A-Level

~ Year 2 ~

Pure Mathematics

PROOF

~ The Art of Absolute Certainty ~

	1	Odd	Square Cube	Triangular
Prime	2	Even		
Prime	3	Odd		Triangular
Composite	4	Even	Square	
Prime	5	Odd		
Composite	6	Even		Triangular
Prime	7	Odd		
Composite	8	Even	Cube	
Composite	9	Odd	Square	
Composite	10	Even		Triangular
Prime	11	Odd		
Composite	12	Even		
Prime	13	Odd		
Composite	14	Even		
Composite	15	Odd		Triangular
Composite	16	Even	Square	
Prime	17	Odd		
	...			

P R O O F
~ The Art of Absolute Certainty ~

Lesson 1

1.1 Consecutive Numbers

Our first foray into proof is going to be mostly working with the natural numbers, $\mathbb{N}$, which are often referred to as "The Counting Numbers",

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, ...\}$$

Notice that zero is not included.

1.2 A Consecutive Sum Example

Three consecutive natural numbers have a sum of 84.
What are the three consecutive natural numbers ?

Method 1 : Form and Solve an Equation

[2 marks]

Method 2 : Make a Simplifying Assumption

[2 marks]

1.3 A Consecutive Product Example

Three consecutive natural numbers have a product of 85140.
What are the three consecutive natural numbers ?

Method 1 : Form and Solve an Equation

[2 marks]

Method 2 : Make a Simplifying Assumption

[2 marks]

1.4 Consecutive Odd Numbers

Complete the following table for the odd number sequence

p	1	2	3	4	...	n	$n+1$	$n+2$	...
T_p	1	3			...				...

[4 marks]

1.5 A First Proof

(a) Calculate each of the following,

(i) 1×3

(ii) 3×5

(iii) 5×7

(iv) 7×9

(v) 9×11

[2 marks]

(b) Prove that the product of two consecutive odd numbers is always one less than a square number.

[4 marks]

1.6 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40 marks

Question 1
Three consecutive natural numbers have a sum of 204
What are the three consecutive natural numbers ?

[2 marks]

Question 2
Three consecutive natural numbers have a product of 778596
What are the three consecutive natural numbers ?

[2 marks]

Question 3
Four consecutive natural numbers have a sum of 78
What are the four consecutive natural numbers ?

[2 marks]

Question 4
Five consecutive natural numbers have a product of 95040
What are the five consecutive natural numbers ?

[3 marks]

Question 5

(a) Calculate each of the following,

(i) $3^2 - 1^2$

(ii) $5^2 - 3^2$

(iii) $7^2 - 5^2$

(iv) $9^2 - 7^2$

(v) $11^2 - 9^2$

[3 marks]

(b) Prove that the difference between two consecutive odd numbers that have been squared is always divisible by 8

[5 marks]

Question 6

Prove that the sum of four consecutive odd numbers is always divisible by 8

[4 marks]

Question 7
Complete the following table for the even number sequence,

p	1	2	3	4	...	$n-1$	n	$n+1$	...
T_p	2	4			...				...

[5 marks]

Question 8
Three consecutive even numbers have a sum of 72.
What are the three consecutive even numbers ?

[4 marks]

Question 9
Prove that the sum of the squares of three consecutive even numbers is always divisible by 4

[5 marks]

Question 10

AS-Level Examination Question from November 2021, Paper 1, Q10 (Edexcel)

A student is investigating the following statement about natural numbers, $\mathbb{N}$,

“$n^3 - n$ is a multiple of 4”

(a) Prove, using algebra, that the statement is true for all odd numbers.

[4 marks]

(b) Use a counterexample to show that the statement is not always true.

[1 mark]

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1.7 Answers

1.7.1 Solutions to Teaching Examples

#1.2 Three consecutive natural numbers have a sum of 84.
What are the three consecutive natural numbers ?

Method 1 : Form and Solve an Equation

$$\begin{aligned}(n-1)+(n)+(n+1) &= 84\\ 3n &= 84\\ n &= 28\end{aligned}$$

$\therefore$ The three consecutive numbers are 27, 28, 29

[2 marks]

Method 2 : Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$\begin{aligned}x+x+x &= 84\\ 3x &= 84\\ x &= 28\end{aligned}$$

The numbers are around 28. Experimentation gives 27, 28, 29

[2 marks]

#1.3 Three consecutive natural numbers have a product of 85140.
What are the three consecutive natural numbers ?

Method 1 : Form and Solve an Equation

$$\begin{aligned}(n-1)(n)(n+1) &= 85140 &&\text{}\\ n\left(n^2-1\right) &= 85140 &&\text{difference of two squares}\\ n^3+0n^2-n-85140 &= 0 &&\text{this method in trouble !}\\ n &= 44 &&\text{using equation solver (cheating)}\end{aligned}$$

so the numbers are 43, 44, 45

[2 marks]

Method 2 : Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$\begin{aligned}x\times x\times x &= 85140\\ x^3 &= 85140\\ x &= \sqrt[3]{85140}\\ &= 43.99\end{aligned}$$

The numbers are around 44. Experimentation gives 43, 44, 45

[2 marks]

#1.4

p	1	2	3	4	...	n	$n + 1$	$n + 2$	...
T_p	1	3	5	7	...	$2n - 1$	$2n + 1$	$2n + 3$	...

[4 marks]

#1.5

(a) **(i)** $1 \times 3 = 3$

(ii) $3 \times 5 = 15$

(iii) $5 \times 7 = 35$

(iv) $7 \times 9 = 63$

(v) $9 \times 11 = 99$

[2 marks]

(b) Let the two consecutive odd numbers be $(2n - 1)$ and $(2n + 1)$

in which case $(2n - 1)(2n + 1) = (2n)^2 - 1^2$

$= m^2 - 1$ where $m = 2n$

which is one less than a square number. $\square$

[4 marks]

1.7.2 Solutions to 1.6 Exercise

Answer 1

Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x + x + x = 204$$

$$3x = 204$$

$$x = 68$$

The numbers are around 68. Experimentation gives 67, 68, 69

[2 marks]

Answer 2

Make a Simplifying Assumption
As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x \times x \times x = 778596$$

$$x^3 = 778596$$

$$x = \sqrt[3]{778596}$$

$$= 91.99$$

The numbers are around 92. Experimentation gives 91, 92, 93

[2 marks]

Answer 3

Make a Simplifying Assumption

As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$x + x + x + x = 78$$
$$4x = 78$$
$$x = 19.5$$

The numbers are around 19.5. Experimentation gives 18, 19, 20, 21

[2 marks]

Answer 4

Make a Simplifying Assumption

As the numbers are consecutive, assume that they are all of approximately the same value, x, in which case,

$$\begin{aligned} x \times x \times x \times x \times x &= 95040 \\ x^5 &= 95040 \\ x &= \sqrt[5]{95040} \\ &= 9.90 \end{aligned}$$

The numbers are around 9.9. Experimentation gives 8, 9, 10, 11, 12

[3 marks]

Answer 5

(a) **(i)** $3^2 - 1^2 = 8$

(ii) $5^2 - 3^2 = 16$

(iii) $7^2 - 5^2 = 24$

(iv) $9^2 - 7^2 = 32$

(v) $11^2 - 9^2 = 40$

[3 marks]

(b) Let the two consecutive odd numbers be $(2n - 1)$ and $(2n + 1)$ in which case,

$$\begin{aligned} (2n + 1)^2 - (2n - 1)^2 &= \left(4n^2 + 4n + 1\right) - \left(4n^2 - 4n + 1\right) \\ &= 4n^2 + 4n + 1 - 4n^2 + 4n - 1 \\ &= 8n \end{aligned}$$

which is divisible by 8 for all $n \in \mathbb{N}$ $\square$

[5 marks]

Answer 6

Let the four consecutive odd numbers be,

$$(2n - 3),\ (2n - 1),\ (2n + 1),\ (2n + 3)$$

in which case,

$$(2n - 3) + (2n - 1) + (2n + 1) + (2n + 3) = 8n$$

which is divisible by 8 for all $n \in \mathbb{N}$ ☐

[4 marks]

Answer 7

p	1	2	3	4	...	$n-1$	n	$n+1$	...
T_p	2	4	6	8	...	$2n-2$	$2n$	$2n+2$	...

[5 marks]

Answer 8

Let the three consecutive even numbers be,

$$(2n - 2),\ (2n),\ (2n + 2)$$

in which case,

$$\begin{aligned}(2n - 2) + (2n) + (2n + 2) &= 72 \\ 6n &= 72 \\ n &= 12\end{aligned}$$

The three consecutive even numbers are, 22, 24, 26

[4 marks]

Answer 9

Let the three consecutive even numbers be,

$$(2n - 2),\ (2n),\ (2n + 2)$$

in which case,

$$\begin{aligned}&(2n - 2)^2 + (2n)^2 + (2n + 2)^2 \\ &= 4n^2 - 8n + 4 + 4n^2 + 4n^2 + 8n + 4 \\ &= 12n^2 + 8 \\ &= 4\left(3n^2 + 2\right) \\ &= 4m \qquad \text{for some natural number } m\end{aligned}$$

which is divisible by 4 for all $m \in \mathbb{N}$ ☐

[5 marks]

Answer 10

(a) Let the odd number n be expressed in the form $2m + 1$ for $m \in \mathbb{N} \cup \{0\}$.
in which case,

$$\begin{aligned} n^3 - n &= n\left[n^2 - 1 \right] \\ &= (2m+1)\left[(2m+1)^2 - 1 \right] \\ &= (2m+1)\left[4m^2 + 4m + 1 - 1 \right] \\ &= (2m+1)\left[4m^2 + 4m \right] \\ &= 4(2m+1)\left[m^2 + m \right] \end{aligned}$$

which is divisible by 4 ☐

[4 marks]

(b) In part (a) the result has been proven true for all odd numbers.
So, in trying to show it's not always true, the counterexample has to be amongst the even numbers.
Let $n = 2$ in which case,

$$\begin{aligned} n^3 - n &= 2^3 - 2 \\ &= 8 - 2 \\ &= 6 \text{ which is not divisible by 4} \end{aligned}$$

$\therefore\ n^3 - n$ is not always divisible by 4 ☐

[1 mark]

Note : Counterexamples are of the form $4m + 2$ for $m \in \mathbb{N} \cup \{0\}$

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Lesson 2

2.1 Divisibility

In mathematics, seemingly simple results can have far reaching consequences. One such result is the following.

The Sum Divisor Rule
Let x, y and n be three natural numbers.
If both $(x + y)$ and x are divisible by n then y must also be divisible by n.

