1.7 Answers to 1.6 Exercise

Answer 1

(i) Crucially, the displacement vector to the first point is a multiple of the displacement vector to the second point and so the two points lie on a line that passes through the origin.
The length scaling factor between these displacement vectors gives an eigenvalue, λ, of the matrix representing the transformation; $\lambda = -3$

[1 mark]

(ii) $\boldsymbol{v} = k\begin{pmatrix}5\\2\end{pmatrix}$ for any real constant k

[2 marks]

(iii) $y = \dfrac{2}{5}x$ (The line through the origin on which both points lie)

[1 mark]

Answer 2

$$det\,(\,\mathbf{A} - \lambda\mathbf{I}\,) = 0$$

$$det\begin{pmatrix} a-\lambda & b \\ -b & a-\lambda \end{pmatrix} = 0$$

$$(\,a-\lambda\,)(\,a-\lambda\,) + b^2 = 0$$

$$\lambda^2 - 2a\lambda + a^2 + b^2 = 0$$

$$\lambda = \frac{2a \pm \sqrt{4\,a^2 - 4(\,a^2 + b^2\,)}}{2}$$

$$= \frac{2a \pm \sqrt{-4\,b^2}}{2}$$

$$= a \pm b\mathbf{i} \qquad \square$$

[4 marks]

Answer 3

(i) For a repeated eigenvalue the discriminant of the characteristic equation must be zero. $det\,(\,\mathbf{A} - \lambda\mathbf{I}\,) = 0$

$$det\left(\begin{pmatrix} 3-\lambda & k \\ 1 & -1-\lambda \end{pmatrix}\right) = 0$$

$$(\,3-\lambda\,)(-1-\lambda\,) - k = 0$$

$$\lambda^2 - 2\lambda - (\,3+k\,) = 0 \qquad \text{The characteristic equation}$$

$$(-2)^2 + 4(\,3+k\,) = 0 \qquad \text{working on the discriminant}$$

$$k = -4$$

[4 marks]

(ii) The characteristic equation with $k = -4$ becomes $\lambda^2 - 2\lambda + 1 = 0$

from which $(\lambda - 1)^2 = 0$ giving $\lambda = 1$

For the corresponding eigenvector, $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow y = \frac{1}{2}x$

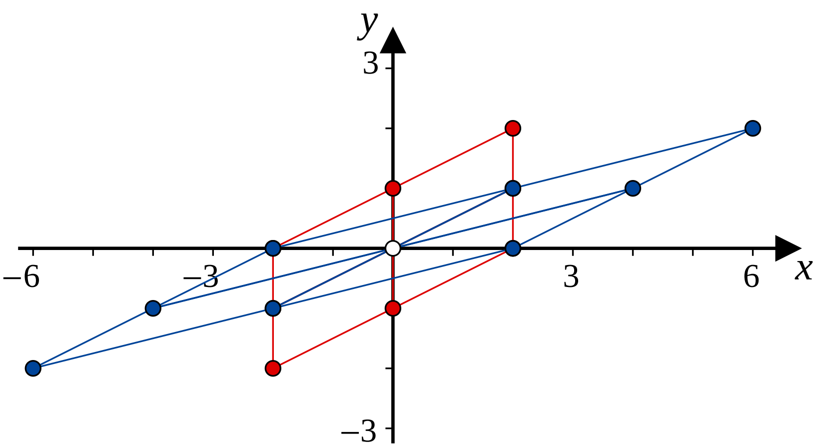
$$\boldsymbol{v} = \mu\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ for any real constant } \mu$$

[3 marks]

(iii)

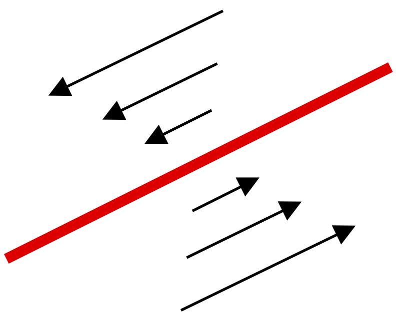
$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 2 & -2 & 0 & 2 & -2 & 0 & 2 \\ 0 & 1 & 2 & -1 & 0 & 1 & -2 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & -4 & -2 & -2 & 0 & 2 & 2 & 4 & 6 \\ -2 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}$$



[4 marks]

(iv) The eigenvalue of 1 along the line $y = \frac{1}{2}x$ means that points on that line do not move. There are no other invariant points or line.
There is a shearing action either side of the point invariant line that increases linearly with the perpendicular distance away from the line.
This can be depicted geometrically as;



[2 marks]

Answer 4

(a)

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\left(\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$det\begin{pmatrix} 3 - \lambda & 2 \\ 2 & 2 - \lambda \end{pmatrix} = 0$$

$$(3 - \lambda)(2 - \lambda) - 4 = 0$$

$$\lambda^2 - 5\lambda + 2 = 0$$

[2 marks]

(b) The Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

$$\mathbf{A}^2 - 5\mathbf{A} + 2 = 0$$

$$\mathbf{A} - 5\mathbf{I} + 2\mathbf{A}^{-1} = 0 \qquad \text{by post-multiplying through by } \mathbf{A}^{-1}$$

$$2\mathbf{A}^{-1} = -\mathbf{A} + 5\mathbf{I}$$

$$\mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A} + \frac{5}{2}\mathbf{I}$$

