5.3 Answers

5.3.1 Solution to 5.1.2 Example #2

Use proof by contradiction to show that if n^2 is even, then n must be even.

Proof

Assume the opposite is true, that there exists a number n such that n^2 is even but n is odd in which case we can write that $n = 2k + 1$ for some $k \in \mathbb{Z}$.

In consequence,

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2\left(2k^2 + 2k\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 2, in other words, an odd number.

This contradicts the assumption that n^2 is even.

Therefore if n^2 is even, n must also be even. $\square$

[3 marks]

5.3.2 Solutions to 5.2 Exercise

Answer 1

Following the hint, assume that the opposite is true, that for some $x, y \in \mathbb{N}$, $x \neq y$, that,

$$\frac{x}{y} + \frac{y}{x} \leqslant 2$$

Multiply through by xy (which we know is positive)

$$\begin{aligned} x^2 + y^2 &\leqslant 2xy \\ x^2 - 2xy + y^2 &\leqslant 0 \\ (x - y)^2 &\leqslant 0 \end{aligned}$$

But this cannot be a true statement as anything squared is greater than zero.

Nor can it equal zero because we are told that $x \neq y$.

Thus, it has been proven that for $x, y \in \mathbb{N}$, $x \neq y$,

$$\frac{x}{y} + \frac{y}{x} > 2 \qquad \square$$

[3 marks]

Answer 2

Assume that the opposite is true, that for $x \in \mathbb{R}$, $0 < x < 1$,

$$\frac{1}{x(1-x)} < 4$$

From $0 < x < 1$ it follows that $x > 0$ and $(1 - x) > 0$ and so $x(1-x) > 0$ which means that it's OK to multiply through by $x(1-x)$ and we know not to reverse the inequality.

$$1 < 4x(1-x)$$

$$1 < 4x - 4x^2$$

$$4x^2 - 4x + 1 < 0$$

$$(2x-1)^2 < 0$$

But this cannot be a true statement as anything squared is greater than zero.

Thus, it has been proven that for $x \in \mathbb{R}$, $0 < x < 1$,

$$\frac{1}{x(1-x)} \geqslant 4 \qquad \square$$

[4 marks]

Answer 3

Assume the opposite is true, integers a and b exist for which $10a + 15b = 17$

Divide both sides by the highest common factor of 10 and 15, that is, 5 to get,

$$2a + 3b = \frac{17}{5}$$

But this clearly cannot be, as any combination of integers that are multiplied, added and subtracted can only yield an integer answer, not a fraction.

Therefore, the original assumption, that integers a and b exist, must be false.

And so, there exist no integers a and b for which $10a + 15b = 17$ $\square$

[3 marks]

Answer 4

(i) Assume the opposite is true, that p and q are integers such that,

pq is even but that both p and q are odd.

This would mean that we could write p and q in the forms,

$p = 2n + 1$ and $q = 2m + 1$ for some $p, q \in \mathbb{Z}$

Then,

$$\begin{aligned} pq &= (2n+1)(2m+1) \\ &= 4nm + 2n + 2m + 1 \\ &= 2(2nm + n + m) + 1 \end{aligned}$$

But this shows that pq is odd and not even.

Therefore, the assumption that p and q are both odd cannot be correct.

At least one of p or q must be even.

[3 marks]

(ii)

$$(x + y)^2 < 9x^2 + y^2$$
$$x^2 + 2xy + y^2 < 9x^2 + y^2$$
$$2xy < 8x^2$$
$$xy < 4x^2$$

Divide both sides by x and reverse the inequality as $x < 0$

$$y > 4x \qquad \square$$

[2 marks]

Answer 5

(i) The condition that $n \in \mathbb{N}$, $n \leqslant 4$, means that $n \in \{1, 2, 3, 4\}$

Working through those possibilities,

$n = 1 : (1 + 1)^3 = 2^3 = 8 > 3^1$

$n = 2 : (2 + 1)^3 = 3^3 = 27 > 3^2$

$n = 3 : (3 + 1)^3 = 4^3 = 64 > 3^3$

$n = 4 : (4 + 1)^3 = 5^3 = 125 > 3^4$

In all cases it has been shown that $(n + 1)^3 > 3^n$, as required. $\square$

[2 marks]

(ii) Suppose that the opposite were true, that $m^3 + 5$ were odd and that m is odd. Writing $m = 2k + 1$ it then follows that,

$$\begin{aligned} m^3 + 5 &= (2k + 1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 2\left(4k^3 + 6k^2 + 3k + 3\right) \end{aligned}$$

But this shows that $m^3 + 5$ is even and not odd.
Therefore, the assumption that m is odd cannot be correct.
It is thus the case that m must be even.