Proof
Let the sum by x and y be z where z will be another natural number.

$$x + y = z$$

Both x and z are divisible by n so I can write that $x = na$ and $z = nb$ for some natural numbers a and b, giving,

$$\begin{aligned} na + y &= nb \\ y &= nb - na \\ &= n(b - a) \end{aligned}$$

which shows that y is divisible by n as the rule claims ☐

2.2 Divisibility by 2

It is easy to tell if a large number is even or odd simply by looking at the rightmost digit, R, of the number.
If $R \in \{0, 2, 4, 6, 8\}$ then the large number is even.
If $R \in \{1, 3, 5, 7, 9\}$ then the large number is odd.

But why does this test work ?

Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits.

Then, $$aR = 10a + R$$

By the Sum Divisor Rule, when aR and $10a$ are both divisible by 2, so must R be.
In other words when $R \in \{0, 2, 4, 6 \text{ or } 8\}$, aR is even, otherwise odd.

Divisibility : How to count the number of sheep in a field.
Step 1 : Count the number of legs.
Step 2 : Divide by four.

2.3 Another Divisibility Test

The test for whether or not a number is divisible by 2 begs the question, “Are there similar, straightforward techniques to determine if a large number is divisible by other primes” ?

The answer is that there are, although they are not as well known.

Divisibility by 7

Subtract twice the last digit from the number formed by the remaining digits.

Repeat as necessary.

If, at any stage, a result is obtained that is obviously divisible by 7 then the original number is also divisible by 7

2.3.1 Example #1

Without using a calculator, show that 1603 is divisible by 7

[2 marks]

$1603 \;\Rightarrow\; 160 - 6 = 154$

$\Rightarrow\; 15 - 8 = 7$ which is obviously divisible by 7

$\therefore$ 1603 is divisible by 7

[2 marks]

2.3.2 Example #2

Without using a calculator, show that 504091 is divisible by 7

[2 marks]

2.3.3 Example #3

Provide a proof for the test for divisibility by 7

[7 marks]

2.4 Another Divisibility Rule

In the proof of the divisibility by 7 test, use was made of the Product Divisor Rule, which you may feel is obvious.

The Product Divisor Rule

Let x, y and n be three natural numbers.
If xy is divisible by n and x is not divisible by n then y must be divisible by n.

Proof

Let the product of x and y be z where z will be another natural number.

$$xy = z$$

As z is divisible by n, I can write that $z = na$ for some $a \in \mathbb{N}$.

$$xy = na$$

$$\frac{xy}{n} = a$$

As a cannot be a fraction, one of x or y must be writable as nb for some $b \in \mathbb{N}$ so that the n on the denominator can be cancelled with an n on the numerator.

But x is not divisible by n and so $x \neq nb$.

Therefore, $y = nb$ which shows y must be divisible by n as claimed. $\square$

2.5 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40 marks

Question 1

(i) Show that 24094 is divisible by 7 using the divisibility by 7 test.

[2 marks]

(ii) Explain why, after part (a), you immediately know that 24094 is divisible by 14

[2 marks]

Question 2
Young schoolchildren often think, mistakenly, that 91 is a prime number.
Show that 91 is divisible by 7 using the divisibility by 7 test.

[1 mark]

Question 3
The four digit numbers 5775, 3223 and 6996 are all of the form *abba* where a and b are natural numbers between 0 and 9 inclusive.
(That is $a, b \in \mathbb{N},\ 0 \leqslant a \leqslant 9,\ 0 \leqslant b \leqslant 9$)
Prove that all such four digit numbers of the form *abba* are divisible by 11.

Hint : $abba = 1000a + 100b + 10b + a$

[4 marks]

Question 4
Prove that the three digit number *aba* when added to the three digit number *bab* is always divisible by 37

[5 marks]

Question 5

(i) Prove that for any four digit number *abcd* to be divisible by 3, the sum of its digits must be divisible by 3

[5 marks]

(ii) Hence explain how you know that 7458 is divisible by 3

[2 marks]

Question 6

(i) Prove that, for any natural number of more than two digits to be divisible by 4, the two rightmost digits must be divisible by 4

[6 marks]

(ii) Hence explain how you know that 290547452 is divisible by 4

[2 marks]

Question 7

Divisibility by 13

Add four times the last digit to the number formed by the remaining digits.

Repeat as necessary.

If at any stage a result is obtained that is obviously divisible by 13 then the original number is divisible by 13

(i) Without using a calculator, show that 119483 is divisible by 13

[2 marks]

(ii) Without using a calculator, show that 504088 is divisible by 13

[2 marks]

(iii) Provide a proof for the test for divisibility by 13

[7 marks]

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2.6 Answers

2.6.1 Solution to 2.3.2 Teaching Example #2

Without using a calculator, show that 504091 is divisible by 7

$$\begin{aligned} 504091 &\Rightarrow 50409 - 2 = 50407 \\ &\Rightarrow 5040 - 14 = 5026 \\ &\Rightarrow 502 - 12 = 490 \text{ which is obviously divisible by 7} \\ &\therefore \ 504091 \text{ is divisible by 7} \end{aligned}$$

[2 marks]

2.6.2 Solution to 2.3.3 Teaching Example #3

Provide a proof for the test for divisibility by 7

Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits.
When aR is divisible by 7,

$$\begin{aligned} aR &= 10a + R && \text{is divisible by 7} \\ &= 3a + 7a + R && \text{is divisible by 7} \\ &= 3a + 7a + R - 7R && \text{is divisible by 7} \\ &= 3a - 6R && \text{is divisible by 7} \\ &= 3(a - 2R) && \text{is divisible by 7} \end{aligned}$$

As 3 is not divisible by 7, $(a - 2R)$ must be. ☐

[7 marks]

2.6.3 Solutions to 2.5 Exercise

Answer 1

(i)

$$\begin{aligned} 24094 &\Rightarrow 2409 - 8 = 2401 \\ &\Rightarrow 240 - 2 = 238 \\ &\Rightarrow 23 - 16 = 7 \text{ which is obviously divisible by 7} \\ &\therefore \ 24094 \text{ is divisible by 7} \end{aligned}$$

[2 marks]

(ii) As 24094 ends in a 4, 24094 is an even number.
Thus, 24094 is divisible by both 2 and by 7
That is, 24094 is divisible by 14

[2 marks]

Answer 2

$91 \;\Rightarrow\; 9 - 2 = 7$ which is obviously divisible by 7

$\therefore$ 91 is divisible by 7

[1 mark]

Answer 3

$$\begin{aligned} abba &= 1000a + 100b + 10b + a \\ &= 1001a + 110b \\ &= 11\,(\,91a + 10b\,) \end{aligned}$$

which is divisible by 11 ☐

[4 marks]

Answer 4

$$\begin{aligned} aba &= 100a + 10b + a &\Rightarrow\quad aba &= 101a + 10b \\ bab &= 100b + 10a + b &\Rightarrow\quad bab &= 10a + 101b \end{aligned}$$

$$\begin{aligned} aba + bab &= 111a + 111b \\ &= 37 \times 3\,(\,a + b\,) \end{aligned}$$

which is divisible by 37 ☐

[5 marks]

Answer 5

(i)

$$\begin{aligned} abcd &= 1000a + 100b + 10c + d \\ &= 999a + a + 99b + b + 9c + c + d \\ &= (\,999a + 99b + 9c\,) + (\,a + b + c + d\,) \\ &= 3\,(\,333a + 33b + 3c\,) + (\,a + b + c + d\,) \end{aligned}$$

Clearly, $3\,(\,333a + 33b + 3c\,)$ is divisible by 3

By the Sum Divisor Rule, when *abcd* and $3\,(\,333a + 33b + 3c\,)$ are both divisible by 3, so must $(\,a + b + c + d\,)$ be.

$\therefore$ *abcd* is divisible by 3 when the sum of the digits is divisible by 3 ☐

[5 marks]

(ii) 7458 is divisible by 3 because 7 + 4 + 5 + 8 = 24 which is divisible by 3

[2 marks]

Answer 6

(i) Observe that any natural number, $\mathbb{N}$, of more than 2 digits can be expressed in algebra as aR where R is the two rightmost digits, and a is all the other digits.

When aR is divisible by 4,

$$aR = 100a + R \qquad \text{is divisible by 4}$$

Clearly $100a$ is divisible by 4

By the Sum Divisor Rule, when aR and $100a$ are both divisible by 4, so must R be. In other words, and natural number of more than 2 digits is divisible by 4 precisely when the two rightmost digits are divisible by 4

[6 marks]

(ii) 290547452 is divisible by 4 because the two rightmost digits, 52, are divisible by 4. That is $52 = 4 \times 13$ (pack of cards).

[2 marks]

Answer 7

(i)
$$\begin{aligned} 119483 &\Rightarrow 11948 + 12 = 11960 \\ &\Rightarrow 1196 + 0 = 1196 \\ &\Rightarrow 119 + 24 = 143 \\ &\Rightarrow 14 + 12 = 26 \text{ which is obviously divisible by 13} \\ &\therefore 119483 \text{ is divisible by 13} \end{aligned}$$

[2 marks]

(ii)
$$\begin{aligned} 504088 &\Rightarrow 50408 + 32 = 50440 \\ &\Rightarrow 5044 + 0 = 5044 \\ &\Rightarrow 504 + 16 = 520 \text{ which is obviously divisible by 13} \\ &\therefore 504088 \text{ is divisible by 13} \end{aligned}$$

[2 marks]

(iii) Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits.