That is, $\lambda = -\dfrac{1}{2}$ and $\mu = \dfrac{5}{2}$

[3 marks]

Answer 5

Statement: The matrix $\mathbf{A}^n$ has eigenvalue λ^n where λ is an eigenvalue of **A**

Proof by induction: Keep in mind the definition of an eigenvalue;

Definition : Eigenvalues and Eigenvectors
An eigenvector of a matrix **A** is a non-zero column vector $\boldsymbol{x}$ which satisfies the equation $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ where λ is a scalar. The value of the scalar λ is the eigenvalue of the matrix **A** corresponding to the eigenvector $\boldsymbol{x}$.

When $n = 1$, this is true by the above definition.
Assume that for some $n = k$ the statement is true.

That is $\mathbf{A}^k$ has eigenvalue λ^k in which case $\mathbf{A}^k\boldsymbol{x} = \lambda^k\boldsymbol{x}$ is assumed true.
Front-multiply both sides of this by **A**;

$$\mathbf{A}\left(\mathbf{A}^k\boldsymbol{x}\right) = \mathbf{A}\left(\lambda^k\boldsymbol{x}\right)$$

$$\mathbf{A}^{k+1}\boldsymbol{x} = \lambda^k(\mathbf{A}\boldsymbol{x}) \qquad \text{as } \lambda \text{ is a scalar and so commutative}$$

$$\mathbf{A}^{k+1}\boldsymbol{x} = \lambda^k(\lambda\boldsymbol{x}) \qquad \text{by the definition}$$

$$\mathbf{A}^{k+1}x = \lambda^{k+1}x$$

which shows that if it is true that λ^k is an eigenvalue of $\mathbf{A}^k$ then λ^{k+1} is an eigenvalue of A^{k+1}. So if true for $n = k$ then the statement is true for $n = k + 1$ and, as it was true for $n = 1$, by mathematical induction it is true for all $n \in \mathbb{Z}^+$

[6 marks]

Answer 6

(a)

$$det\left(\mathbf{A} - \lambda\mathbf{I} \right) = 0$$

$$\begin{vmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0$$

Expanding along the top row...

$$(1-\lambda)((-4-\lambda)(3-\lambda)-2) - k(2(3-\lambda)-1) - 2(4-(-4-\lambda)) = 0$$

$$\left(1-\lambda\right)\left(\lambda^2+\lambda-14\right) - k\left(5-2\lambda\right) - 2\left(8+\lambda\right) = 0$$

$$\lambda^2 + \lambda - 14 - \lambda^3 - \lambda^2 + 14\lambda - 5k + 2\lambda k - 16 - 2\lambda = 0$$

$$-\lambda^3 + \lambda(13+2k) - 30 - 5k = 0$$

$$\lambda^3 - \lambda(2k+13) + 5(k+6) = 0 \quad \square$$

[3 marks]

(b) **(i)** $\begin{vmatrix} 1-\lambda & k & -2 \\ 2 & -4-\lambda & 1 \\ 1 & 2 & 3-\lambda \end{vmatrix}$ with $\lambda = 0$ becomes $\begin{vmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

so solving $\begin{vmatrix} 1 & k & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 5$

is the same as solving $-5(k+6) = 5$

(The minus sign is undoing the sign flip in the last step of part (a))

$$\therefore \ k = -7$$

[2 marks]

(ii) By the Cayley-Hamilton theorem, $\mathbf{M}^3 + \mathbf{M} - 5\mathbf{I} = \mathbf{0}$

Post-multiplying by $\mathbf{M}^{-1}$ gives $\mathbf{M}^2 + \mathbf{I} - 5\mathbf{M}^{-1} = \mathbf{0}$

$$\mathbf{M}^{-1} = \frac{1}{5}\left(\mathbf{M}^2 + \mathbf{I}\right)$$

$$= \frac{1}{5}\left(\begin{pmatrix} 1 & -7 & -2 \\ 2 & -4 & 1 \\ 1 & 2 & 3 \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{5}\left(\begin{pmatrix} -15 & 17 & -15 \\ -5 & 4 & -5 \\ 8 & -9 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{5}\begin{pmatrix} -14 & 17 & -15 \\ -5 & 5 & -5 \\ 8 & -9 & 10 \end{pmatrix}$$

[5 marks]

Answer 7

Finding the generalised characteristic equation...

$$det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

$$det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$det\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

The Cayley-Hamilton theorem now claims that,

$$\mathbf{M}^2 - (a + d)\mathbf{M} + (ad - bc)\mathbf{I} = \mathbf{0}$$

$$\begin{aligned}
\text{LHS} &= \mathbf{M}^2 - (a + d)\mathbf{M} + (ad - bc)\mathbf{I} \\
&= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a + d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix} + \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\
&= \begin{pmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab - bd + 0 \\ ac + cd - ac - cd + 0 & bc + d^2 - ad - d^2 + ad - bc \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \mathbf{0} \\
&= \text{RHS} \qquad \square
\end{aligned}$$

[8 marks]