}$$

Now. we know that an odd number multiplied by an odd number is an odd number (proven in the lesson's introduction) so as *x* and *y* are odd, we have that their squares are also odd.

When these squares are added together, we're adding an odd number to an odd number which is an even number (proven in question 1).

So the sum of the squares of *x* and *y* is even.

For this even number to be a perfect square it must be divisible by 4 from part (i) of this question.

But we have just shown that the sum of the squares of *x* and *y* is 2 more than a multiple of 4 which cannot be a multiple of 4

$\therefore\; x^2 + y^2$ can not be a perfect square if both *x* and *y* are odd.

[6 marks]

Answer 9

One way to do this question is to show that $n^3 + 5n$ is divisible by 2 and also by 3

To show $n^3 + 5n$ is divisible by 2

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^3 + 5n &= (2k)^3 + 5(2k) \\ &= 8k^3 + 10k \\ &= 2\left(4k^3 + 5k\right) \end{aligned}$$

- Secondly, suppose n is odd in which case $n = 2k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^3 + 5n &= (2k+1)^3 + 5(2k+1) \\ &= 8k^3 + 12k^2 + 6k + 1 + 10k + 5 \\ &= 2\left(4k^3 + 6k^2 + 8k + 3\right) \end{aligned}$$

It is deduced that $n^3 + 5n$ is always divisible by 2 for any $n \in \mathbb{N}$.

To show $n^3 + 5n$ is divisible by 3

Any natural number n must be of one of the forms $3k$, $3k + 1$, or $3k + 2$, where $k \in \mathbb{N}$

- Case $n = 3k$: $$\begin{aligned} n^3 + 5n &= (3k)^3 + 5(3k) \\ &= 27k^3 + 15k \\ &= 3\left(9k^3 + 5k\right) \end{aligned}$$
- Case $n = 3k + 1$: $$\begin{aligned} n^3 + 5n &= (3k+1)^3 + 5(3k+1) \\ &= 27k^3 + 27k^2 + 9k + 1 + 15k + 5 \\ &= 3\left(9k^3 + 9k^2 + 8k + 2\right) \end{aligned}$$
- Case $n = 3k + 2$: $$\begin{aligned} n^3 + 5n &= (3k+2)^3 + 5(3k+2) \\ &= 27k^3 + 54k^2 + 36k + 8 + 15k + 10 \\ &= 3\left(9k^3 + 18k^2 + 17k + 6\right) \end{aligned}$$

It is deduced that $n^3 + 5n$ is always divisible by 3 for any $n \in \mathbb{N}$.

Overall, we deduce that $n^3 + 5n$ is divisible by 6 $\square$

[7 marks]