[4 marks]

Answer 6

Assume that there are positive integers p and q such that,

$$4p^2 - q^2 = 25$$

in which case, using the difference of two squares,

$$(2p + q)(2p - q) = 25$$

Now, each of $(2p + q)$ and $(2p - q)$ must be an integer factor of 25

One possibility is that each equals 5, for then we have $5 \times 5 = 25$

$$\begin{array}{rl} 2p + q &= 5 \\ \text{ADD } 2p - q &= 5 \\ \hline 4p &= 10 \\ p &= \dfrac{5}{2} \end{array}$$

But this contradicts the requirement that p is an integer.

Another possibility is that the one is 25 and the other 1, for then $25 \times 1 = 25$

$$\begin{array}{rl} 2p + q &= 25 \\ \text{ADD } 2p - q &= 1 \\ \hline 4p &= 26 \\ p &= \dfrac{13}{2} \end{array} \qquad \begin{array}{rl} 2p + q &= 1 \\ \text{ADD } 2p - q &= 25 \\ \hline 4p &= 26 \\ p &= \dfrac{13}{2} \end{array}$$

But this contradicts the requirement that p is an integer.

Having exhausted the possibilities, and always obtained a contradiction it must be the case that there are no positive integers p and q such that,

$$4p^2 - q^2 = 25 \qquad \square$$

[4 marks]

Answer 7

(i) Assume that the opposite is true, that n^2 is a multiple of 3, and n is not a multiple of 3, in which case either $n = 3k + 1$ or $n = 3k + 2$

If $n = 3k + 1$ then,

$$\begin{aligned} n^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3\left(3k^2 + 2k\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 3 and so not a multiple of 3

If $n = 3k + 2$ then,

$$\begin{aligned} n^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3\left(3k^2 + 4k + 1\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 3 and so not a multiple of 3

In both cases the assumption that n is not a multiple of 3 has given rise to a contradiction and so the assumption must be false.
Therefore, n is a multiple of 3. □

[3 marks]

(ii) Assume the opposite; that $\sqrt{3}$ is a rational number.
Then, by the definition of what a rational number is,

$$\sqrt{3} = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z}, q \neq 0$$

Furthermore, assume that this fraction cannot be reduced further.
That is p and q have no factors in common.
(In other words, they are coprime with $hcf\,\{p, q\} = 1$)

In consequence, $$3 = \frac{p^2}{q^2}$$

$$p^2 = 3\,q^2$$

The 3 on the RHS tells us that it is a multiple of 3.
Therefore the LHS must be a multiple of 3.
From part (i), if p^2 is a multiple of 3 then so too is p
Therefore p can be expressed in the form $3a$ for some integer a.
So we now have that,

$$(3a)^2 = 3\,q^2$$

$$9\,a^2 = 3\,q^2$$

$$3\,a^2 = q^2$$

The 3 on the LHS tells us that it is a multiple of 3.
Therefore the RHS must be a multiple of 3.
Again, from part (i), if q^2 is a multiple of 3, then so too is q
If p and q are both multiples of 3, they will have a common factor of 3.
This contradicts the statement that p and q have no common factors.
The inescapable conclusion is that $\sqrt{3}$ is an irrational number. □

[6 marks]

Answer 8

We want to prove that $x + y$ is irrational given that x is irrational and y is rational.
For the sake of contradiction, let $x + y$ be rational.
Call this rational sum z such that $x + y = z$ and hence $x = z - y$
We know (or could easily prove) that, since both z and y are rational, their difference $z - y$ must also be a rational number.
However, if $z - y$ is rational, then so too must be x (since $x = z - y$).
This directly contradicts our starting premise that x is irrational.
As assuming $x + y$ is rational leads to a contradiction, that assumption must be false.
Therefore, $x + y$ must be irrational. □

[6 marks]