When aR is divisible by 13,

$$\begin{aligned} aR &= 10a + R && \text{is divisible by 13} \\ &= 10a - 13a + R - 13R && \text{is divisible by 13} \\ &= -3a - 12R && \text{is divisible by 13} \\ &= (-3)(a + 4R) && \text{is divisible by 13} \end{aligned}$$

As (-3) is not divisible by 13, $(a + 4R)$ must be. $\square$

[7 marks]

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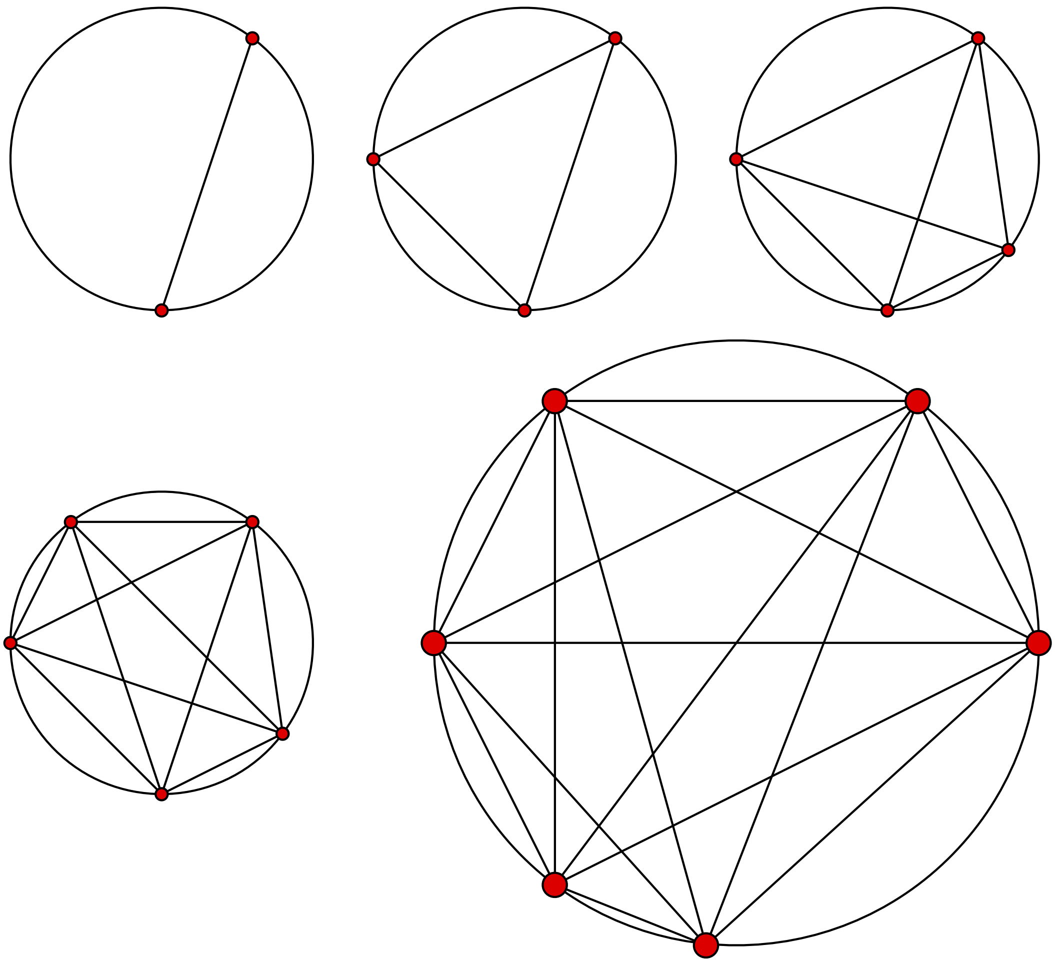
3.1 Always, Sometimes, Never

In mathematical investigations, a lookout is kept for patterns be they numerical, geometrical or algebraic. Once a pattern is suspected, mathematics requires that a proof be found; a reason why the pattern holds and will continue to hold.

3.2 Finding a Counterexample

A pattern that seems to be there may not, in fact, always hold. The easiest way to show that an alleged pattern is an illusion is to find a counterexample.

For example, suppose that a circle has n distinct points on its circumference, each joined to the others in all possible ways by straight lines. It is conjectured that the interior of the circle then has a maximum number of regions given by 2^{n-1}. Investigate this conjecture with the aid of the following diagrams.



n	2	3	4	5	6
2^{n-1}	2	4			
From Diagram					

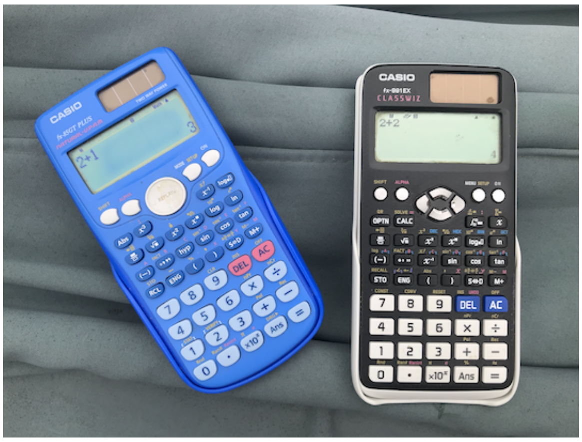
[6 marks]

3.3 When a Counterexample Can't Be Found

Either prove, or find a counterexample to the conjecture that, “if you add 1 to the product of four consecutive natural numbers the answer is always a perfect square”.

[6 marks]

~ Weapons of Mass Deduction ~



Photograph by Martin Hansen

3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 30 marks

Question 1

Give a counterexample to prove that the following statement is false;

For any pair of real numbers, x and y , if $x^2 = y^2$ then $x = y$

[2 marks]

Question 2

Give a counterexample to prove that the following statement is false;

For $a, b \in \mathbb{R}$, if $a^2 - b^2 > 0$ then $a - b > 0$

(Note that $a, b \in \mathbb{R}$ is saying that a and b are real numbers)

[2 marks]

Question 3

Give a counterexample to prove that the following statement is false;

For $x \in \mathbb{R}$, $0° \leqslant x \leqslant 360°$, $\cos x \leqslant 1 + \sin x$

[2 marks]

Question 4

For $x, y \in \mathbb{Q}$, $x \neq 0$, $y \neq 0$, give a counterexample to the following claim;

$$x^2 + y^2 \neq 1$$

(Note that the information $x, y \in \mathbb{Q}$ is saying that a and b are rational numbers)

[2 marks]

Question 5

AS-Level Examination Question from May 2018. Paper 1, Q2 (b), (Edexcel)

"If I add 3 to a number and square the sum, the result is greater than the square of the original number".

State if the above statement is always true, sometimes true or never true.

Give a reason for your answer.

[2 marks]

Question 6

Give a counterexample to prove that the following statement is false;

For any real numbers, x, if $|2x + 1| \leqslant 5$ then $|x| \leqslant 2$

[2 marks]

Question 7

Consider the following statement;

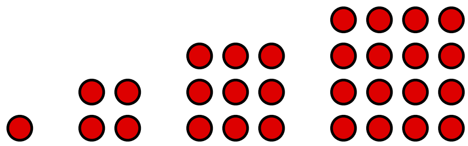
"For all $n \in \mathbb{N}$, $n^2 - n + 3$ is a prime number"

If true, prove it is true or, if false, find a counterexample.

[2 marks]

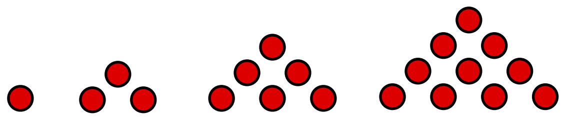
Question 8

The square numbers are so called because they are numbers that can be visualised as dots arranged into squares,



1 4 9 16

Similarly, the triangular numbers can be visualised as arrangements of dots into equilateral triangles,



1 3 6 10

The formulae for square numbers, S, and triangular numbers T are,

$$S_n = n^2 \qquad T_n = \frac{1}{2}n(n+1) \quad \text{where } n \in \mathbb{N}$$

Consider the following statement;

"For all $n \in \mathbb{N}$, $8T_n + 1$ is always a square number"

If true, prove it is true or, if false, find a counterexample.

[5 marks]

Question 9

Consider the following statement;

"The difference between the squares of two consecutive triangular numbers is always a cube"

If true, prove it is true or, if false, find a counterexample.

[5 marks]

Question 10

Examination Question from 2008, Q11 AH Maths (SQA)

For each of the following statements, decide whether it is true or false.

In each case, either prove it is true or find a counterexample.

(i) For $n \in \mathbb{N}$, if m^2 is divisible by 4 then m is divisible by 4.

[3 marks]

(ii) For $p, q \in \mathbb{N}$, the cube of any odd integer p plus the square of any even integer q is always odd.

[3 marks]

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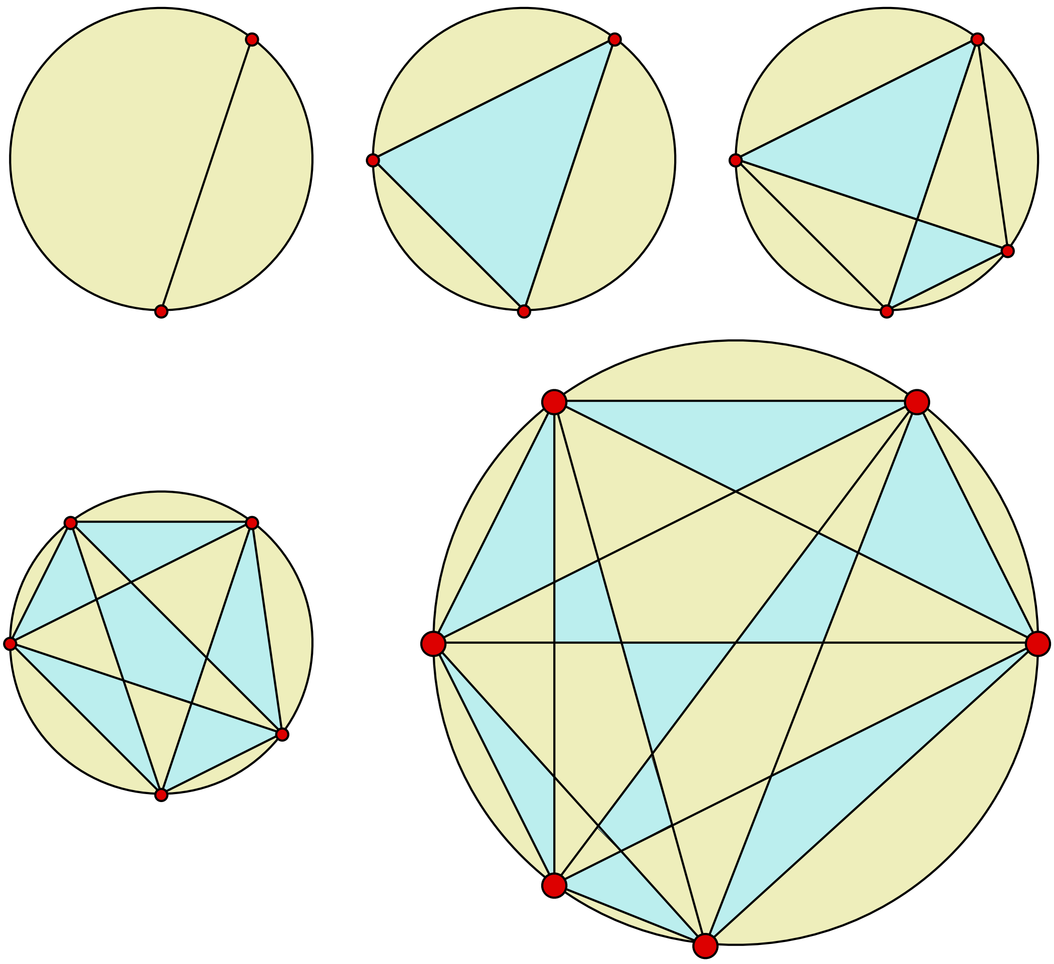
3.5 Answers

3.5.1 Solution to 3.2 Finding a Counterexample

The following colouring of the diagrams makes it easier to count the number of regions.



n	2	3	4	5	6
2^{n-1}	2	4	8	16	32
From Diagram	2	4	8	16	31

An excellent illustration of why proof is needed before a pattern that seems to be there is accepted into mathematics !

