Answer 1

Marcus may have forgotten about the odd number 1.

$1 \times \text{Even} \square = \text{Even} \square$ For example $1 \times 4 = 4$

This is part of a more general result where,

$\text{Odd} \square \times \text{Even} \square = \text{Even} \square$ For example, $9 \times 4 = 36$

[3 marks]

Answer 2

Let the odd number be $2k - 1$ for some $k \in \mathbb{N}$.

Squaring this odd number gives,

$$\begin{aligned}(2k - 1)^2 &= 4k^2 - 4k + 1 \\ &= 4\left(k^2 - k\right) + 1 \\ &= 4\left(k(k - 1)\right) + 1\end{aligned}$$

The $k(k - 1)$ piece of this represents two consecutive integers.

One of these must be even.

Therefore $k(k - 1) = 2w$ for some $w \in \mathbb{N}$.

$$\begin{aligned}(2k - 1)^2 &= 4(2w) + 1 \\ &= 8w + 1\end{aligned}$$

Which is always one more than a multiple of 8, as expected. $\square$

[5 marks]

Answer 3

(i) Let the two consecutive even numbers be, $2n$ and $2n + 2$ in which case,

$$\begin{aligned}(2n + 2)^2 - (2n)^2 &= 4(n + 1)^2 - 4n^2 \\ &= 4\left((n + 1)^2 - n^2\right)\end{aligned}$$

which is a multiple of 4 and so divisible by 4 $\square$

[3 marks]

(ii) It is true for any two consecutive odd numbers.

Proof

Let the two consecutive odd numbers be, $2n - 1$ and $2n + 1$ in which case,

$$\begin{aligned}(2n + 1)^2 - (2n - 1)^2 &= 4n^2 + 4n + 1 - 4n^2 + 4n - 1 \\ &= 8n\end{aligned}$$

which is a multiple of 8 and so divisible by 4 $\square$

[3 marks]

Answer 4

(i) $3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 135$

[1 mark]

(ii) Let the five consecutive integers and their sum be,

$$(n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 5n$$

which is a multiple of five and so divisible by 5

[2 marks]

(iii) Working out the mean of the squares...

$$\frac{(n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2}{5}$$

$$= \frac{n^2 - 4n + 4 + n^2 - 2n + 1 + n^2 + n^2 + 2n + 1 + n^2 + 4n + 4}{5}$$

$$= \frac{5n^2 + 10}{5}$$

$$= n^2 + 2$$

The median of the five consecutive integers is n and so it has been shown that if five consecutive integers are squared, the mean of the squares does indeed exceed the square of there median by 2 □

[4 marks]

(iv) Hence, $\dfrac{98^2 + 99^2 + 100^2 + 101^2 + 102^2}{5} = 100^2 + 2$

$$= 10002$$

[2 marks]

Answer 5

(a) Assume that the opposite is true, that for all positive values of a and b

$$\frac{4a}{b} + \frac{b}{a} < 4$$

Multiply through by ab

(which we know is positive as so does not reverse the inequality)

$$4a^2 + b^2 < 4ab$$

$$4a^2 - 4ab + b^2 < 0$$

$$(2a - b)^2 < 0$$

But this cannot be a true statement because anything squared is greater than or equal to zero. Via this contradiction it has been proven that, for all positive values of a and b

$$\frac{4a}{b} + \frac{b}{a} \geqslant 4$$ □

[4 marks]

(b) Let $a = -1$ and $b = 1$ in which case,

$$\frac{4a}{b} + \frac{b}{a} = -4 - 1$$
$$= -5$$

which fails to be greater than or equal to 4
(Any suitable counterexample will suffice, and there are many, easily found)

[1 mark]

Answer 6

$abcabc$ represents $100000a + 10000b + 1000c + 100a + 10b + c$

$$= 100100a + 10010b + 1001c$$
$$= 1001(100a + 10b + c)$$
$$= 7 \times 11 \times 13(100a + 10b + c)$$

Thus, $abcabc$ is divisible by 7, by 11 and by 13 ☐

[5 marks]

Answer 7

(i) $2331 \Rightarrow 233 - 2 = 231$

$\Rightarrow 23 - 2 = 21$ which is obviously divisible by 7

$\therefore$ 2331 is divisible by 7

[2 marks]

(ii) $515011 \Rightarrow 51501 - 2 = 51499$

$\Rightarrow 5149 - 18 = 5131$

$\Rightarrow 513 - 2 = 511$

$\Rightarrow 51 - 2 = 49$ which is obviously divisible by 7

$\therefore$ 515011 is divisible by 7

[2 marks]

(iii) Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits.
When aR is divisible by 7,

$aR = 10a + R$ is divisible by 7

$= 3a + 7a + R$ is divisible by 7

$= 3a + 7a + R - 7R$ is divisible by 7

$= 3a - 6R$ is divisible by 7

$= 3(a - 2R)$ is divisible by 7

As 3 is not divisible by 7, $(a - 2R)$ must be. ☐

[7 marks]

Answer 8

(i) True

Proof

If n is a multiple of 9 then it can be expressed in the form $9m$, where $m \in \mathbb{N}$

$$\text{Then, } n^2 = (9m)^2$$
$$= 9\left(9m^2\right)$$

which shows that $\sqrt{n^2}$ is a multiple of 9 $\square$

[3 marks]

(ii) False

Counterexample

If $n^2 = 9$ then n^2 is a multiple of 9 but $n = 3$ is not.

[3 marks]