7.2 Answers

Answer 1

Some examples of counterexamples are given below.
To answer the question, only one counterexample is required.

$1 \times 24 + 1 = 25 = 5^2$

$2 \times 24 + 1 = 49 = 7^2$

$3 \times 24 + 1 = 73$ which is not a square <== counterexample

$4 \times 24 + 1 = 97$ which is not a square <== counterexample

$5 \times 24 + 1 = 121 = 11^2$

$6 \times 24 + 1 = 145$ which is not a square <== counterexample

$7 \times 24 + 1 = 169 = 13^2$

15 is the next number that is not a counterexample

[2 marks]

Answer 2

Let the two consecutive integers be n and $(n + 1)$ in which case,

$$n^2 + (n + 1)^2 = n^2 + n^2 + 2n + 1$$
$$= 2\left(n^2 + n\right) + 1$$

which is one more than a multiple of 2, an odd number

$\therefore$ The sum of the squares of any two consecutive integers is an odd number. ☐

[4 marks]

Answer 3

ababab represents $100000a + 10000b + 1000a + 100b + 10a + b$

$$= 101010a + 10101b$$
$$= 10101(10a + b)$$
$$= 3 \times 7 \times 13 \times 37(10a + b)$$

Thus, *ababab* is divisible by 3, by 7, by 13 and by 37 ☐

[4 marks]

Answer 4

Let the two consecutive even numbers be $2m$ and $2m + 2$ for some $m \in \mathbb{N}$.

The product is given by,

$$\begin{aligned}(2m)(2m + 2) &= 4m^2 + 4m \\ &= 4\left(m^2 + m\right) \\ &= 4\left(m(m + 1)\right)\end{aligned}$$

The $m(m + 1)$ piece of this represents two consecutive integers.

One of these must be even.

Therefore $m(m + 1) = 2w$ for some $w \in \mathbb{N}$.

$$\begin{aligned}(2m)(2m + 2) &= 4(2w) \\ &= 8w\end{aligned}$$

Which is always a multiple of 8, as expected.
It is proven the product of two consecutive even numbers is divisible by 8 □

[6 marks]

Answer 5

(i) $T_n = \dfrac{n(n + 1)}{2}$

$\therefore\ T_7 = 28,\ T_8 = 36$ and so $T_7 + T_8 = 64$

[2 marks]

(ii) $T_{n+1} = \dfrac{(n + 1)(n + 2)}{2}$

[2 marks]

(iii) The sum of any two consecutive triangular numbers is given by,

$$\begin{aligned}T_n + T_{n+1} &= \frac{n(n + 1)}{2} + \frac{(n + 1)(n + 2)}{2} \\ &= \frac{(n + 1)}{2}\left(n + n + 2\right) \\ &= \frac{(n + 1)(2n + 2)}{2} \\ &= (n + 1)(n + 1) \\ &= (n + 1)^2\end{aligned}$$

which is a square number. □

[4 marks]

Answer 6

(i) $3531 \Rightarrow 353 - 1 = 352$

$\Rightarrow 35 - 2 = 33$ which is obviously divisible by 11

$\therefore$ 3531 is divisible by 11

[2 marks]

(ii) $1984422 \Rightarrow 198442 - 2 = 198440$

$\Rightarrow 19844 - 0 = 19844$

$\Rightarrow 1984 - 4 = 1980$

$\Rightarrow 198 - 0 = 198$

$\Rightarrow 19 - 8 = 11$ which is obviously divisible by 11

$\therefore$ 1984422 is divisible by 11

[2 marks]

(iii) Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits. When aR is divisible by 11,

$$\begin{aligned} aR &= 10a + R && \text{is divisible by 11} \\ &= 10a + R - 11R && \text{is divisible by 11} \\ &= 10a - 10R && \text{is divisible by 11} \\ &= 10(a - R) && \text{is divisible by 11} \end{aligned}$$

As 10 is not divisible by 11, $(a - R)$ must be. ☐

[7 marks]

Answer 7

(i) Assume that the opposite is true, that n^3 is even, and n is not even, in which case $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Then
$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2\left(4k^3 + 6k^2 + 3k\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 2 and so is an odd number.

The assumption that n is odd has given rise to a contradiction and so the assumption must be false.

Therefore, n is even. □

[3 marks]

(ii) Assume the opposite; that $\sqrt[3]{2}$ is a rational number.

Then, by the definition of what a rational number is,

$$\sqrt[3]{2} = \frac{p}{q} \text{ for some } p,\, q \in \mathbb{Z},\, q \neq 0$$

Furthermore, assume that this fraction cannot be reduced further.

That is p and q have no factors in common.

(In other words, they are coprime with $hcf\,\{p,\, q\} = 1$)

In consequence, $$2 = \frac{p^3}{q^3}$$

$$p^3 = 2q^3$$

The 2 on the RHS tells us that it is a multiple of 2.

Therefore the LHS must be a multiple of 2.

From part (i), if p^3 is a multiple of 2 then so too is p

Therefore p can be expressed in the form $2a$ for some integer a.

So we now have that,

$$\begin{aligned} (2a)^3 &= 2q^3 \\ 8a^3 &= 2q^3 \\ 4a^3 &= q^3 \end{aligned}$$

The 4 on the LHS tells us that it is a multiple of 2.

Therefore the RHS must be a multiple of 2.

Again, from part (i), if q^3 is a multiple of 2, then so too is q

If p and q are both multiples of 2, they will have a common factor of 2.

This contradicts the statement that p and q have no common factors.

The inescapable conclusion is that $\sqrt[3]{2}$ is an irrational number. □

[6 marks]

Answer 8

For the pattern to always hold, it needs to be proven that, for $n \in \mathbb{N}$,

$$n^2 + (n+1)^2 + (n(n+1))^2 = m^2 \text{ for some } m \in \mathbb{N}.$$

$$\begin{aligned}
\text{LHS} &= n^2 + (n+1)^2 + (n(n+1))^2 \\
&= n^2 + n^2 + 2n + 1 + n^2(n+1)^2 \\
&= 2n^2 + 2n + 1 + n^2\left(n^2 + 2n + 1\right) \\
&= 2n^2 + 2n + 1 + n^4 + 2n^3 + n^2 \\
&= n^4 + 2n^3 + 3n^2 + 2n + 1 \\
&= \left(n^2 + n + 1\right)^2 \\
&= m^2 \quad \text{where } m = n^2 + n + 1 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[6 marks]