[6 marks]

3.5.2 Solution to 3.3 When a Counterexample Can't Be Found

Either prove, or find a counterexample to the conjecture that, “if you add 1 to the product of four consecutive natural numbers the answer is always a perfect square”.

Exploring some numerical cases to see if a counterexample emerges...

$$1 \times 2 \times 3 \times 4 + 1 = 25 = 5^2$$

$$2 \times 3 \times 4 \times 5 + 1 = 121 = 11^2$$

$$3 \times 4 \times 5 \times 6 + 1 = 361 = 19^2$$

$$4 \times 5 \times 6 \times 7 + 1 = 841 = 29^2$$

The claim, expressed in algebra, is that,

$$(n-1)n(n+1)(n+2) + 1 = m^2 \quad \text{for } n, m \in \mathbb{N}$$

$$\begin{aligned}
\text{LHS} &= (n-1)n(n+1)(n+2) + 1 \\
&= \left(n^2 - 1\right)\left(n^2 + 2n\right) + 1 \\
&= n^4 + 2n^3 - n^2 - 2n + 1 \\
&= \left(n^2 + an \pm 1\right)^2 && \text{for some } a \in \mathbb{N} \\
&= \left(n^2 + n - 1\right)^2 \\
&= m^2 && \text{where } m = n^2 - n + 1 \\
&= \text{RHS} && \square
\end{aligned}$$

	n^2	n	-1
n^2	n^4	n^3	$-n^2$
n	n^3	n^2	$-n$
-1	$-n^2$	$-n$	1

3.5.3 Solutions to 3.4 Exercise

Answer 1

$$x^2 = y^2$$
$$x^2 - y^2 = 0$$
$$(x - y)(x + y) = 0 \qquad \text{Difference of two squares}$$
$$\text{Either } x - y = 0 \text{ in which case } x = y$$
$$\text{Or } x + y = 0 \text{ in which case } x = -y$$
$$\therefore \text{ Any counterexample that satisfies } x = -y \text{ will suffice}$$

[2 marks]

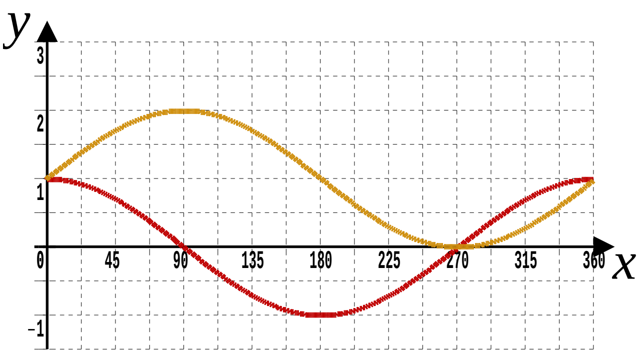
Answer 2

Any counterexample where $|a| > |b|$ with $a < 0$ and $b > 0$
For example,

$$a = -3, \quad b = 2 \text{ gives } a^2 - b^2 = 9 - 4 > 0 \quad \text{with } a - b < 0$$

[2 marks]

Answer 3



In red : $y = \cos x$

In gold : $y = 1 + \sin x$

From the graph it can be seen that any counterexample will have to be between $x = 270°$ and $x = 360°$

[2 marks]

Answer 4

Counterexamples can be found by taking any Pythagorean integer triple, for example, $3^2 + 4^2 = 5^2$ and dividing through by, in this case, the 5^2. This then gives the counterexample $x = \frac{3}{5}, \ y = \frac{4}{5}$ for which $x^2 + y^2 = 1$

[2 marks]

Answer 5

"If I add 3 to a number and square the sum, the result is greater than the square of the original number".

Let the original number be x in which case the above statement claims that,

$$(3 + x)^2 > x^2$$

from which is deduced that, $9 + 6x + x^2 > x^2$

$$9 + 6x > 0$$

$$6x > -9$$

$$x > -\frac{3}{2}$$

The statement would be true if x was a natural number.

However, the question does not say this so if x was a negative integer less than negative 1 then the statement would be false, for example if $x = -2$

[2 marks]

Answer 6

One approach is to use a bit of common sense to pick a suitable value for x such as $x = -\frac{5}{2}$ for then $|2x + 1| = \left|2 \times \left(-\frac{5}{2}\right) + 1\right|$

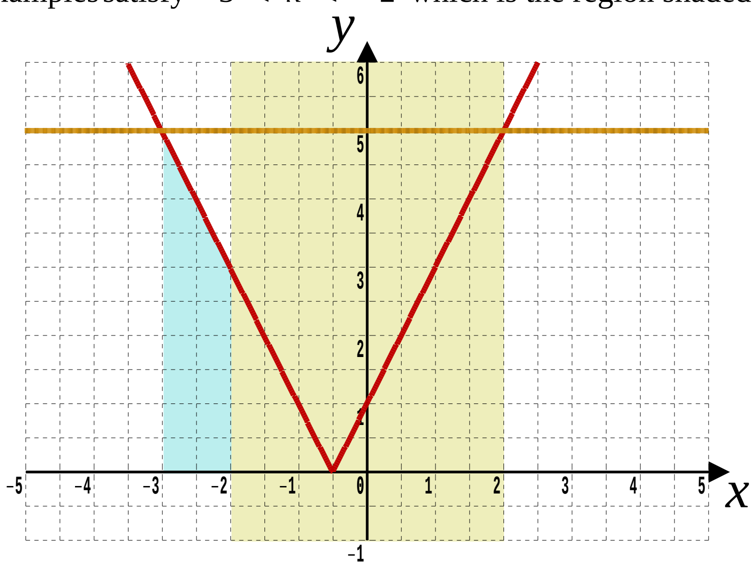
$$= |-4|$$

$$= 4$$

which is less than 5 as required but with $|x|$ greater than 2

The bigger picture is given by the following graph where $y = |2x + 1|$ is in red, $y = 5$ in gold, such that $|2x + 1| \leqslant 5$ is the part of the red graph that's under or on the gold graph. $|x| \leqslant 2$ is the region shaded yellow.

Counterexamples satisfy $-3 \leqslant x < -2$ which is the region shaded blue.



[2 marks]

Answer 7

For $n = 1$: $n^2 - n + 3 = 3$ prime

For $n = 2$: $n^2 - n + 3 = 5$ prime

For $n = 3$: $n^2 - n + 3 = 9$ not prime

$\therefore$ $n = 3$ is the smallest counterexample (there are infinitely more...)

[2 marks]

Answer 8

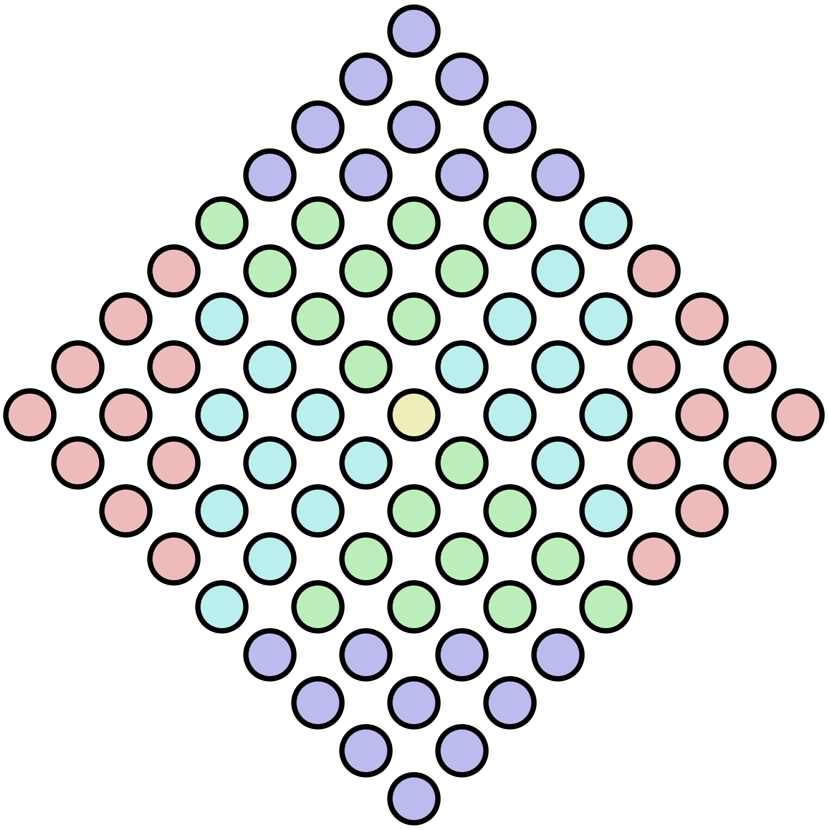
"For all $n \in \mathbb{N}$, $8T_n + 1$ is always a square number"

Algebraic Proof

$$
\begin{aligned}
8T_n + 1 &= 8 \times \frac{1}{2} n(n+1) + 1 \\
&= 4n(n+1) + 1 \\
&= 4n^2 + 4n + 1 \\
&= (2n+1)^2 \\
&= m^2 \qquad \text{where } m = 2n + 1
\end{aligned}
$$

which is a square number, as required. $\square$

Geometric Proof (without words)



[5 marks]

Answer 9

"The difference between the squares of two consecutive
triangular numbers is always a cube"

Proof

$$\left(\frac{1}{2} n (n + 1) \right)^2 - \left(\frac{1}{2} (n - 1) n \right)^2 = \frac{1}{4} n^2 \left((n + 1)^2 - (n + 1)^2 \right)$$
$$= \frac{1}{4} n^2 \left(n^2 + 2n + 1 - n^2 - 2n - 1 \right)$$
$$= \frac{1}{4} n^2 \left(4n \right)$$
$$= n^3$$

which is a cube number, as required. ☐

[5 marks]

Answer 10

(i) As m^2 is divisible by 4, $m^2 = 4n$ for some $m \in \mathbb{N}$
Taking positive square roots, $m = 2\sqrt{n}$ and so m will only be divisible by 4 if n contains a factor of 4.
Thus, although sometimes true, in general this conjecture is false.

[3 marks]

(ii) For $p, q \in \mathbb{N}$, the cube of any odd integer p plus the square of any even integer q is always odd.

Proof

Let $p = 2m + 1$ be an odd integer for some $m \in \mathbb{Z}$

Let $q = 2n$ be an even integer for some $n \in \mathbb{Z}$

$$p^3 + q^2 = (2m + 1)^3 + (2n)^2$$
$$= 8m^3 + 12m^2 + 6m + 1 + 4n^2$$
$$= \left(8m^3 + 12m^2 + 6m + 4n^2 \right) + 1$$
$$= 2\left(4m^3 + 6m^2 + 3m + 2n^2 \right) + 1$$
$$= 2w + 1 \quad \text{for } w = 4m^3 + 6m^2 + 3m + 2n^2$$

which is an odd number, as required. ☐

[3 marks]

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4.1 Non-Consecutive Numbers

Previously, in lesson 1, proofs about situations involving consecutive numbers were examined. However, suppose you wanted to prove a more general result such as the seemingly obvious fact that when any two odd numbers multiply, another odd number results. How do we drop the restriction that the two odd numbers be consecutive ?

4.2 The Non-Consecutive Odd Number Sequence

(i) Complete the following table for the odd number sequence,

p	1	2	3	4	...	m	...	n	...
T_p	1	3			...		...		...

[2 marks]

(ii) Hence prove that when any two odd numbers multiply, another odd number is the result.

[3 marks]

4.3 A Less Obvious use of Odd and Even

AS-Level Examination Question from June 2019, Paper 1, Q15, (Edexcel)

Given $n \in \mathbb{N}$, prove that $n^3 + 2$ is not divisible by 8

This proof technique is known as "proof by exhaustion".

[4 marks]

4.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40 marks

Question 1

Show that the sum of any two odd numbers is always an even number.

[2 marks]

Question 2

Show that 4 will always divide the product of any two even numbers.

[2 marks]

Question 3

For any natural number *n* use proof by exhaustion to show that $n^2 + 5$ is always either 1 or 2 more than a multiple of 4

[4 marks]

Question 4

A-Level Examination Question from June 2019, Paper 1, Q10 (i) (Edexcel)

Prove that for all $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4

[4 marks]

Question 5

For any natural number n show that $n^2 + n + 2$ is not divisible by 3 by reasoning that n must be one of either $3k$ or $3k + 1$ or $3k + 2$ and then working through each of these possibilities in turn (a proof by exhaustion on divisibility by 3).

[4 marks]

Question 6

A-Level Examination Question from October 2020, Paper 2, Q16 (Edexcel)

Use algebra to prove that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3

[4 marks]

Question 7

By making use of the result from question 6, prove that for any natural number, n, $3n^2 - 1$ is never a perfect square.

[4 marks]

Question 8

(i) Explain why a perfect square that is even must be divisible by 4

[3 marks]

(ii) If x and y are odd numbers, prove that $x^2 + y^2$ can not be a perfect square.

[6 marks]

Question 9
*** Challenge Question ***

Prove that $n^3 + 5n$ is divisible by 6

[7 marks]

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4.5.1 Solution to 4.2 Non-Consecutive Odd Number Sequence

(i)

p	1	2	3	4	...	m	...	n	...
T_p	1	3	5	7	...	$2m-1$	...	$2n-1$	...

[2 marks]

(ii) Let the two odd numbers be $(2m-1)$ and $(2n-1)$ for some $m, n \in \mathbb{N}$, in which case,

$$\begin{aligned}(2m-1)(2n-1) &= 4mn - 2m - 2n + 1 \\ &= 2(2mn - m - n) + 1\end{aligned}$$

which is one more than a multiple of 2, (an odd number) $\square$

[3 marks]

4.5.2 Solution to 4.3 A Less Obvious Use of Odd and Even

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^3 + 2 &= (2k)^3 + 2 \\ &= 8k^3 + 2\end{aligned}$$

Clearly, 2 more than a multiple of 8 is not a multiple of 8 so

So $n^3 + 2$ does not divide by 8 when n is even.

- Secondly, suppose n is odd in which case $n = 2k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^3 + 2 &= (2k+1)^3 + 2 \\ &= \quad 1 \times (2k)^3 \times 1^0 \\ &\quad + 3 \times (2k)^2 \times 1^1 \\ &\quad + 3 \times (2k)^1 \times 1^2 \\ &\quad + 1 \times (2k)^0 \times 1^3 + 2 \qquad \text{(Using the binomial theorem)} \\ &= 8k^3 + 12k^2 + 6k + 3 \\ &= 2\left(4k^3 + 6k^2 + 3k + 1\right) + 1\end{aligned}$$

This is one more than a multiple of 2, an odd number, and not a multiple of 8.

So $n^3 + 2$ does not divide by 8 when n is odd.

Overall, we deduce that $n^3 + 2$ is not divisible by 8 for any natural number n.

[4 marks]

4.5.3 Solutions to 4.4 Exercise

Answer 1

Let the two odd numbers be $(2m - 1)$ and $(2n - 1)$ for some $m, n \in \mathbb{N}$, in which case,

$$\begin{aligned}(2m - 1) + (2n - 1) &= 2m + 2n - 2 \\ &= 2(m + n - 1)\end{aligned}$$

which is a multiple of 2, in other words an even number. ☐

[2 marks]

Answer 2

Let the two even numbers be $(2m)$ and $(2n)$ for some $m, n \in \mathbb{N}$, in which case,

$$(2m)(2n) = 4mn$$

which is a multiple of 4 and therefore divisible by 4, as we were asked to show. ☐

[2 marks]

Answer 3

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^2 + 5 &= (2k)^2 + 5 \\ &= 4k^2 + 4 + 1 \\ &= 4\left(k^2 + 1\right) + 1\end{aligned}$$

  which is 1 more than a multiple of 4

- Secondly, suppose n is odd in which case $n = 2k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned}n^2 + 5 &= (2k + 1)^2 + 5 \\ &= 4k^2 + 4k + 1 + 4 + 1 \\ &= 4\left(k^2 + k + 1\right) + 2\end{aligned}$$

  which is 2 more than a multiple of 4

Overall, we deduce that $n^2 + 5$ is either 1 or 2 more than a multiple of 4 ☐

[4 marks]

Answer 4

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$n^2 + 2 = (2k)^2 + 2$$
$$= 4k^2 + 2$$

  which is 2 more than a multiple of 4 and so not divisible by 4

- Secondly, suppose n is odd in which case $n = 2k + 1$ for some $k \in \mathbb{N}$.

$$n^2 + 2 = (2k + 1)^2 + 2$$
$$= 4k^2 + 4k + 1 + 2$$
$$= 4\left(k^2 + k\right) + 3$$

  which is 3 more than a multiple of 4 and so not divisible by 4

Overall, we deduce that $n^2 + 2$ is never divisible 4 $\square$

[4 marks]

Answer 5

Either the natural number n is a multiple of 3 or one more than a multiple of 3 or 2 more than a multiple of 3.

- Firstly, suppose n is a multiple of 3 in which case $n = 3k$ for some $k \in \mathbb{N}$.

$$n^2 + n + 2 = (3k)^2 + (3k) + 2$$
$$= 9k^2 + 3k + 2$$
$$= 3\left(3k^2 + k\right) + 2$$

  which is 2 more than a multiple of 3 and so not divisible by 3

- Secondly, suppose n is one more than a multiple of 3 in which case $n = 3k + 1$ for some $k \in \mathbb{N}$.

$$n^2 + n + 2 = (3k + 1)^2 + (3k + 1) + 2$$
$$= 9k^2 + 6k + 1 + 3k + 1 + 2$$
$$= 3\left(3k^2 + 3k + 1\right) + 1$$

  which is 1 more than a multiple of 3 and so not divisible by 3

- Thirdly, suppose n is two more than a multiple of 3 in which case $n = 3k + 2$ for some $k \in \mathbb{N}$.

$$n^2 + n + 2 = (3k + 2)^2 + (3k + 2) + 2$$
$$= 9k^2 + 12k + 4 + 3k + 2 + 2$$
$$= 3\left(3k^2 + 5k + 2\right) + 2$$

  which is 2 more than a multiple of 3 and so not divisible by 3

Overall, we deduce that $n^2 + n + 2$ is never divisible 3 $\square$

[4 marks]

Answer 6

Either the natural number n is a multiple of 3 or one more than a multiple of 3 or 2 more than a multiple of 3.

- Firstly, suppose n is a multiple of 3 in which case $n = 3k$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 &= (3k)^2 \\ &= 9k^2 \end{aligned}$$

which is a multiple of 3

- Secondly, suppose n is one more than a multiple of 3 in which case $n = 3k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3\left(3k^2 + 2k\right) + 1 \end{aligned}$$

which is one more than a multiple of 3

- Thirdly, suppose n is two more than a multiple of 3 in which case $n = 3k + 2$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3\left(3k^2 + 4k + 1\right) + 1 \end{aligned}$$

which is one more than a multiple of 3

Overall, it has been shown that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3, as required. □

[4 marks]

Answer 7

To show $3n^2 - 1$ is a not perfect square show that it cannot be written in the form $3k$ or $3k+1$, using the result from question 6.

In other words, show $3n^2 - 1$ is of the form $3k + 2$

Let $n = m + 1$, for some integer m, in which case,

$$\begin{aligned} 3n^2 - 1 &= 3(m + 1)^2 - 1 \\ &= 3m^2 + 6m + 3 - 1 \\ &= 3\left(m^2 + 2m\right) + 2 \\ &= 3k + 2 \end{aligned}$$

Thus, $3n^2 - 1$ cannot ever be a perfect square for any $n \in \mathbb{N}$. □

[4 marks]

Answer 8

(i) From being told that we have a perfect square that is even we can write that,

$$n^2 = 2k \qquad \text{for some integer } k$$

From this we have that,

$$n = \sqrt{2k} \qquad \text{ignoring the negative square root an } n \in \mathbb{N}$$

The $\sqrt{2k}$ must be a natural number and so k must be of the form

$2\,m^2$, for some $m \in \mathbb{N}$.

Thus, $\quad n^2 = 2 \times 2\,m^2$

$\qquad\qquad = 4\,m^2$

showing that n^2 is divisible by 4, as requested. $\quad\square$

[3 marks]

(ii) Let the two odd numbers be $x = (2m - 1)$ and $y = (2n - 1)$ for some $m, n \in \mathbb{N}$, in which case,

$$x^2 + y^2 = (2m - 1)^2 + (2n - 1)^2$$

$$= 4\,m^2 - 4m + 1 + 4\,n^2 - 4n + 1$$

$$= 4\left(m^2 - m + n^2 - n\right) + 2$$

Now. we know that an odd number multiplied by an odd number is an odd number (proven in the lesson's introduction) so as x and y are odd, we have that their squares are also odd.

When these squares are added together, we're adding an odd number to an odd number which is an even number (proven in question 1).

So the sum of the squares of x and y is even.

For this even number to be a perfect square it must be divisible by 4 from part (i) of this question.

But we have just shown that the sum of the squares of x and y is 2 more than a multiple of 4 which cannot be a multiple of 4

$\therefore\ x^2 + y^2$ can not be a perfect square if both x and y are odd.

[6 marks]

Answer 9

One way to do this question is to show that $n^3 + 5n$ is divisible by 2 and also by 3

To show $n^3 + 5n$ is divisible by 2

Either the natural number n is even or it is odd.

- Firstly, suppose n is even in which case $n = 2k$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^3 + 5n &= (2k)^3 + 5(2k) \\ &= 8k^3 + 10k \\ &= 2\left(4k^3 + 5k\right) \end{aligned}$$

- Secondly, suppose n is odd in which case $n = 2k + 1$ for some $k \in \mathbb{N}$.

$$\begin{aligned} n^3 + 5n &= (2k + 1)^3 + 5(2k + 1) \\ &= 8k^3 + 12k^2 + 6k + 1 + 10k + 5 \\ &= 2\left(4k^3 + 6k^2 + 8k + 3\right) \end{aligned}$$

It is deduced that $n^3 + 5n$ is always divisible by 2 for any $n \in \mathbb{N}$.

To show $n^3 + 5n$ is divisible by 3

Any natural number n must be of one of the forms $3k$, $3k + 1$, or $3k + 2$, where $k \in \mathbb{N}$

- Case $n = 3k$: $n^3 + 5n = (3k)^3 + 5(3k)$

$$\begin{aligned} &= 27k^3 + 15k \\ &= 3\left(9k^3 + 5k\right) \end{aligned}$$

- Case $n = 3k + 1$: $n^3 + 5n = (3k + 1)^3 + 5(3k + 1)$

$$\begin{aligned} &= 27k^3 + 27k^2 + 9k + 1 + 15k + 5 \\ &= 3\left(9k^3 + 9k^2 + 8k + 2\right) \end{aligned}$$

- Case $n = 3k + 2$: $n^3 + 5n = (3k + 2)^3 + 5(3k + 2)$

$$\begin{aligned} &= 27k^3 + 54k^2 + 36k + 8 + 15k + 10 \\ &= 3\left(9k^3 + 18k^2 + 17k + 6\right) \end{aligned}$$

It is deduced that $n^3 + 5n$ is always divisible by 3 for any $n \in \mathbb{N}$.

Overall, we deduce that $n^3 + 5n$ is divisible by 6 $\square$

[7 marks]

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5.1 Proof by Contradiction

Proof by contradiction is a clever method of proving that a statement is true by assuming that if the opposite were true, this would leads to a nonsensical situation. Given that this opposite is not true, the original statement must be true.

This piece of logical thinking takes a little bit of getting used to, and so we will look at three classic examples.

5.1.1 Example #1

Use proof by contradiction to show that there exist no integers a and b for which

$$21a + 14b = 1$$

Proof

Assume the opposite is true, integers a and b exist for which $21a + 14b = 1$

Divide both sides by the highest common factor of 21 and 14, that is, 7 to get,

$$3a + 2b = \frac{1}{7}$$

But this clearly cannot be, as any combination of integers that are multiplied, added and subtracted can only yield an integer answer, not a fraction.

Therefore, the original assumption, that integers a and b exist, must be false.

And so, there exist no integers a and b for which $21a + 14b = 1$ □

[3 marks]

5.1.2 Example #2

Use proof by contradiction to show that if n^2 is even, then n must be even.

Proof

[3 marks]

5.1.3 Example #3

Use proof by contradiction to show that $\sqrt{2}$ is an irrational number.

Proof

Firstly, to be clear, a number n is rational if it can be written in the form $\dfrac{p}{q}$ where p and q are integers, and q is not equal to zero. Mathematicians' write, $n \in \mathbb{Q}$.
An irrational number is a number that cannot be written in this form.
Assume the opposite of what is wanted is true; that $\sqrt{2}$ is a rational number. Then, by the definition of what a rational number is,

$$\sqrt{2} = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z}, q \neq 0$$

Furthermore, assume that this fraction cannot be reduced further, p and q have no factors in common. In other words, they are coprime with $hcf\{p, q\} = 1$

In consequence, $\quad 2 = \dfrac{p^2}{q^2}$

$$p^2 = 2q^2$$

The 2 on the RHS tells us that it is even. Therefore the LHS must be even.
From example #2, if p^2 is even then p is also even and so can be expressed in the form $p = 2a$ for some integer a. So we now have that,

$$(2a)^2 = 2q^2$$

$$4a^2 = 2q^2$$

$$2a^2 = q^2$$

The 2 on the LHS tells us that it is even. Therefore the RHS must be even.
Again, from example #2, if q^2 is even, then q is also even.
If p and q are both even, they will have a common factor of 2,
This contradicts the statement that p and q have no common factors.
The inescapable conclusion is that $\sqrt{2}$ is an irrational number. $\square$

[6 marks]

The Inescapable Conclusion

5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40 marks

Question 1

For $x, y \in \mathbb{N}$, $x \neq y$, use proof by contradiction to show that,

$$\frac{x}{y} + \frac{y}{x} > 2$$

Hint : Assume that the opposite is true, that $\frac{x}{y} + \frac{y}{x} \leqslant 2$

Do algebra on this inequality to derive a statement that can't be true.

[3 marks]

Question 2

For $x \in \mathbb{R}$, $0 < x < 1$, use proof by contradiction to show that,

$$\frac{1}{x(1 - x)} \geqslant 4$$

Hint : From $0 < x < 1$ it follows that $x > 0$ and $(1 - x) > 0$ and so $x(1 - x) > 0$

[4 marks]

Question 3
Use proof by contradiction to show there are no integers a and b such that,

$$10a + 15b = 17$$

[3 marks]

Question 4
A-Level Examination Question from June 2022, Paper 1, Q7 (Edexcel)

(i) Given that p and q are integers such that,

$$pq \text{ is even}$$

use algebra to prove by contradiction that at least one of p or q is even.

[3 marks]

(ii) Given that x and y are integers such that,

- $x < 0$
- $(x + y)^2 < 9x^2 + y^2$

show that $y > 4x$

[2 marks]

Question 5

A-Level Examination Question from October 2021, Paper 1, Q15 (Edexcel)

(i) Use proof by exhaustion to show that for $n \in \mathbb{N}, n \leqslant 4$

$$(n + 1)^3 > 3^n$$

[2 marks]

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

[4 marks]

Question 6

A-Level Examination Question from October 2020, Paper 1, Q16 (Edexcel)

Prove by contradiction that there are no positive integers p and q such that,

$$4p^2 - q^2 = 25$$

[4 marks]

Question 7

(i) Prove by contradiction that if n^2 is a multiple of 3, then n is a multiple of 3

Hint : Consider numbers in the form $3k + 1$ and $3k + 2$

[3 marks]

(ii) Hence prove by contradiction that $\sqrt{3}$ is an irrational number.

[6 marks]

Question 8
Let x be an irrational number and y be a rational number.
Use proof by contradiction to show that $x + y$ is irrational.

[6 marks]

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5.3 Answers

5.3.1 Solution to 5.1.2 Example #2

Use proof by contradiction to show that if n^2 is even, then n must be even.

Proof

Assume the opposite is true, that there exists a number n such that n^2 is even but n is odd in which case we can write that $n = 2k + 1$ for some $k \in \mathbb{Z}$.

In consequence,

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2\left(2k^2 + 2k\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 2, in other words, an odd number.
This contradicts the assumption that n^2 is even.
Therefore if n^2 is even, n must also be even. $\square$

[3 marks]

5.3.2 Solutions to 5.2 Exercise

Answer 1

Following the hint, assume that the opposite is true, that for some $x, y \in \mathbb{N}$, $x \neq y$, that,

$$\frac{x}{y} + \frac{y}{x} \leqslant 2$$

Multiply through by xy (which we know is positive)

$$\begin{aligned} x^2 + y^2 &\leqslant 2xy \\ x^2 - 2xy + y^2 &\leqslant 0 \\ (x - y)^2 &\leqslant 0 \end{aligned}$$

But this cannot be a true statement as anything squared is greater than zero.
Nor can it equal zero because we are told that $x \neq y$.
Thus, it has been proven that for $x, y \in \mathbb{N}$, $x \neq y$,

$$\frac{x}{y} + \frac{y}{x} > 2 \qquad \square$$

[3 marks]

Answer 2

Assume that the opposite is true, that for $x \in \mathbb{R}$, $0 < x < 1$,

$$\frac{1}{x(1-x)} < 4$$

From $0 < x < 1$ it follows that $x > 0$ and $(1 - x) > 0$ and so $x(1-x) > 0$ which means that it's OK to multiply through by $x(1-x)$ and we know not to reverse the inequality.

$$1 < 4x(1-x)$$

$$1 < 4x - 4x^2$$

$$4x^2 - 4x + 1 < 0$$

$$(2x-1)^2 < 0$$

But this cannot be a true statement as anything squared is greater than zero.

Thus, it has been proven that for $x \in \mathbb{R}$, $0 < x < 1$,

$$\frac{1}{x(1-x)} \geqslant 4 \qquad \square$$

[4 marks]

Answer 3

Assume the opposite is true, integers a and b exist for which $10a + 15b = 17$

Divide both sides by the highest common factor of 10 and 15, that is, 5 to get,

$$2a + 3b = \frac{17}{5}$$

But this clearly cannot be, as any combination of integers that are multiplied, added and subtracted can only yield an integer answer, not a fraction.

Therefore, the original assumption, that integers a and b exist, must be false.

And so, there exist no integers a and b for which $10a + 15b = 17$ $\square$

[3 marks]

Answer 4

(i) Assume the opposite is true, that p and q are integers such that,

pq is even but that both p and q are odd.

This would mean that we could write p and q in the forms,

$p = 2n + 1$ and $q = 2m + 1$ for some $p, q \in \mathbb{Z}$

Then,

$$\begin{aligned} pq &= (2n+1)(2m+1) \\ &= 4nm + 2n + 2m + 1 \\ &= 2(2nm + n + m) + 1 \end{aligned}$$

But this shows that pq is odd and not even.

Therefore, the assumption that p and q are both odd cannot be correct.

At least one of p or q must be even.

[3 marks]

(ii)

$$(x + y)^2 < 9x^2 + y^2$$

$$x^2 + 2xy + y^2 < 9x^2 + y^2$$

$$2xy < 8x^2$$

$$xy < 4x^2$$

Divide both sides by x and reverse the inequality as $x < 0$

$$y > 4x \qquad \square$$

[2 marks]

Answer 5

(i) The condition that $n \in \mathbb{N}, n \leqslant 4$, means that $n \in \{1, 2, 3, 4\}$

Working through those possibilities,

$n = 1 : (1 + 1)^3 = 2^3 = 8 > 3^1$

$n = 2 : (2 + 1)^3 = 3^3 = 27 > 3^2$

$n = 3 : (3 + 1)^3 = 4^3 = 64 > 3^3$

$n = 4 : (4 + 1)^3 = 5^3 = 125 > 3^4$

In all cases it has been shown that $(n + 1)^3 > 3^n$, as required. $\square$

[2 marks]

(ii) Suppose that the opposite were true, that $m^3 + 5$ were odd and that m is odd. Writing $m = 2k + 1$ it then follows that,

$$\begin{aligned} m^3 + 5 &= (2k + 1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 2\left(4k^3 + 6k^2 + 3k + 3\right) \end{aligned}$$

But this shows that $m^3 + 5$ is even and not odd.

Therefore, the assumption that m is odd cannot be correct.

It is thus the case that m must be even.

[4 marks]

Answer 6

Assume that there are positive integers p and q such that,

$$4p^2 - q^2 = 25$$

in which case, using the difference of two squares,

$$(2p + q)(2p - q) = 25$$

Now, each of $(2p + q)$ and $(2p - q)$ must be an integer factor of 25

One possibility is that each equals 5, for then we have $5 \times 5 = 25$

$$\begin{array}{rr} & 2p + q = 5 \\ \text{ADD} & 2p - q = 5 \\ \hline & 4p = 10 \\ & p = \frac{5}{2} \end{array}$$

But this contradicts the requirement that p is an integer.

Another possibility is that the one is 25 and the other 1, for then $25 \times 1 = 25$

$$\begin{array}{rr} & 2p + q = 25 \\ \text{ADD} & 2p - q = 1 \\ \hline & 4p = 26 \\ & p = \frac{13}{2} \end{array} \qquad \begin{array}{rr} & 2p + q = 1 \\ \text{ADD} & 2p - q = 25 \\ \hline & 4p = 26 \\ & p = \frac{13}{2} \end{array}$$

But this contradicts the requirement that p is an integer.

Having exhausted the possibilities, and always obtained a contradiction it must be the case that there are no positive integers p and q such that,

$$4p^2 - q^2 = 25 \qquad \square$$

[4 marks]

Answer 7

(i) Assume that the opposite is true, that n^2 is a multiple of 3, and n is not a multiple of 3, in which case either $n = 3k + 1$ or $n = 3k + 2$

If $n = 3k + 1$ then,

$$\begin{aligned} n^2 &= (3k + 1)^2 \\ &= 9k^2 + 6k + 1 \\ &= 3\left(3k^2 + 2k\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 3 and so not a multiple of 3

If $n = 3k + 2$ then,

$$\begin{aligned} n^2 &= (3k + 2)^2 \\ &= 9k^2 + 12k + 4 \\ &= 3\left(3k^2 + 4k + 1\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 3 and so not a multiple of 3

In both cases the assumption that n is not a multiple of 3 has given rise to a contradiction and so the assumption must be false.
Therefore, n is a multiple of 3. □

[3 marks]

(ii) Assume the opposite; that $\sqrt{3}$ is a rational number.
Then, by the definition of what a rational number is,

$$\sqrt{3} = \frac{p}{q} \text{ for some } p,\ q \in \mathbb{Z},\ q \neq 0$$

Furthermore, assume that this fraction cannot be reduced further.
That is p and q have no factors in common.
(In other words, they are coprime with $hcf\ \{p,\ q\} = 1$)

In consequence, $$3 = \frac{p^2}{q^2}$$

$$p^2 = 3q^2$$

The 3 on the RHS tells us that it is a multiple of 3.
Therefore the LHS must be a multiple of 3.
From part (i), if p^2 is a multiple of 3 then so too is p
Therefore p can be expressed in the form $3a$ for some integer a.
So we now have that,

$$(3a)^2 = 3q^2$$

$$9a^2 = 3q^2$$

$$3a^2 = q^2$$

The 3 on the LHS tells us that it is a multiple of 3.
Therefore the RHS must be a multiple of 3.
Again, from part (i), if q^2 is a multiple of 3, then so too is q
If p and q are both multiples of 3, they will have a common factor of 3.
This contradicts the statement that p and q have no common factors.
The inescapable conclusion is that $\sqrt{3}$ is an irrational number. □

[6 marks]

Answer 8

We want to prove that $x + y$ is irrational given that x is irrational and y is rational.
For the sake of contradiction, let $x + y$ be rational.
Call this rational sum z such that $x + y = z$ and hence $x = z - y$
We know (or could easily prove) that, since both z and y are rational, their difference $z - y$ must also be a rational number.
However, if $z - y$ is rational, then so too must be x (since $x = z - y$).
This directly contradicts our starting premise that x is irrational.
As assuming $x + y$ is rational leads to a contradiction, that assumption must be false.
Therefore, $x + y$ must be irrational. □

[6 marks]

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Lesson 6

A-Level Pure Mathematics, Year 2

Proof I : The Art of Absolute Certainty

6.1 Revision

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50 marks

Question 1

Marcus has a theory.

He believes that the product of an odd and an even number is never a perfect square.

Show that Marcus is wrong by finding a counterexample to disprove his theory.

[3 marks]

Question 2

Prove that when any odd integer is squared, the result is always one more than a multiple of 8.

[5 marks]

Question 3

(i) Prove that the difference of the squares of two consecutive even numbers is always divisible by 4.

[3 marks]

(ii) Is the above statement true for odd numbers ?
Give a reason for your answer.

[3 marks]

Question 4

(i) Work out, $3^2 + 4^2 + 5^2 + 6^2 + 7^2$

[1 mark]

(ii) Prove that the sum of the squares of any five consecutive integers is divisible by 5

[2 marks]

(iii) Prove that if five consecutive integers are squared, the mean of the squares exceeds the square of their median by 2

[4 marks]

(iv) Hence, without using a calculator, write down the value of

$$\frac{98^2 + 99^2 + 100^2 + 101^2 + 102^2}{5}$$

[2 marks]

Question 5

AS-Level Examination Question from October 2020, Paper 1, Q13 (Edexcel)

(a) Prove that for all positive values of a and b

$$\frac{4a}{b} + \frac{b}{a} \geqslant 4$$

[4 marks]

(b) Prove, by counter example, that this is not true for all values of a and b.

[1 mark]

Question 6

Write down any 3-digit number *abc* then repeat the digits to form the 6-digit number

abcabc

Explain why the 6-digit number will be divisible by 7, by 11 and by 13

[5 marks]

Question 7

Divisibility by 7

Subtract twice the last digit from the number formed by the remaining digits.

Repeat as necessary.

If, at any stage, a result is obtained that is obviously divisible by 7 then the original number is also divisible by 7

(i) Without using a calculator, show that 2331 is divisible by 7

[2 marks]

(ii) Without using a calculator, show that 515011 is divisible by 7

[2 marks]

(iii) Provide a proof that explains why the test for divisibility by 7 works

[7 marks]

Question 8

Examination Question from 2013, Q12 AH Maths (SQA)

Let n be a natural number.

For each of the following statements, decide whether it is true or false.

If true, give a proof; if false, give a counterexample.

(i) If n is a multiple of 9 then so is n^2

[3 marks]

(ii) If n^2 is a multiple of 9 then so is n

[3 marks]

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Answer 1

Marcus may have forgotten about the odd number 1.

$$1 \times \text{Even} \,\square = \text{Even} \,\square \qquad \text{For example } 1 \times 4 = 4$$

This is part of a more general result where,

$$\text{Odd} \,\square \times \text{Even} \,\square = \text{Even} \,\square \qquad \text{For example, } 9 \times 4 = 36$$

[3 marks]

Answer 2

Let the odd number be $2k - 1$ for some $k \in \mathbb{N}$.

Squaring this odd number gives,

$$\begin{aligned}(2k - 1)^2 &= 4k^2 - 4k + 1 \\ &= 4\left(k^2 - k\right) + 1 \\ &= 4\left(k(k - 1)\right) + 1\end{aligned}$$

The $k(k - 1)$ piece of this represents two consecutive integers.

One of these must be even.

Therefore $k(k - 1) = 2w$ for some $w \in \mathbb{N}$.

$$\begin{aligned}(2k - 1)^2 &= 4(2w) + 1 \\ &= 8w + 1\end{aligned}$$

Which is always one more than a multiple of 8, as expected. □

[5 marks]

Answer 3

(i) Let the two consecutive even numbers be, $2n$ and $2n + 2$ in which case,

$$\begin{aligned}(2n + 2)^2 - (2n)^2 &= 4(n + 1)^2 - 4n^2 \\ &= 4\left((n + 1)^2 - n^2\right)\end{aligned}$$

which is a multiple of 4 and so divisible by 4 □

[3 marks]

(ii) It is true for any two consecutive odd numbers.

Proof

Let the two consecutive odd numbers be, $2n - 1$ and $2n + 1$ in which case,

$$\begin{aligned}(2n + 1)^2 - (2n - 1)^2 &= 4n^2 + 4n + 1 - 4n^2 + 4n - 1 \\ &= 8n\end{aligned}$$

which is a multiple of 8 and so divisible by 4 □

[3 marks]

Answer 4

(i) $3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 135$

[1 mark]

(ii) Let the five consecutive integers and their sum be,

$$(n-2)+(n-1)+n+(n+1)+(n+2) = 5n$$

which is a multiple of five and so divisible by 5

[2 marks]

(iii) Working out the mean of the squares...

$$\frac{(n-2)^2+(n-1)^2+n^2+(n+1)^2+(n+2)^2}{5}$$

$$= \frac{n^2-4n+4+n^2-2n+1+n^2+n^2+2n+1+n^2+4n+4}{5}$$

$$= \frac{5n^2+10}{5}$$

$$= n^2 + 2$$

The median of the five consecutive integers is *n* and so it has been shown that if five consecutive integers are squared, the mean of the squares does indeed exceed the square of there median by 2 ☐

[4 marks]

(iv) Hence, $\dfrac{98^2+99^2+100^2+101^2+102^2}{5} = 100^2 + 2$

$$= 10002$$

[2 marks]

Answer 5

(a) Assume that the opposite is true, that for all positive values of *a* and *b*

$$\frac{4a}{b} + \frac{b}{a} < 4$$

Multiply through by *ab*

(which we know is positive as so does not reverse the inequality)

$$4a^2 + b^2 < 4ab$$

$$4a^2 - 4ab + b^2 < 0$$

$$(2a-b)^2 < 0$$

But this cannot be a true statement because anything squared is greater than or equal to zero. Via this contradiction it has been proven that, for all positive values of *a* and *b*

$$\frac{4a}{b} + \frac{b}{a} \geqslant 4$$ ☐

[4 marks]

(b) Let $a = -1$ and $b = 1$ in which case,

$$\frac{4a}{b} + \frac{b}{a} = -4 - 1$$
$$= -5$$

which fails to be greater than or equal to 4

(Any suitable counterexample will suffice, and there are many, easily found)

[1 mark]

Answer 6

$abcabc$ represents $100000a + 10000b + 1000c + 100a + 10b + c$

$$= 100100a + 10010b + 1001c$$
$$= 1001(100a + 10b + c)$$
$$= 7 \times 11 \times 13(100a + 10b + c)$$

Thus, $abcabc$ is divisible by 7, by 11 and by 13 ☐

[5 marks]

Answer 7

(i) $2331 \Rightarrow 233 - 2 = 231$

$\Rightarrow 23 - 2 = 21$ which is obviously divisible by 7

$\therefore$ 2331 is divisible by 7

[2 marks]

(ii) $515011 \Rightarrow 51501 - 2 = 51499$

$\Rightarrow 5149 - 18 = 5131$

$\Rightarrow 513 - 2 = 511$

$\Rightarrow 51 - 2 = 49$ which is obviously divisible by 7

$\therefore$ 515011 is divisible by 7

[2 marks]

(iii) Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits. When aR is divisible by 7,

$aR = 10a + R$ is divisible by 7

$= 3a + 7a + R$ is divisible by 7

$= 3a + 7a + R - 7R$ is divisible by 7

$= 3a - 6R$ is divisible by 7

$= 3(a - 2R)$ is divisible by 7

As 3 is not divisible by 7, $(a - 2R)$ must be. ☐

[7 marks]

Answer 8

(i) True

Proof

If n is a multiple of 9 then it can be expressed in the form $9m$, where $m \in \mathbb{N}$

$$\text{Then, } n^2 = (9m)^2$$
$$= 9\left(9m^2\right)$$

which shows that $\sqrt{n^2}$ is a multiple of 9 □

[3 marks]

(ii) False

Counterexample

If $n^2 = 9$ then n^2 is a multiple of 9 but $n = 3$ is not.

[3 marks]

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7.1 Test

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50 marks

Question 1

Find a counterexample that disproves the statement, "All numbers which are one greater than a multiple of 24 are the squares of prime numbers"

[2 marks]

Question 2

Prove than the sum of the squares of any two consecutive integers is an odd number.

[4 marks]

Question 3

Write down any 2-digit number *ab* then repeat the digits twice to form the 6-digit number *ababab*

Explain why the 6-digit number will be divisible by 3, by 7, by 13 and by 37

[4 marks]

Question 4

Prove that the product of two consecutive even numbers is divisible by 8

[6 marks]

Question 5

The n^{th} triangular number is of the form, $T_n = \dfrac{n(n+1)}{2}$

(i) Calculate $T_7 + T_8$

[2 marks]

(ii) Write down the formula for the $(n + 1)^{\text{th}}$ triangular number

[2 marks]

(iii) Prove that the sum of any two consecutive triangular numbers is a square number

[4 marks]

Question 6

Divisibility by 11

Subtract the last digit from the number formed by the remaining digits.

Repeat as necessary.

If, at any stage, a result is obtained that is obviously divisible by 11 then the original number is also divisible by 11

(i) Without using a calculator, show that 3531 is divisible by 11

[2 marks]

(ii) Without using a calculator, show that 1984422 is divisible by 11

[2 marks]

(iii) Provide a proof that explains why the test for divisibility by 11 works

[7 marks]

Question 7

(i) Prove by contradiction that if n^3 is even, then n is even.

[3 marks]

(ii) Hence prove by contradiction that $\sqrt[3]{2}$ is an irrational number.

[6 marks]

Question 8

Consider the following sequence of calculations,

$$x_1 = 1^2 + 2^2 + (1 \times 2)^2$$
$$x_2 = 2^2 + 3^2 + (2 \times 3)^2$$
$$x_3 = 3^2 + 4^2 + (3 \times 4)^2$$
$$x_4 = 4^2 + 5^2 + (4 \times 5)^2$$
$$\dots$$

Show that x_n will always be a perfect square.

[6 marks]

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Answer 1

Some examples of counterexamples are given below.
To answer the question, only one counterexample is required.

$1 \times 24 + 1 = 25 = 5^2$

$2 \times 24 + 1 = 49 = 7^2$

$3 \times 24 + 1 = 73$ which is not a square <== counterexample

$4 \times 24 + 1 = 97$ which is not a square <== counterexample

$5 \times 24 + 1 = 121 = 11^2$

$6 \times 24 + 1 = 145$ which is not a square <== counterexample

$7 \times 24 + 1 = 169 = 13^2$

15 is the next number that is not a counterexample

[2 marks]

Answer 2

Let the two consecutive integers be n and $(n + 1)$ in which case,

$$n^2 + (n + 1)^2 = n^2 + n^2 + 2n + 1$$
$$= 2\left(n^2 + n\right) + 1$$

which is one more than a multiple of 2, an odd number

$\therefore$ The sum of the squares of any two consecutive integers is an odd number. ☐

[4 marks]

Answer 3

ababab represents $100000a + 10000b + 1000a + 100b + 10a + b$

$$= 101010a + 10101b$$
$$= 10101(10a + b)$$
$$= 3 \times 7 \times 13 \times 37(10a + b)$$

Thus, *ababab* is divisible by 3, by 7, by 13 and by 37 ☐

[4 marks]

Answer 4

Let the two consecutive even numbers be $2m$ and $2m + 2$ for some $m \in \mathbb{N}$.

The product is given by,

$$\begin{aligned}(2m)(2m+2) &= 4m^2 + 4m \\ &= 4\left(m^2 + m\right) \\ &= 4\left(m(m+1)\right)\end{aligned}$$

The $m(m+1)$ piece of this represents two consecutive integers.

One of these must be even.

Therefore $m(m+1) = 2w$ for some $w \in \mathbb{N}$.

$$\begin{aligned}(2m)(2m+2) &= 4(2w) \\ &= 8w\end{aligned}$$

Which is always a multiple of 8, as expected.
It is proven the product of two consecutive even numbers is divisible by 8 □

[6 marks]

Answer 5

(i) $T_n = \dfrac{n(n+1)}{2}$

$\therefore\ T_7 = 28,\ T_8 = 36$ and so $T_7 + T_8 = 64$

[2 marks]

(ii) $T_{n+1} = \dfrac{(n+1)(n+2)}{2}$

[2 marks]

(iii) The sum of any two consecutive triangular numbers is given by,

$$\begin{aligned}T_n + T_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)}{2}\left(n + n + 2\right) \\ &= \frac{(n+1)(2n+2)}{2} \\ &= (n+1)(n+1) \\ &= (n+1)^2\end{aligned}$$

which is a square number. □

[4 marks]

Answer 6

(i) $3531 \;\Rightarrow\; 353 - 1 = 352$

$\Rightarrow\; 35 - 2 = 33$ which is obviously divisible by 11

$\therefore$ 3531 is divisible by 11

[2 marks]

(ii) $1984422 \;\Rightarrow\; 198442 - 2 = 198440$

$\Rightarrow\; 19844 - 0 = 19844$

$\Rightarrow\; 1984 - 4 = 1980$

$\Rightarrow\; 198 - 0 = 198$

$\Rightarrow\; 19 - 8 = 11$ which is obviously divisible by 11

$\therefore$ 1984422 is divisible by 11

[2 marks]

(iii) Observe that any natural number, $\mathbb{N}$, can be expressed in algebra as aR where R is the single rightmost digit, and a is all the other digits. When aR is divisible by 11,

$$\begin{aligned} aR &= 10a + R && \text{is divisible by 11} \\ &= 10a + R - 11R && \text{is divisible by 11} \\ &= 10a - 10R && \text{is divisible by 11} \\ &= 10\,(\,a - R\,) && \text{is divisible by 11} \end{aligned}$$

As 10 is not divisible by 11, $(\,a - R\,)$ must be. $\square$

[7 marks]

Answer 7

(i) Assume that the opposite is true, that n^3 is even, and n is not even, in which case $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Then
$$\begin{aligned} n^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2\left(4k^3 + 6k^2 + 3k\right) + 1 \end{aligned}$$

But this is 1 more than a multiple of 2 and so is an odd number.

The assumption that n is odd has given rise to a contradiction and so the assumption must be false.

Therefore, n is even. □

[3 marks]

(ii) Assume the opposite; that $\sqrt[3]{2}$ is a rational number.

Then, by the definition of what a rational number is,

$$\sqrt[3]{2} = \frac{p}{q} \text{ for some } p, q \in \mathbb{Z}, q \neq 0$$

Furthermore, assume that this fraction cannot be reduced further.

That is p and q have no factors in common.

(In other words, they are coprime with $hcf\,\{p, q\} = 1$)

In consequence, $$2 = \frac{p^3}{q^3}$$

$$p^3 = 2q^3$$

The 2 on the RHS tells us that it is a multiple of 2.

Therefore the LHS must be a multiple of 2.

From part (i), if p^3 is a multiple of 2 then so too is p

Therefore p can be expressed in the form $2a$ for some integer a.

So we now have that,

$$\begin{aligned} (2a)^3 &= 2q^3 \\ 8a^3 &= 2q^3 \\ 4a^3 &= q^3 \end{aligned}$$

The 4 on the LHS tells us that it is a multiple of 2.

Therefore the RHS must be a multiple of 2.

Again, from part (i), if q^3 is a multiple of 2, then so too is q

If p and q are both multiples of 2, they will have a common factor of 2.

This contradicts the statement that p and q have no common factors.

The inescapable conclusion is that $\sqrt[3]{2}$ is an irrational number. □

[6 marks]

Answer 8

For the pattern to always hold, it needs to be proven that, for $n \in \mathbb{N}$,

$$n^2 + (n + 1)^2 + (n(n + 1))^2 = m^2 \text{ for some } m \in \mathbb{N}.$$

$$\begin{aligned}
\text{LHS} &= n^2 + (n + 1)^2 + (n(n + 1))^2 \\
&= n^2 + n^2 + 2n + 1 + n^2(n + 1)^2 \\
&= 2n^2 + 2n + 1 + n^2\left(n^2 + 2n + 1\right) \\
&= 2n^2 + 2n + 1 + n^4 + 2n^3 + n^2 \\
&= n^4 + 2n^3 + 3n^2 + 2n + 1 \\
&= \left(n^2 + n + 1\right)^2 \\
&= m^2 \text{ where } m = n^2 + n + 1 \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[6 marks]

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